Math 105: Review to Stokes

Purpose is to review material at course end conclude with statement of at least one of Green/Gauss/Stokes Thm.

This is a massive generalization of the Fundamental Calc

Briefly, moving derivative from "function" to "region"

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \]

where \( \partial C \) is boundary of \( S \)

and \( d\mathbf{r} \) is derivative of \( \mathbf{F} \)

Sec 4.2: Arc Length

If \( \mathbf{C}(t) = (x(t), y(t), z(t)) \)

Then \( \mathbf{C}'(t) = (x'(t), y'(t), z'(t)) \) is speed

and length of path \( C(t) \) from \( a \) to \( t \) is

\[ s = \int_a^t \| \mathbf{C}'(t) \| \, dt = \int_a^t \sqrt{C_x'(t)^2 + C_y'(t)^2 + C_z'(t)^2} \, dt \]

In most of the time these integrals are hard to do

So reduces to a Calc-II problem!

So comes down to parameterizing a curve

- Stokeses -(-)
Sec 4.2: Arc Length (cont)

Justification for formula

Polyhedral approximation
Chop into many small time steps

\[ t_i, t_{i+1} = t_i + \Delta t, C(t_i) \]

Length of red line segment is \( \| C(t_{i+1}) - C(t_i) \| \)

Using Riemann sums, note

\[ X \text{-coord is } X(t_{i+1}) - X(t_i) = X'(t_i) \Delta t \approx X'(t_i) \Delta t \]
\[ Y(t_{i+1}) - Y(t_i) = Y'(t_i) \Delta t \approx Y'(t_i) \Delta t \]
\[ Z(t_{i+1}) - Z(t_i) = Z'(t_i) \Delta t \approx Z'(t_i) \Delta t \]

Yields red line segment has length \( \approx \sum \| C'(t_i) \| \Delta t \)

So length \( \approx \sum \int_{t_i}^{t_{i+1}} \| C'(t) \| dt \rightarrow \int_{t_0}^{t_f} \| C'(t) \| dt \)

What does this review?

- Parametrizing curves
- Riemann sums
- \( \text{MVT} \)

\[ \text{Suggested Problems: } #12, #18 \]
Section 7.1: The Path Integral

Definition: The path integral, or integral of \( f \) along the path \( C \) and denoted \( \int_C f \) ds is \( \int_a^b f(c(t)) \|C'(t)\| \, dt \) where \( C: [a,b] \to \mathbb{R}^3 \) is a \( C^1 \) map.

Guide: If \( f = 1 \) get arc-length.
Think of \( f \) as a density.

\( \Rightarrow \) See pg 424 for interpretation if \( C: [a,b] \to \mathbb{R}^2 \)
\( \Rightarrow \) gives area of fence or height \( \phi(t) \) at \( C(t) \)

Ex: \( f(x,y,z) = x+y+z \), \( C(t) = (\sin t, \cos t, t) \) \( \text{as } t \leq 2\pi \)

What Does This Review

\( \Rightarrow \) Parametrizing curves
\( \Rightarrow \) Derivatives and integrals

\[ \int_{L^2} \Phi \cdot n \, ds \]
\[ \int_{L^2} \Phi \cdot n \, ds \]
Section 7.2: Line Integrals

Definition: The line integral of a vector field \( \mathbf{F} \) along a path \( C \), is

\[
\int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(c(t)) \cdot c'(t) \, dt
\]

Interpretation: pages 429 - 430: Work done

Let if take \( \mathbf{F} \) to be \( \mathbf{c}'(t)/\|\mathbf{c}(t)\| \) (unit tangent)
Then get arc length

Notation: \( \mathbf{F} = (F_1, F_2, F_3) \), other write

\[
\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C F_1 \, dx + \int_C F_2 \, dy + \int_C F_3 \, dz
\]

which is \( \int_C \left( \frac{F_1}{dt} + \frac{F_2}{dt} + \frac{F_3}{dt} \right) \, dt \)

\[ = \int_C \mathbf{F}(c(t)) \cdot c'(t) \, dt \]

Example: \( c(t) = (t, t^2, 1) \), \( \mathbf{F}(x, y, z) = (x^2, xy, 1) \), \( 0 \leq t \leq 1 \)

So \( \mathbf{c}'(t) = (1, 2t, 0) \)
\( \mathbf{F}(c(t)) = (t^2, t^3, 1) \)
\[
\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (t^2 \cdot 1 + t^3 \cdot 2t + 1) \, dt = \frac{11}{15}
\]

Key Theorem: \( \int_C \nabla f \cdot d\mathbf{s} = f(c(b)) - f(c(a)) \)

Proof: If \( F(t) = f(c(t)) \), then \( F'(t) = (\nabla f) (c(t)) \cdot c'(t) \)

By FTC in 1-var: \( \int_a^b F'(t) \, dt = F(b) - F(a) \)

What We Reviewed: Chain Rule, Derivatives, FTC, paths
GREEN'S THM: Let $D$ be a simple region with boundary $C$. Suppose $P, Q : D \to \mathbb{R}$ are $C^1$ functions. Then we have

$$
\oint_{C^+} P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy
$$

where $C^+$ is the oriented boundary (let the region $D$ be on the right as you travel).

Let $F = (P, Q, 0)$, let $\partial D$ be boundary of $D$.

Rewrite: $\oint_{\partial D} F \cdot d\vec{s} = \iint_D (D \times F) \cdot \hat{k} \, dx \, dy$

Proof: First do for a rectangle $k$

$$(a, b) \quad (b, d) \quad (b, c) \quad (a, c)$$

$C_1 \quad C_2 \quad C_3 \quad C_4$

Note on $C_1, C_3$ have $y'(t) = 0$

Note on $C_2, C_4$ have $x'(t) = 0$

$$
\oint_{C^+} P \, dx = \int_{C_1} P \, dx + \int_{C_3} P \, dx
$$

$$= \int_a^b P(x, c) \, dx + \int_c^d P(x, d) \, dx$$

$$= \int_a^b \left[ P(x, c) - P(x, d) \right] \, dx$$

$$= \left[ \int_a^b \int_c^d P \, dy \right] \, dx$$

Similarly get $\oint_{C^+} Q \, dy = \int_{C_2} Q \, dy + \int_{C_4} Q \, dy$

$$= \int_a^b \int_c^d \frac{\partial P}{\partial x} \, dx \, dy$$

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Section 8.1: Green's Theorem (cont)

Proof in general:

Interior line integrals cancel in pairs
Left with line integral over boundary

=> Control errors

What Did We Review?
- Double integrals
- De-luxtures
- Cross Product / Partial Derivatives

Application: \( \text{Area}(D) = \frac{1}{2} \int\int_D x \, dy - y \, dx \)

See example 2, page 524

See Thm 4, page 527

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