MATH 105: PRACTICE PROBLEMS FOR CHAPTER 5: SPRING 2010

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Question 1 : Define the following terms:

(1) What does it mean for a function $f : \mathbb{R}^2 \to \mathbb{R}$ to be bounded?
(2) Define a simple region.

Solution: (1) A function f is bounded if there is some B such that for all points in the domain, the absolute value of f at that point is at most B: $|f(x, y)| \le B$ or $-B \le f(x, y) \le B$. (2) A region $D \subset \mathbb{R}^2$ is x-simple if there are continuous ψ_1 and ψ_2 defined on $[c, d]$ such that

$$\psi_1(y) \ \le \ \psi_2(y)$$

and

$$D \ = \ \{(x, y) : \psi_1(y) \le x \le \psi_2(y) \ \text{ and } \ c \le y \le d\};$$

similarly, D is y-simple if there are continuous functions $\phi_1(x)$ and $\phi_2(x)$ such that

$$\phi_1(x) \ \le \ \phi_2(x)$$

and

$$D \ = \ \{(x, y) : \phi_1(x) \le y \le \phi_2(x) \ \text{ and } \ a \le x \le b\}.$$

If D is both x-simple and y-simple then we say D is simple.

Question 2 : Compute

$$\int_{x=1}^{4} \int_{y=\sqrt{x}}^{x^2} y dy dx.$$

Interchange the order of integration. Write down what the new bounds of integration are, and compute the new double integral.

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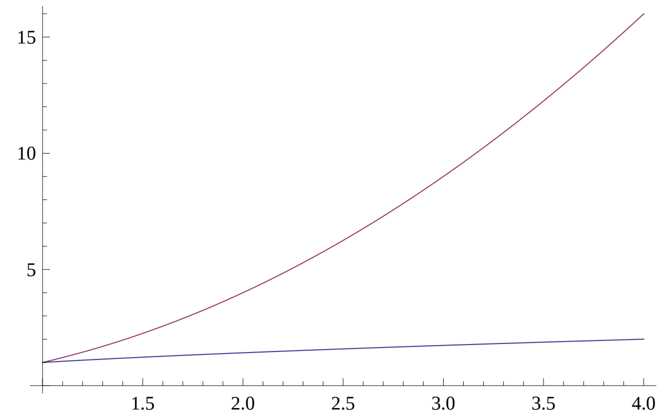

FIGURE 1. Plot of region given by $1 \le x \le 4$ and $\sqrt{x} \le y \le x^2$.

Solution:

$$\begin{aligned}
\int_{x=1}^{4}\int_{y=\sqrt{x}}^{x^2} y\,dy\,dx &= \int_{x=1}^{4} \left.\frac{y^2}{2}\right|_{\sqrt{x}}^{x^2} dx \\
&= \int_{x=1}^{4}\left(\frac{x^4}{2}-\frac{x}{2}\right)dx \\
&= \int_{x=1}^{4}\frac{x^4}{2}dx - \int_{x=1}^{4}\frac{x}{2}dx \\
&= \left.\frac{x^5}{10}\right|_1^4 - \left.\frac{x^2}{4}\right|_1^4 \\
&= \left(\frac{4^5}{10}-\frac{1}{10}\right) - \left(\frac{4^2}{4}-\frac{1}{4}\right) \\
&= \frac{4^5}{10}-\frac{1}{10}-\frac{4^2}{4}+\frac{1}{4} \\
&= \frac{1971}{20}.
\end{aligned}$$

For this problem, it is set up as a y-simple region. We have the lower boundary curve $\phi_1(x) = \sqrt{x}$ and the upper bounding curve $\phi_2(x) = x^2$.

What if we try and do the problem the other way? It is an x-simple region – we need to find the left most boundary $\psi_1(y)$ and the rightmost boundary $\psi_2(y)$. We display the plot in Figure 1.

We see that $\psi_1(y) = \sqrt{y}$ for all y in our range. This arises from the relation $y = x^2$; as we want to solve for x in terms of y, this becomes $x = \sqrt{y}$. What of $\psi_2(y)$? The functional form changes depending on whether or not $y \le 2$. We have

$$\psi_2(y) \;=\; \begin{cases} y^2 & \text{if } 1 \le y \le 2 \\ 4 & \text{if } 2 \le y \le 16. \end{cases}$$

To see this is the answer, note that for $y \leq 2$ the right boundary curve is given by $y = \sqrt{x}$, which when we solve for x in terms of y becomes $x = y^2$. Once $y \geq 2$ then the largest value x takes on becomes 4.

This leads to

$$\begin{aligned}
\int_{y=1}^{16} \int_{x=\psi_1(y)}^{\psi_2(y)} y dx dy &= \int_{y=1}^{2} \int_{x=\sqrt{y}}^{y^2} y dx dy + \int_{y=2}^{16} \int_{x=\sqrt{y}}^{4} y dx dy \\
&= \int_{y=1}^{2} y(y^2 - \sqrt{y}) dy + \int_{y=2}^{16} y(4 - \sqrt{y}) dy \\
&= \left.\frac{y^4}{4}\right|_1^2 - \left.\frac{y^{5/2}}{5/2}\right|_1^2 + \left.2y^2\right|_2^{16} - \left.\frac{y^{5/2}}{5/2}\right|_2^{16} \\
&= \frac{1971}{20}.
\end{aligned}$$

Question 3 : Let f be a function of class C^2. Must

$$\int_{x=0}^{\infty} \int_{y=0}^{\infty} f(x,y) dy dx \;=\; \int_{y=0}^{\infty} \int_{x=0}^{\infty} f(x,y) dx dy,$$

or could there be a function f such that this fails?

Solution: There is an f such that these two examples are not equal. While we saw an example in class and the additional comments, the important point to note is that we are not satisfying all of the conditions of Fubini's theorem, but rather only some of them. Specifically, for Fubini's theorem we need to be integrating over a bounded region (i.e., a region living inside a ginat but finite rectangle).

Question 4 : Let D be the region in the plane where $x, y \geq 0$ and $3x \geq 2y \geq x$. Write down (but do not evaluate) the integral for the function e^{xy} over this region.

Solution: If we were integrating over a finite region, say $1 \leq x \leq 2$, the problem would be a lot harder. We have a y-simple region. For a given value of x, y starts at $x/2$ (this is from $2y \geq x$) and increases to $3x/2$ (this is from $3x = 2y$). Thus the answer is

$$\int_{x=0}^{\infty} \int_{y=x/2}^{3x/2} e^{xy} dy dx.$$

Question 5 : Prove

$$2 \;\leq\; \int_{x=0}^{1} \int_{y=0}^{2} e^{x+y} dy dx \;\leq\; 2e^3.$$

Solution: Let $f(x,y) = e^{x+y}$. As $x \leq 1$ and $y \leq 2$, we are integrating over a rectangle The smallest smallest value is when $x = y = 0$, namely 1; the largest is when $x = 1$ and $y - 2$, or Professor e^3. The claim now follows by taking the values of the upper and lower bounds and multiplying by the area of the region.