ITERATED PARTIALS (SECTION 3.1)

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1. Introduction to Iterated Partials

Typically in mathematics, order does tend to matter. It would be great if this were not the case, as then we would not have to be careful in how we evaluate expressions. For example, in a perfect world we would have:

$$f(g(x)) = g(f(x))$$

This would hold for any two functions of f and g. Thus, we would not need to worry about the order in which we evaluate the composition. As is readily seen from some quick searches, this fails for most choices of f and g. For example, in a slight abuse of notation, the square-root of a sum is not the sum of the square-roots:

$$\sqrt{x^2 + y^2} \neq \sqrt{x^2} + \sqrt{y^2}$$

To assume that the sum of a square root will always equal the components of the square root is false, even though it may hold for some special values of x and y. Similarly, $\cos(x^2)$ is typically not $\cos^2(x)$.

Fortunately, there are many instances where order does not matter. One of the most important is in iterated partial derivatives. We will see later that if the second order derivatives of f are continuous, then

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

In other words, we obtain the same answer if we first differentiate with respect to x and then with respect to y as we would if we first differentiate with respect to y and then with respect to x.

Before stating the general result, let's look at an example. Consider the function

$$f(x, y) = \cos(xy^2).$$

If we take the derivatives with respect to x and y we find these are the partial derivatives.

$$\frac{\partial f}{\partial x} = -y^2 - \sin(xy^2)$$

$$\frac{\partial f}{\partial y} = -2xy\sin(xy^2)$$

If we take this even further, and we talk the partials of x and y again with respect to the partial derivatives we have already computed, we get another set of four partial derivatives.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = -y^4\cos(xy^2)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -2x\sin(xy^2) - 2xy^2\cos(xy^2)$$

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$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -2y\sin(xy^2) - 2xy^3\cos(xy^2)$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -2y\sin(xy^2) - 2xy^3\cos(xy^2)$$

Note that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. For this function, the order of differentiation does not matter: we may first differentiate with respect to x and then with respect to y, or first with respect to y and then with respect to x.

2. Equality of Mixed Partials

Definition 2.1. *We say f is $\mathcal{C}^2$ (or of class $\mathcal{C}^2$) if all partial derivatives up to the second order exist and are continuous.*

Theorem 2.2 (Equality of Mixed Partials)**.** *If f is $\mathcal{C}^2$, then*

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

The Law of Equality of Mixed Partials holds for any function of class $\mathcal{C}^2$; however, if f is not $\mathcal{C}^2$ then its conclusions need not hold, as the following example shows. Consider

$$f(x) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0). \end{cases}$$

Calculating the mixed partials, we find

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1 \qquad \frac{\partial^2 f}{\partial y \partial x}(0,0) = -1.$$

To see that these are the answers, one must return to the definition of the derivative for the values at $(0,0)$, though we may differentiate directly away from $(0,0)$. We leave it to the reader to compute the partial derivatives and see that they are not continuous, and thus this does not violate the theorem as the conditions are not satisfied.

3. Proof of the Theorem of Equality of Mixed Partials

Proof of Theorem 2.2. Suppose we have a 2 dimensional square on the x,y coordinate plane. The upper left corner of the rectangle's coordinates are $(x_0, y_0 + \Delta y)$, the upper right corner of the rectangle's coordinates are $(x_0 + \Delta x, y_0 + \Delta y)$, the lower left corner has coordinates (x_0, y_0) while the lower right corner is $(x_0 + \Delta x, y_0)$. See Figure 1 for a picture. We consider the following function

$$S(\Delta x, \Delta y) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) - f(x_0, y_0 + \Delta y) + f(x_0, y_0);$$

note we are evaluating our function f at the four corners of the rectangle, taking it with positive signs at the upper right and lower left and with minus signs at the other two corners. The goal is to compute

$$\lim_{\Delta x, \Delta y \to 0} \frac{S(\Delta x, \Delta y)}{\Delta x \Delta y}$$

two different ways; one way will give $\frac{\partial^2 f}{\partial x \partial y}$ and the other will give $\frac{\partial^2 f}{\partial y \partial x}$. This will prove the theorem because if the two mixed partials are equal to the same limit, they must be equal to each other. We first

