Differential Eqs:

Goal: given how y changes, find y

↳ $y'(t)$ or $y'(x)$

Physics: $x'(t) = v(t)$

$v'(t) = a(t) \Rightarrow x''(t) = a(t)$

$F = ma$

$$a(t) = 9.8\ m/sec^2$$

$$v'(t) = a(t) = 9.8$$

$$v(t) = 9.8t + v(0)$$

* true for all t
set $t = 0$ so $v(0) = (9.8)\cdot 0 + v(0)$

$$\frac{dv}{dt} = 9.8 \implies dv = 9.8\,dt$$

$$\int_{v_i}^{v_f} dv = \int_{t_i}^{t_f} 9.8\,dt$$

$$v_f - v_i = 9.8(t_f - t_i)$$

$(t_i = 0)$
$v_i = v(0)$

$$v(t) = 9.8t + v(0)$$

Given: $y'(t) = 2$

then $y(t) = 2t + y(0)$

Thought: if you have one deriv,

need one initial condition

Say $x''(t) = 9.8$

Have two derivs, need two initial conditions

↳ natural: $x(0)$ and $x'(0)$

$$x''(t) = 9.8$$

constant

$$x'(t) = 9.8t + x'(0)$$

$$x(t) = 9.8 \cdot \frac{1}{2}t^2 + x'(0)t + C$$

take $t=0$ get

$$x(0) = 4.9 \cdot 0^2 + x'(0) \cdot 0 + C \Rightarrow C = x(0)$$

Thus $x(t) = \frac{1}{2}9.8t^2 + \underline{\underline{x'(0)}}\,t + \underline{\underline{x(0)}}$

free parameters

Ex: $y'(x) = y(x)$

Ex: if $y(x_0) = 4$ then $y'(x_0) = 4$

$y(x_1) = 7$ then $y'(x_1) = 7$

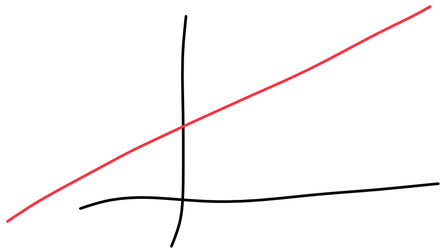
Linear linear $y'(x) = m$

constant rate of change: linear growth

$y(x) = mx + b$

$\Delta x = .01$ then $\Delta y = .01\,m$

$\Delta x = .1$ then $\Delta y = .1\,m$



more than linear

Solve $y'(x) = y(x)$

What fn is its own derivative? e^x no free parameter

Another: $(e^x + b)' = e^x$: NO

$(e^{2x})' = 2e^{2x}$: NO

$(ae^x)' = ae^x$ has a free parameter

"a" is our one free parameter

$$y'(x) = y(x)$$

$$\frac{dy}{dx} = y(x) \quad \text{or} \quad \frac{dy}{dx} = y$$

Separate if possible: all y's on left, x's on right

$$\int \frac{dy}{y} = \int dx$$

$$\ln(y) = x + C$$

$$y = e^{x+c} = e^x e^c \qquad a = e^c$$

$$y = a e^x$$

$$\frac{dy}{dx} = y'(x) = f(x)$$

$$\int dy = \int f(x)\,dx$$

$$y = F(x) + C \quad \text{where } F' = f$$

need to be able to find F

Harder: $\frac{dy}{dx} = y'(x) = f(x, y)$

cannot have all y's on left, all x's on right

Differential Eqs

$$\frac{dy}{dx} = f(x) \quad \text{Then} \quad \int dy = \underbrace{\int f(x)\,dx}_{\text{If we can solve, golden!}}$$

(1) Solve exactly

(2) Approximate: Euler's Method

$$\frac{dy}{dx} = y'(x) = f(x) \quad \text{where } f \text{ is a poly in } x$$

$$y'(x) = 3x^2 - 17x + 8$$

$$\int dy = \int (3x^2 - 17x + 8)\, dx$$

$$y = \underbrace{x^3 - \frac{17}{2}x^2 + 8x}_{\text{vanishes at } x=0} + \underbrace{C}_{y(0)} = \underbrace{y(x)}_{y(0) \text{ when } x=0}$$

