MATH 105: PRACTICE PROBLEMS FOR CHAPTER 6 AND SEQUENCES AND SERIES: SPRING 2010

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Question 1 : State the change of variable theorem in the plane. How does the element $dxdy$ transform in polar coordinates? How does $dxdydz$ transform in cylindrical and spherical coordinates? Let D be the disk of radius 3 centered at the origin. Evaluate the integral of $f(x,y) = x^2 + y^3$.

Solution: **Change of Variables Theorem:** Let S be an elementary region in the xy-plane (such as a disk or parallelogram for example). Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible and differentiable mapping, and let $T(S)$ be the image of S under T. Then

$$\int\int_S 1 \cdot dxdy \;=\; \int\int_{T(S)} 1 \cdot \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial y}\right| dudv,$$

or more generally

$$\int\int_S f(x,y) \cdot dxdy \;=\; \int\int_{T(S)} f\left(T^{-1}(u,v)\right) \cdot \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial y}\right| dudv.$$

The volume elements transform as: Polar:

$$dxdy \;=\; rdrd\theta,$$

Cylindrical:

$$dxdydz \;=\; rdrd\theta dz$$

and in Spherical we have

$$dxdydz \;=\; \rho^2 \sin\phi d\rho d\theta d\phi.$$

In Cartesian coordinates the integral is

$$\int\int_D f(x,y)dydx \quad = \quad \int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^2+y^3)dydx.$$

As we are integrating over a disk, it is natural to convert to polar coordinates. Before doing so, however, it is worth noting that the y^3 piece integrates to zero. The easiest way to see this is that this is an odd function over a region that is symmetric. More formally, the integral of y^3dy becomes $y^4/4$, and evaluating at the boundary points gives zero. Thus it suffices to integrate x over the region. This analysis is extremely common and important – often one can greatly simplify the computations to be done by a little inspection in the beginning.

We are now left with

$$\int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x^2 dydx.$$

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If we integrate as written, we would have

$$\int_{x=-3}^{3} 2x^2\sqrt{9-x^2},$$

and while doable this is not a function whose integral we remember. If we shift to polar coordinates we have this integral equals

$$\int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\cos^2(\theta)\cdot r dr d\theta.$$

While the r-integral is relatively straightforward, we need the anti-derivative of $\cos^2(\theta)$ (or more precisely, we need to know the definite integral of this from 0 to 2π). The following trick simplifies our life. By symmetry, integrating x^2 over the disk of radius 3 gives the same contribution as integrating the function y^2. Thus integrating x^2 gives the same answer as integrating $\frac{x^2+y^2}{2}$, and this is very nice in polar coordinates, being just $r^2/2$. We thus have

$$\begin{aligned}
\int\int_D f(x,y)dydx &= \int\int_D (x^2+y^3)dydx \\
&= \int\int_D x^2 dydx \\
&= \int\int_D \frac{x^2+y^2}{2}dydx \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{3}\frac{r^2}{2}\cdot r dr d\theta \\
&= \int_{\theta=0}^{2\pi}\left[\int_{r=0}^{3}\frac{r^3}{2}dr\right]d\theta \\
&= \int_{\theta=0}^{2\pi}\left.\frac{r^4}{8}\right|_0^3 d\theta \\
&= \int_{\theta=0}^{2\pi}\frac{1}{8}d\theta \ = \ \frac{2\pi}{8} \ = \ \frac{\pi}{4}.
\end{aligned}$$

We urge the reader to try and do the polar integrations directly. The following trig identity may be useful:

$$\cos(2\theta) \ = \ \cos(\theta)\cos(\theta)-\sin(\theta)\sin(\theta) \ = \ 2\cos^2(\theta)-1,$$

or

$$\cos^2(\theta) \ = \ \frac{\cos(2\theta)+1}{2}.$$

Question 2 : Integrate the function z over the unit ball (i.e., all points (x,y,z) with $x^2+y^2+z^2 \le 1$).