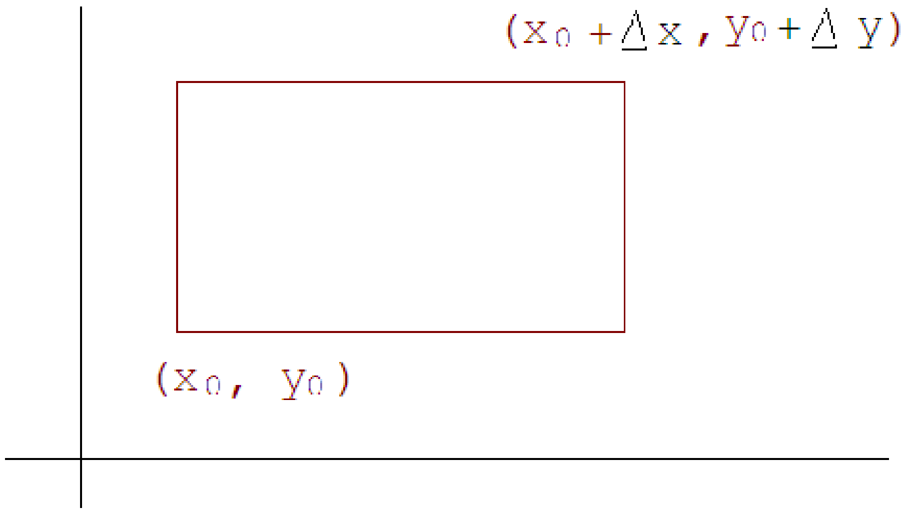

FIGURE 1. This is the figure that was made in reference to the proof of the Equality of Mixed Partials. As we can see, we have a visualization of the points of the square, and knowledge of each point that denotes either a negative or a positive value.

analyze the quotient $S(\Delta x, \Delta y)/\Delta x \Delta y$ and see that in the limit it equals $\frac{\partial^2 f}{\partial y \partial x}$. To do so we introduce the function

$$g(x) = f(x, y_0 + \Delta y) - f(x, y_0),$$

and we notice that

$$S(\Delta x, \Delta y) = g(x + \Delta x) - g(x).$$

Using the Mean Value Theorem[1], we have

$$g(x + \Delta x) - g(x) = \frac{\partial g}{\partial x}(\tilde{x})\Delta x$$

for some $\tilde{x} \in (x_0, x_0 + \Delta x)$, which implies

$$S(\Delta x, \Delta y) = g(x + \Delta x) - g(x) = \frac{\partial g}{\partial x}(\tilde{x})\Delta x\ .$$

As g is the difference of f evaluated at two points, we have for $\tilde{x} \in (x_0, x_0 + \Delta x)$

$$S(\Delta x, \Delta y) = \frac{\partial g}{\partial x}(\tilde{x})\Delta x \ = \left[\frac{\partial f}{\partial x}(\tilde{x}, y_0 + \Delta y) - \frac{\partial f}{\partial x}(\tilde{x}, y_0)\right]\Delta x.$$

We apply the Mean Value Theorem to the function $\frac{\partial f}{\partial x}$, noting that the x-coordinate is the same and the y-coordinate is varying. Thus in using the Mean Value Theorem for our expression above, the derivative of $\frac{\partial f}{\partial x}$ that enters is the derivative with respect to y (as the first coordinate is fixed, we are just applying

[1]Recall the Mean Value Theorem says the following: if $h(x)$ is a differentiable function, then $\frac{h(b)-h(a)}{b-a} = h'(c)$ for some c in $[a, b]$.

the one-dimensional Mean Value Theorem to $\frac{\partial f}{\partial x}$, viewing it as only a function of the second coordinate, y). We find for some $\tilde{y} \in [y_0, y_0 + \Delta y]$ that

$$S(\Delta x, \Delta y) = \left[\frac{\partial f}{\partial x}(\tilde{x}, y_0 + \Delta y) - \frac{\partial f}{\partial x}(\tilde{x}, y_0)\right] \Delta x = \frac{\partial^2 f}{\partial y \partial x}(\tilde{x}, \tilde{y}) \Delta x \Delta y.$$

Dividing by $\Delta x \Delta y$ yields

$$\frac{\partial^2 f}{\partial y \partial x}(\tilde{x}, \tilde{y}) = \frac{S(\Delta x, \Delta y)}{\Delta x \Delta y}.$$

We now take the limit as $\Delta x, \Delta y \to 0$. We finally use our assumption that f is of class $\mathcal{C}^2$ (*we should have to use this somewhere!*). We have

$$\lim_{\Delta x, \Delta y \to 0} \frac{\partial^2 f}{\partial y \partial x}(\tilde{x}, \tilde{y}) = \lim_{\Delta x, \Delta y \to 0} \frac{S(\Delta x, \Delta y)}{\Delta x \Delta y}.$$

By our continuity assumption, the left hand side is just $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$, because $\tilde{x} \to x$ and $\tilde{y} \to y$. We have thus shown that

$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \lim_{\Delta x, \Delta y \to 0} \frac{S(\Delta x, \Delta y)}{\Delta x \Delta y}.$$

As $S(\Delta x, \Delta y)$ is symmetric, similar calculations shown above is also $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$, thus completing the proof. □