$$x y'(x) + y(x) = 3x^2$$

$$\underbrace{x \frac{dy}{dx} + y(x)} = 3x^2$$

$$x y'(x) + y(x)$$

$$= 1 \cdot y(x) + x \cdot y'(x)$$

$$= \underbrace{f'(x) y(x) + f(x) y'(x)}$$

$$f(x) = x$$

$$f'(x) = 1$$

$$[f(x) y(x)]' = [x y(x)]'$$

$$[x\,y(x)]' = 3x^2$$

$$\frac{d}{dx}[x\,y(x)] = 3x^2$$

$$\frac{d(x\,y(x))}{dx} = 3x^2$$

$$\int d(x\,y(x)) = \int 3x^2\,dx$$

$$x\,y(x) = x^3 + C$$

$$x=0:\ 0\cdot y(0) = 0^3 + C$$

$$C = 0 \quad \text{so } y(x) = x^2$$

Check

$$x\,y'(x) + y(x) = 3x^2$$

if $y(x) = x^2$

$$y'(x) = 2x$$

$$x\cdot 2x + x^2 \stackrel{?}{=} 3x^2$$

YES!

Had $\quad x y'(x) + y(x) = f(x)$

$$\int [x y(x)]' dx = \int f(x)\, dx$$

$$x y(x) = F(x) + C$$

$$y'(x) + p(x)\,y(x) = q(x)$$

If had $P(x)$, with $P' = p$, $[P(x)\,y(x)]'$

multiply by $\mu(x)$

$$\underbrace{\mu(x)\,y'(x) + \mu(x)\,p(x)\,y(x)}_{[\mu(x)\,y(x)]'} = q(x)\,\mu(x)$$

$$= \mu(x)\,y'(x) + \mu'(x)\,y(x)$$

Need

$\mu(x)\,p(x)$

$= \mu'(x)$

Solve $\mu'(x) = \mu(x)p(x)$

Guess $\mu(x) = e^{g(x)}$

$\mu'(x) = e^{g(x)} g'(x)$

So $g'(x) = p(x)$

So $g(x) = \int p(x)dx = P(x) + C$

Answer: $\mu(x) = e^{\int p(x)dx} = e^{P(x)+C}$

$= e^{P(x)} \cdot e^{C}$

$= \underbrace{\text{Const}}_{1} e^{P(x)}$

Start: $y'(x) + p(x)y(x) = q(x)$

mult by $\mu(x)$: $\mu(x)y'(x) + \mu(x)p(x)y(x) = q(x)\mu(x)$

So $[\mu(x)y(x)]' = q(x)\mu(x)$ if $\mu(x) = 1 \cdot e^{P(x)}$

where $P'(x) = p(x)$

$$\int [\mu(x)y(x)]'\,dx = \int q(x)\mu(x)\,dx$$

$$\mu(x)y(x) = \int q(x)e^{P(x)}\,dx + C$$

Can divide by $\mu(x)$ as $e^{P(x)} \neq 0$

$$y(x) = \frac{1}{e^{P(x)}} * \left[\int q(x)e^{P(x)}\,dx + C\right]$$

$\mu(x)$ is called the integrating factor

Example 3

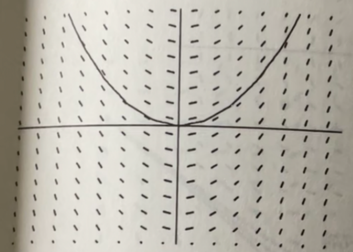
Match each slope field in Figure 9.3 with the proper d.e. from the following set. Find the general solution for each d.e. The particular solution that goes through (0,0) has been sketched in.

(A) $y' = \cos x$

(B) $\dfrac{dy}{dx} = 2x$

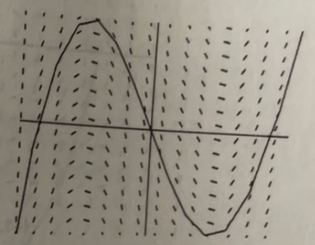
(C) $\dfrac{dy}{dx} = 3x^2 - 3$

(D) $y' = -\dfrac{\pi}{2}$



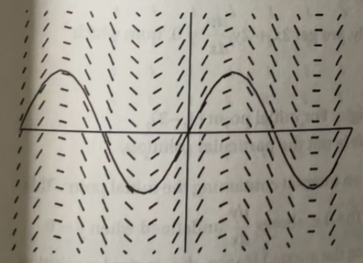
$[-2,2] \times [-2,2]$

Figure 9.3a



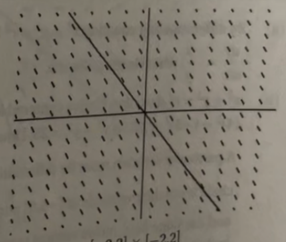
$[-2,2] \times [-2,2]$

Figure 9.3b



$[-2\pi,2\pi] \times [-2,2]$

Figure 9.3c



$[-2,2] \times [-2,2]$

Figure 9.3d

Solutions

(a) goes with Figure 9.3c. The solution curves in the family $y = \sin x + C$ are quite obvious.

