Math 105: Multivariable Calculus
Seventeenth Lecture (3/17/10)

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Summary for the Day

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- Fast Taylor Series.
- Critical Points and Extrema.
- Constrained Maxima and Minima.

Fast Taylor Series

Notation: f twice differentiable function

- Gradient: $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$.
- Hessian:

$$Hf = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}.$$

Second Order Taylor Expansion at $\vec{x}_0$

$$f(\vec{x}_0) + (\nabla f)(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) + \frac{1}{2}(\vec{x} - \vec{x}_0)^{\mathrm{T}}(Hf)(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

where $(\vec{x} - \vec{x}_0)^{\mathrm{T}}$ is the row vector which is the transpose of $\vec{x} - \vec{x}_0$.

Example

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and

$$(Hf)(x,y) \ = \ \begin{pmatrix} -\sin(x+y) + 6(x+1)^2 y & -\sin(x+y) + 3(x+1)^2 \\ -\sin(x+y) + 3(x+1)^2 & -\sin(x+y) \end{pmatrix},$$

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so

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which implies the second order Taylor expansion is

$$0 + (1,2) \cdot (x,y) + \frac{1}{2}(x,y) \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \ x + 2y + \frac{1}{2}(x,y) \begin{pmatrix} 3y \\ 3x \end{pmatrix} \ = \ x + 2y + 3xy.$$

Idea: Use Taylor expansions in one-variable to simplify expansions in several variables.

Key observation: Second Order Taylor Series involves combinations of $1, x, y, x^2, xy, y^2$; any higher order terms do not appear (such as x^3, x^2y, xy^2, y^3).

Method: Expand as a function of one variable, keeping only the appropriate order, and then substitute.

Applicability: Works for quantities such as $\sin(x+y)$ or $\log(1+x^2y)$, but not $\sin(\sqrt{x}+\sqrt[3]{y})$.

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Take $u = x+y$, $v = x$ and find

$$\sin(x+y)+(x+1)^3 y \;=\; (x+y+\cdots)+(1+3x+3x^2+\cdots)y$$

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Take $u = x + y$, $v = x$ and find

$$\sin(x+y)+(x+1)^3 y = (x+y+\cdots)+(1+3x+3x^2+\cdots)y$$

We now multiply out, keeping only terms of degree 2:

$$x + y + y + 3xy = x + 2y + 3xy.$$

Note recover previous answer with less work!

Critical Points and Extrema

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Candidates for Extrema

Let $f : [a, b] \to \mathbb{R}$. If f has an extrema at c then either $f'(c) = 0$ (so c is a critical point) or $c = a$ or $c = b$.

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Note: Major real analysis theorem that any continuous function on a closed, bounded set attains its maxima and minima; can fail for functions on open sets: $f(x) = \frac{1}{x} + \frac{1}{x-1}$.

Definitions

Local extrema

A function f has a local maximum at $\vec{x}_0$ if there is a ball B about $\vec{x}_0$ such that

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Critical point

A point $\vec{x}_0$ is a critical point of f if

$$(Df)(\vec{x}_0) = (\nabla f)(\vec{x}_0) = \vec{0}.$$

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Soln: Solve $\nabla f = \overrightarrow{0}$. We have

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We thus have $y = -2x/3$ from the first equation and $y = -3x/2$ from the second. Thus the only solution is $x = y = 0$. Alternatively, note $5x + 5y = 0$ so $x = -y$ and then $-2y + 3y = 0$ yields $y = 0$.

Find the critical points of $f(x,y) = x^2 - 3xy + 5x - 2y + 6y^2 + 8$.

Soln: Solve $\nabla f = \overrightarrow{0}$. Have

$$(\nabla f)(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x - 3y + 5, -3x - 2 + 12y),$$

so to equal the zero vector must have

$$2x - 3y + 5 = 0, \quad \text{and} \quad -3x + 12y - 2 = 0.$$

Two equations in two unknowns. We have

$$2x - 3y = -5, \quad -3x + 12y = 2.$$

Many ways to solve. We could multiply the first equation by 4 and add it to the second. Cancels all y terms, leaves us with $8x - 3x = -20 + 2$, or $5x = -18$ or $x = -18/5$. As $y = \frac{2x+5}{3}$, this implies $y = -\frac{11}{15}$.
Another way is to isolate y as a function of x using the first equation, and substitute this into the second. We find $2x - 3y = -5$, so $y = \frac{2x+5}{3}$. Substituting this into the second equation yields

$$-3x - 2 + 12\frac{2x+5}{3} = 0,$$

which implies

$$-3x - 2 + 8x + 20 = 0,$$

or

$$x = -\frac{18}{5},$$

exactly as before.

