

MATH 105: QUIZ 4

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NOTE: Write your name *and* section number. Each question is worth 10 points. You have 30 minutes to do the quiz, which is closed book. Do not use computers or any other devices.

Question 1: Let $f(x, y)$ be a continuously differentiable function. Define what it means for f to have a local extremum at $(2, 3)$, and give a condition to find local extrema.

Solution: Our function f has a local extremum at $(2, 3)$ if it has a local maximum or a local minimum at $(2, 3)$. For example, if it has a local maximum then this means that for all (x, y) sufficiently close to $(2, 3)$, $f(2, 3) \geq f(x, y)$. A condition for a local extremum is that the gradient must vanish at that point, or in this case

$$(\nabla f)(2, 3) = \left(\frac{\partial f}{\partial x}(2, 3), \frac{\partial f}{\partial y}(2, 3) \right) = (0, 0).$$

Question 2: Find the minimum of $x^2 + y^2 - 4xy$ subject to $x^2 + y^2 = 1$.

Solution: Letting $f(x, y) = x^2 + y^2 - 4xy$ and $g(x, y) = x^2 + y^2 = 1$, using the Method of Lagrange Multipliers we find

$$\nabla f = (2x - 4y, 2y - 4x) = \lambda(2x, 2y) = \lambda \nabla g.$$

We thus have

$$2x - 4y = 2\lambda x, \quad 2y - 4x = 2\lambda y,$$

which implies

$$2(1 - \lambda)x = 4y, \quad 2(1 - \lambda)y = 4x.$$

If $\lambda = 1$ then $x = y = 0$, which does not satisfy the constraint $x^2 + y^2 = 1$. Thus we may assume $\lambda \neq 1$. Taking the ratio of these two equations gives $\frac{x}{y} = \frac{y}{x}$, or $x^2 = y^2$. Thus $x = \pm y$. As $x^2 + y^2 = 1$, we find $2x^2 = 1$ or $x = \pm\sqrt{2}/2$. The extrema are thus the four points $(\pm\sqrt{2}/2, \pm\sqrt{2}/2)$, and we see the minimum is when $x = y$. Substituting $x = y = \pm\sqrt{2}/2$ gives the minimum value of $x^2 + y^2 - 4xy$ is 3.

Alternatively, we could have solved this problem by noting that since $x^2 + y^2 = 1$, instead of minimizing $x^2 + y^2 - 4xy$ we might as well just minimize $-4xy$, as the $x^2 + y^2$ part is just 1. This leads to the simpler Lagrange multiplier problem with $h(x, y) = -4xy$ and $g(x, y) = x^2 + y^2 = 1$. We find

$$\nabla h = (-4y, -4x) = \lambda(2x, 2y) = \lambda \nabla g.$$

Clearly $\lambda \neq 0$, and thus taking ratios gives $\frac{-4y}{-4x} = \frac{2\lambda x}{2\lambda y}$, so once again $x^2 = y^2$.

Find $\int_0^1 ye^{xy} dx$.

Solution: We are integrating with respect to x ; thus we should treat y as a constant. As $\frac{\partial}{\partial x} e^{xy} = ye^{xy}$, we see that

$$\int_0^1 ye^{xy} dx = e^{xy} \Big|_{x=0}^1 = e^y - 1.$$