(b) goes with Figure 9.3a. The general solution is the family of parabolas $y = x^2 + C$.

(c) goes with Figure 9.3b. The general solution is the family of cubics

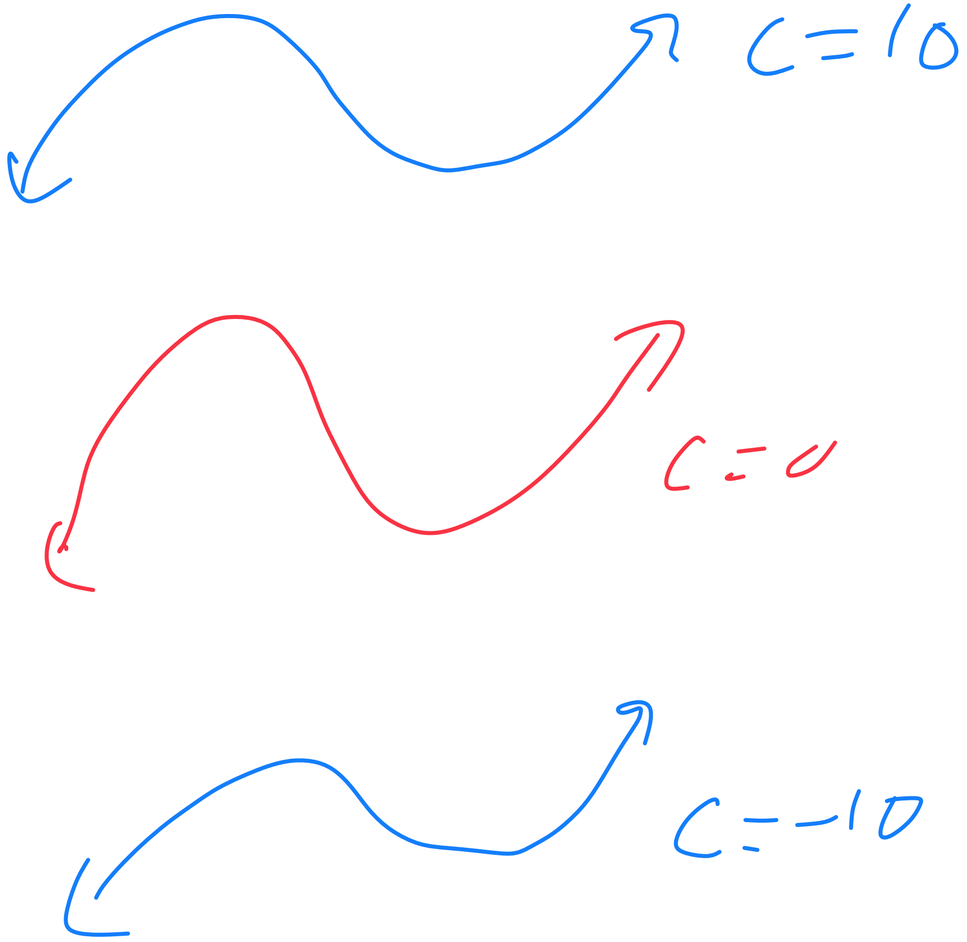
$$y'(x) = 3x^2 - 3$$

$$\frac{dy}{dx} = 3x^2 - 3$$

$$\int dy = \int (3x^2 - 3)\,dx$$

$$y = x^3 - 3x + C$$

If know $y(0)$, then can find C



Euler Method

Approx solns

$a(t) = v'(t)$ $\quad v(t) = x'(t)$ so $a(t) = x''(t)$

$F = ma$

$\hookrightarrow F(t) = ma(t) = m x''(t) = \dfrac{d^2 x}{dt^2}$

$$\frac{d}{dt}\left(\frac{d}{dt} x(t)\right) = \frac{d^2}{dt^2} x(t) = \frac{d^2 x}{dt^2}$$