Solution: In Cartesian coordinates we have

$$\begin{aligned}\int\int\int_S z dz dy dx &= \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z=-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} z dz dy dx \\ &= \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left[\int_{z=-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} z\right] dz dy dx \\ &= \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{z^2}{2}\Big|_{\sqrt{1-x^2-y^2}}^{-\sqrt{1-x^2-y^2}} = 0.\end{aligned}$$

To practice using spherical coordinates, we give that solution as well. We have

$$\int\int\int_S z dz dy dx = \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{1} \rho\cos\phi\cdot\rho^2\sin\phi d\rho d\theta d\phi,$$

where we used $z = \rho\cos\phi$. The θ-integral is just 2π and the ρ-integral is just $1/4$. Here we are using Fubini's theorem to do these integrals first. We can as our function is continuous and, in spherical coordinates, the sphere becomes a box, and thus it is easy to justify interchanging orders. We have

$$\begin{aligned}\int\int\int_S z dz dy dx &= 2\pi\cdot\frac{1}{4}\int_{\phi=0}^{\pi}\cos\phi\cdot\sin\phi d\phi \\ &= \frac{\pi}{2}\left[-\cos^2\phi\Big|_0^{\pi}\right] = 0.\end{aligned}$$

We could also argue by symmetry that the integral of z over the unit ball (or sphere) vanishes, as it is an odd function and we're integrating over a symmetric region with respect to its oddness.

Note by symmetry the integrals of x or y over the unit sphere also vanishes, though the angular integrals would be a bit more involved, they won't be too bad.

Question 3 : Compute the limits of the following sequences, or prove they do not exist: (1) $a_n = \frac{n^2+3}{n^3+2}$; (2) $b_n = \frac{\cos(n^2)}{n}$; (3) $c_n = \frac{n^2+3}{n!}$.

Solution: We may use L'Hopital's rule for (1), as the limit is infinity over infinity (and thus we cannot use the limit of a quotient is the quotient of a limit). We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n}{3n^2} = \lim_{n\to\infty}\frac{2}{6n}.$$

At this point we no longer have infinity over infinity, and we can evaluate it immediately and find it equals zero.

For (2), note $-1 \le \cos(n^2) \le 1$, so as the numerator is bounded by 1 and the denominator tends to infinity, the sequence clearly tends to 0.

For (3), we can use the comparison test. Note $n^2 + 3 \le 3n^2$ for n large. Further, $n! > 6n^3$ for n large, which gives $|c_n| \le 3/6n$, and thus this sequence too converges to zero.

Question 4 : State whether or not the following converge, justifying your reasons: (1) $\sum_{n=0}^{\infty} \frac{2^n}{3^n}$; (2) $\sum_{n=0}^{\infty} \frac{n!}{n!^2}$; (3) $\sum_{n=0}^{\infty} \frac{n!}{(2n)!}$; (4) $\sum_{n=0}^{\infty} \frac{2n}{e^{n^2}}$.

Solution: For (1), note this is the same as $\sum_{n=0}^{\infty}(2/3)^n$. This is just a geometric series with ratio of $r = 2/3$; as $|r| < 1$, the series converges.

For (2), the denominator is $n! \cdot n!$. We may cancel one of the $n!$ and find we have $\sum_{n=0}^{\infty} 1/n!$. This is just e, as

$$e^x \ = \ \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We could also see that it converges by using the ratio test.

For (3), note $0 \le n!/(2n)! \le 1/n!$, and thus by the comparison test the series converges. Why is this true? There are $2n$ terms in $(2n)!$, and we thus have

$$\frac{n!}{(2n)!} \ = \ \frac{n!}{(2n)\cdot(2n-1)\cdots(n+1)\cdot n!} \ = \ \frac{1}{(2n)\cdot(2n-1)\cdots(n+1)}.$$

There are thus n terms in the denominator here, and since $2n > n$, $2n-1 > n-1$, $2n-2 > n-2$ and so on down to $n+1 > 1$, we have $(2n)\cdot(2n-1)\cdots(n+1) > n\cdot(n-1)\cdots 1$, or

$$\frac{n!}{(2n)!} \ = \ \frac{1}{(2n)\cdot(2n-1)\cdots(n+1)} \ \le \ \frac{1}{n!}.$$

If we use the ratio test we find

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \ = \ \lim_{n\to\infty} \frac{(n+1)!/(2n+2)!}{n!/(2n)!} \ = \ \lim_{n\to\infty} \frac{(n+1)!(2n)!}{n!(2n+2)!} \ = \ \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} \ = \ 0;$$

thus the series converges.

Finally, for (4) we use the integral test. For n large the terms are decreasing, and the series converges / diverges depending on whether or not the integral

$$\int_1^{\infty} \frac{2x}{e^{x^2}} dx$$

converges or diverges. The integral equals

$$\int_1^{\infty} e^{-x^2} 2x dx \ = \ -e^{-x^2}\Big|_0^{\infty} \ = \ 0-(-1) \ = \ 1$$

which is finite, implying the series converges.