First Derivative Test for Local Extrema

Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable on an open set $U \subset \mathbb{R}^n$. If $\overrightarrow{x}_0 \in U$ is a local extremum then $(\nabla f)(\overrightarrow{x}_0) = \overrightarrow{0}$.

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As function of one variable, max implies $A'(0) = 0$, thus $(\nabla f)(\vec{x}_0) \cdot \vec{v} = \vec{0}$ for all $\vec{v}$ and hence $(\nabla f)(\vec{x}_0) = \vec{0}$ (take $\vec{v} = (\nabla f)(\vec{x}_0)$).

There is a second derivative test, but without linear algebra it looks like magic.

We'll discuss some special cases on Friday.

Constrained Extrema
and Lagrange Multipliers

Method of Lagrange Multipliers

Let $f, g : U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^n$. Let S be the level set of value c for the function g, and let $f|_S$ be the function f restricted to S (in other words, we only evaluate f at $\overrightarrow{x} \in U$). Assume $(\nabla g)(\overrightarrow{x}_0) \neq \overrightarrow{0}$. Then $f|_S$ has an extremum at $\overrightarrow{x}_0$ if and only if there is a λ such that $(\nabla f)(\overrightarrow{x}_0) = \lambda(\nabla g)(\overrightarrow{x}_0)$.

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S level set, path $c(t)$ in S with $c(0) = \vec{x}_0$ and $c'(0) = \vec{v}$:

$$\frac{d}{dt} g(c(t))\Big|_{t=0} = (\nabla g)(c(0))c'(0) \;=\; (\nabla g)(\vec{x}_0) \cdot \vec{v} \;=\; 0,$$

where it vanishes as $g(c(t))$ is constant on S.

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Thus $(\nabla g)(\vec{x}_0)$ and $(\nabla f)(\vec{x}_0)$ perpendicular to all tangent directions, *only one direction left and thus parallel!*.

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Thus $(\nabla g)(\vec{x}_0)$ and $(\nabla f)(\vec{x}_0)$ perpendicular to all tangent directions, *only one direction left and thus parallel!*.

Interpretation: $(\nabla g)(\vec{x}_0)$ is normal to surface, says at max/min $(\nabla f)(\vec{x}_0)$ is normal to surface, else increases by flowing in appropriate direction.

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$$\nabla f = (1, -1, 1), \quad \nabla g = (2x, 2y, 2z).$$

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We find

$$2\lambda x \;=\; 1, \quad 2\lambda y \;=\; -1, \quad 2\lambda z \;=\; 1.$$

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As $\lambda \neq 0$, we have $x = z = -y$. We still have another equation to use, namely $g(x,y,z) = 2$. There are several ways to proceed. We can solve and find $x = z = 1/2\lambda$, $y = -1/2\lambda$, and thus

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$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 2,$$

which implies $3/4\lambda^2 = 2$ or $\lambda^2 = 3/8$, which yields $\lambda = \pm\sqrt{3/8}$ and points

$$(1/2\sqrt{3/8}, -1/2\sqrt{3/8}, 1/2\sqrt{3/8}), \quad (-1/2\sqrt{3/8}, 1/2\sqrt{3/8}, -1/2\sqrt{3/8}).$$

Find the extrema of $f(x, y) = x$ subject to $g(x, y) = x^2 + 2y^2 = 3$.

Soln: Have $\nabla f = (1, 0)$, $\nabla g = (2x, 4y)$ and at an extremum

$$(1, 0) \;=\; \lambda(2x, 4y).$$

Implies $1 = 2\lambda x$ and $0 = 4\lambda y$. Thus $y = 0$, but don't know x and λ, only their product (which is 1/2). All is not lost as know $x^2 + 2y^2 = 3$. As $y = 0$, we find $x^2 = 3$ so $x = \pm\sqrt{3}$. We could now find λ (it is $\pm 1/2\sqrt{3}$); however, there is no need. Only care about λ b/c helps us find where f has an extremum. As know the x and y coordinates, have all the needed info. Thus the extrema occur at $x = \pm\sqrt{3}$.

We could have predicted this. Function depends only on x and is constrained to lie on an ellipse. Want x-extension as large as possible, means taking $y = 0$ and being at the extremes of the major-axis.