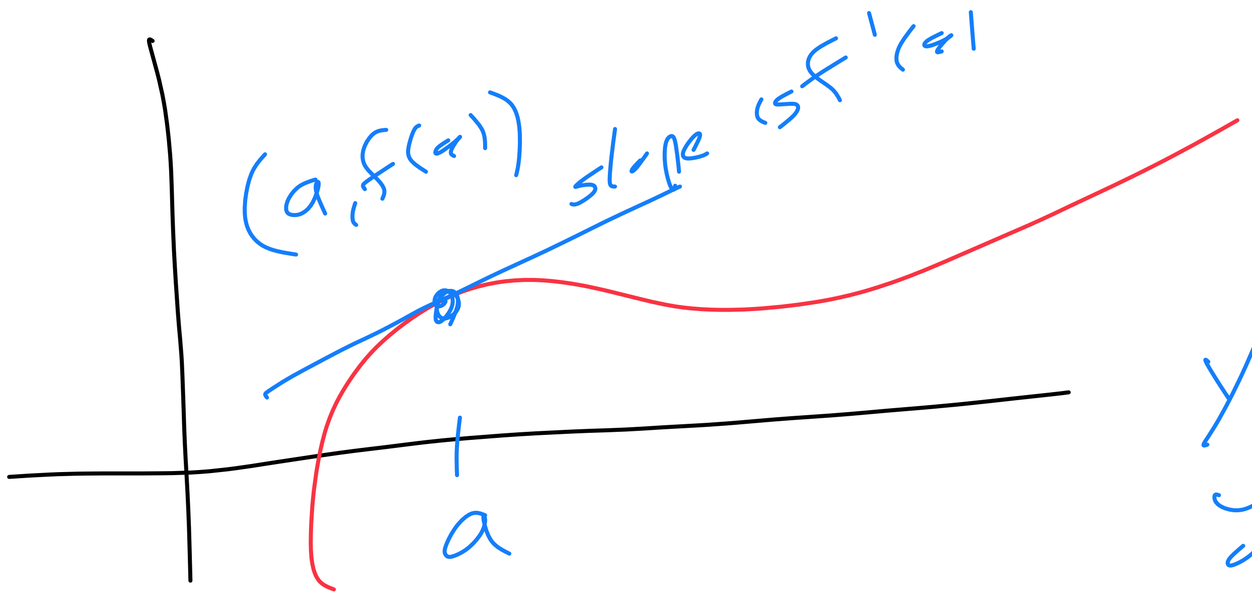
$$F(t) = m v'(t) = m \frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{1}{m} F(t)$$

$$\int dv = \underbrace{\int \frac{1}{m} F(t)\, dt}$$

Need to be able to integrate!

Euler: Numerically approximate



Tangent Line

Slope: $m = f'(a)$

Point: $(a, f(a))$

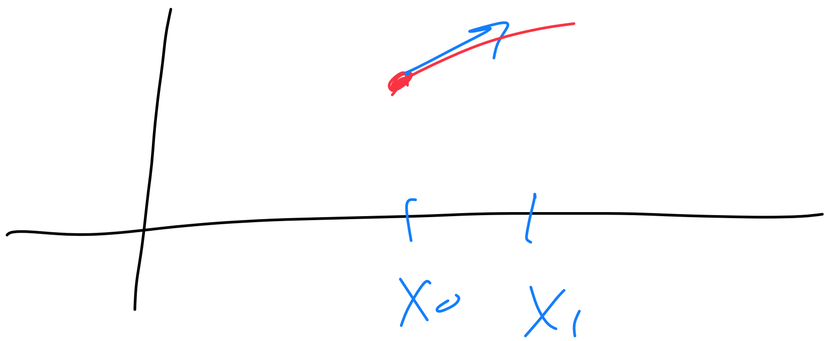
$$y - f(a) = f'(a)(x-a)$$

$$y = \underbrace{f(a)}_{\text{start}} + \underbrace{f'(a)}_{\text{speed}}\underbrace{(x-a)}_{\text{elapsed time}}$$

Exact if speed is constant

Euler's Method

$$\frac{dy}{dx} = f(x,y)$$



Start at (x_0, y_0)

Compute $\frac{dy}{dx}(x_0)$: find deriv at x_0: equals $f(x_0, y_0)$

evaluate at

$$y_1 = \underbrace{y_0}_{\text{start}} + \underbrace{\frac{dy}{dx}(x_0)}_{\text{speed}} * \underbrace{(x_1 - x_0)}_{\text{elapsed time}} = \underbrace{y_0 + f(x_0, y_0)(x_1 - x_0)}_{y_1}$$

