MATH 105: QUIZ 3

INSTRUCTOR: STEVEN MILLER (SJM1@WILLIAMS.EDU)

NOTE: Write your name *and* section number. Each question is worth 10 points. You have 30 minutes to do the quiz, which is closed book. Do not use computers or any other devices.

Question 1: Find the limit as (x, y) tends to $(0, 0)$ of $4 + xy^2$.

Solution: As the limit of a sum is the sum of the limits and a limit of a product is the product of the limits, we have

$$\begin{aligned} \lim_{(x,y)\to(0,0)} (4 + xy^2) &= \lim_{(x,y)\to(0,0)} 4 + \lim_{(x,y)\to(0,0)} xy^2 \\ &= 4 + \lim_{(x,y)\to(0,0)} x \lim_{(x,y)\to(0,0)} y \lim_{(x,y)\to(0,0)} y \\ &= 4 + 0^3 \ = \ 4. \end{aligned}$$

Question 2: State whether or not each function is continuous or discontinuous at the stated point; no proofs required: (a) Let $f(x) = 1/x$ if x is not zero and 0 if x is zero; is this function continuous at $x = 0$? (b) Is $g(x, y) = x^3y^2 - 2x + 5$ continuous at $(x, y) = (0, 0)$? (c) Let $h(x, y)$ equal $x\cos(1/y)$ if y is not zero and 0 if y is zero; is this function continuous at $(x, y) = (0, 0)$?

Solution: (a) The function is not continuous. As x approaches zero from above, $f(x)$ tends to ∞, while as x approaches zero from below, $f(x)$ tends to $-\infty$. The left and right hand limits are unequal, and clearly not equal to $f(0)$ which is zero. (b) This function is continuous. The fastest way to see this is that sums, differences, products and multiples of continuous functions are continuous. (c) Thus function is continuous. No matter what value of (x, y) we take, we have $|h(x, y)| \le |x|$. The reason is that $|\cos(u)| \le 1$ for any u. Thus if $y \neq 0$ we have $|h(x, y)| = |x| \cdot |\cos(1/y)| \le |x|$, while if $y = 0$ we have $|h(x, y)| = 0$. As $(x, y) \to (0, 0)$ we clearly have $|x| \to 0$, and thus $h(x, y)$ tends to 0. As $h(0, 0) = 0$, we see $h(x, y)$ is continuous at $(0, 0)$.

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Solution: As $x \to 1$ the numerator tends to 1-3+3-1 = 0, and the denominator tends to 2-2 = 0. Thus one solution is to use L'Hopital's rule, which tells us that this limit is the same as $\lim_{x\to 1} \frac{3x^2-6x+3}{2} = 0$. Alternatively, we could notice that the numerator is just $(x-1)^3$ and the denominator is $2(x-1)$. Thus our quotient is the same as $(x-1)^3/2(x-1) = (x-1)^2/2$. We can now easily take the limit as $x \to 1$, and obtain $0^2/2 = 0$.

Question 3: Let $f(x,y,z) = x^2y^2z^2 + xyz$. Find the partial derivatives of f with respect to x, y and z; in other words, compute $\partial f/\partial x, \partial f/\partial y, \partial f/\partial z$.

Solution: When taking the partial derivatives, we keep all variables constant but one, the variable we are differentiating with respect to. For example, for our problem y and z are to be considered constant when computing $\partial f/\partial x$. Let us write a for y and b for z to remind ourselves that these are constants. Then our problem is the same as computing $dg/dx = g'(x)$ for $g(x) = a^2b^2x^2 + abx$. We have

$$\frac{dg}{dx} = 2a^2b^2x + ab,$$

which gives

$$\frac{\partial f}{\partial x} = 2xy^2z^2 + yz.$$

We could compute the other two partial derivatives similarly, but it is faster to note that our original function is symmetric with respect to the three variables, and thus a similar computation gives

$$\frac{\partial f}{\partial y} = 2x^2yz + xz, \quad \frac{\partial f}{\partial z} = 2x^2y^2z + xy.$$