Euler's Method

$$y_2 = y_1 + \frac{dy}{dx}(x_1) * (x_2 - x_1)$$

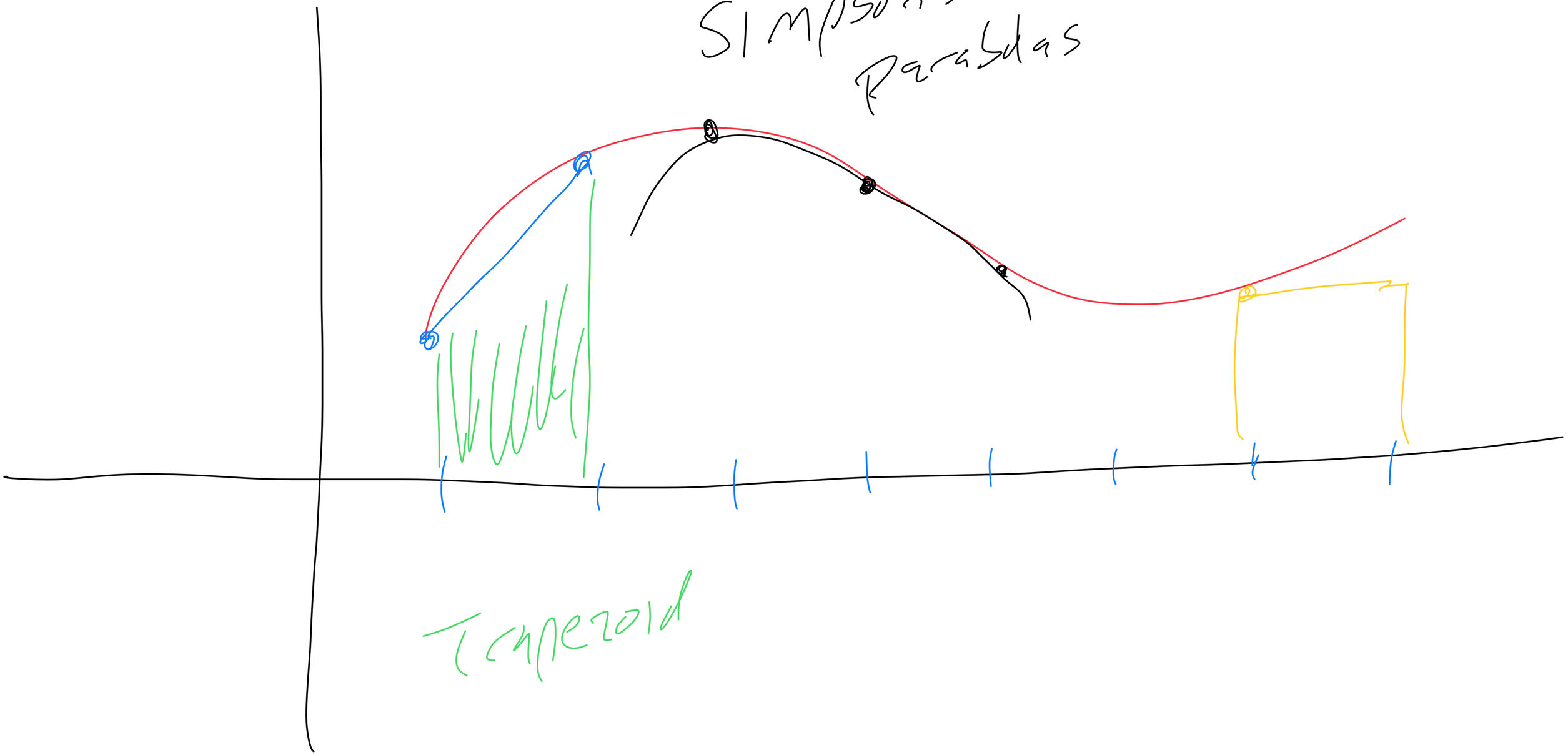
$$= y_1 + \underline{\underline{f(x_1, y_1)}} * (x_2 - x_1)$$

update slope as now at x_1 not x_0

↳ use the info we have

Often $x_{k+1} - x_k = \Delta x =$ constant (fixed step size)

↳ sometimes $h = \Delta x =$ step size

Smaller h, less error: how does error depend on h?

Simpson's rule:
Parabolas