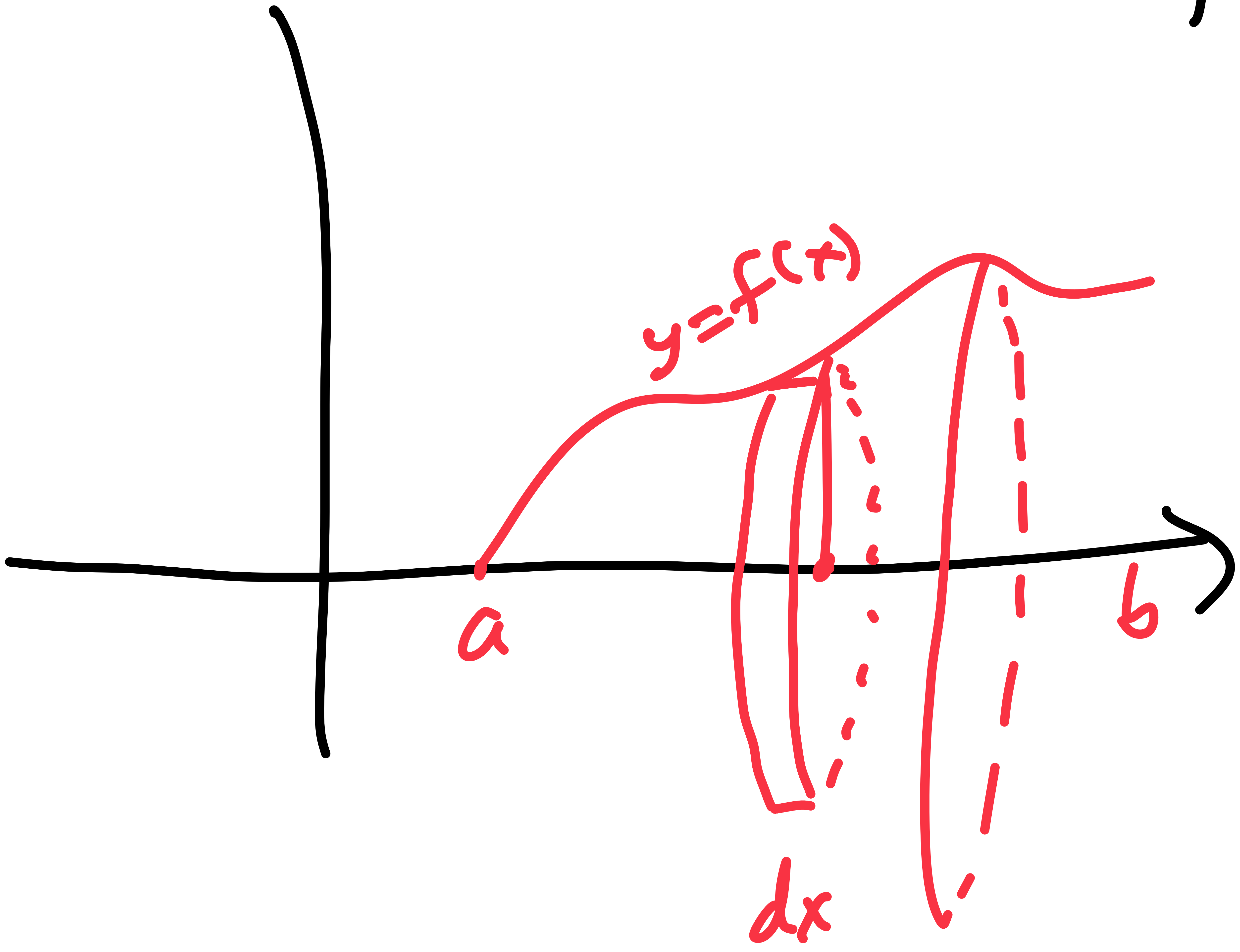
Areas/Volumes

Rotate about x-axis



Volume

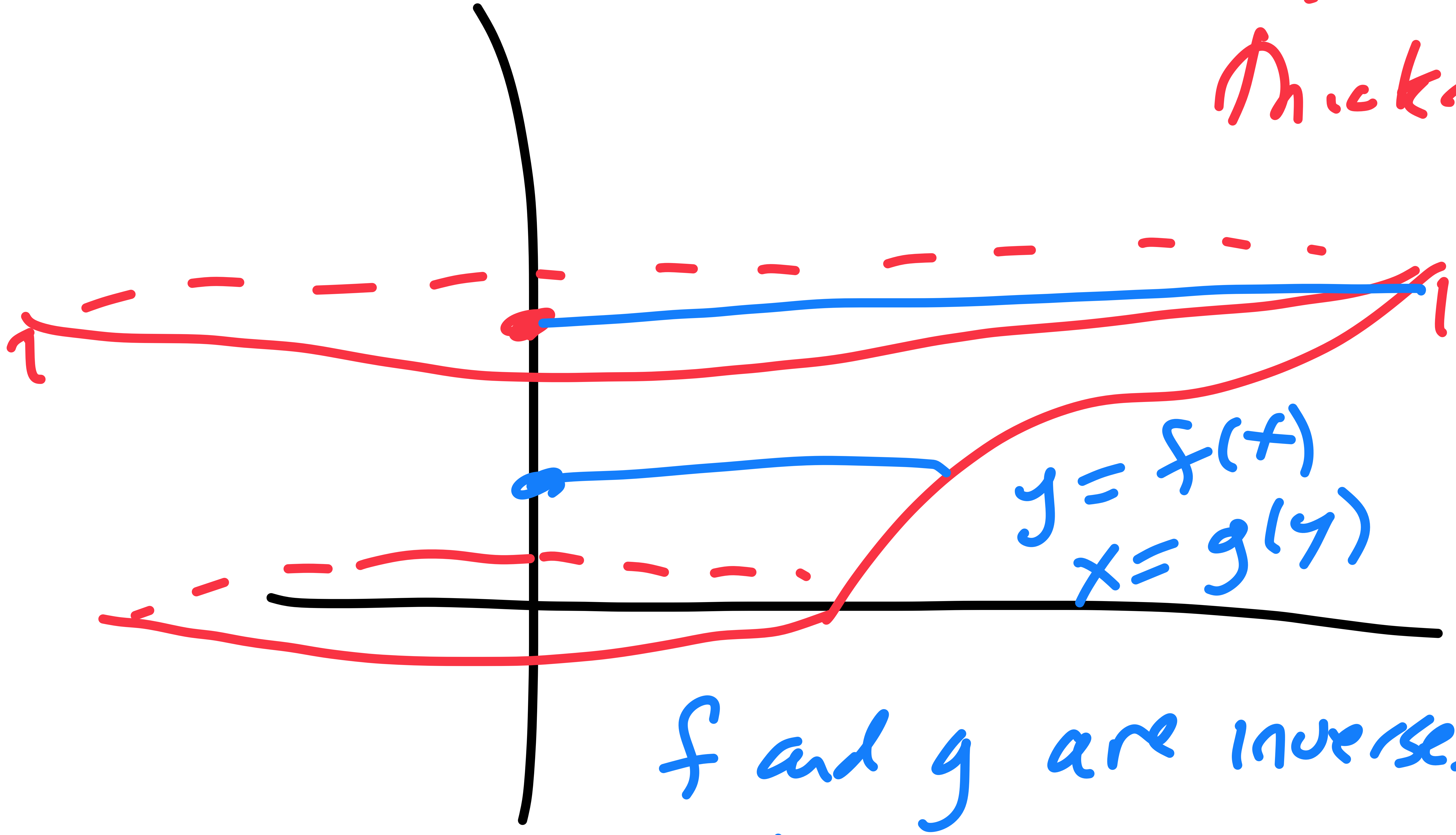
$$= \int_a^b \pi f(x)^2 \, dx$$

$$= \pi \int_a^b f(x)^2 \, dx$$

Rotating about y-axis

thickness dy

disc area is πx^2



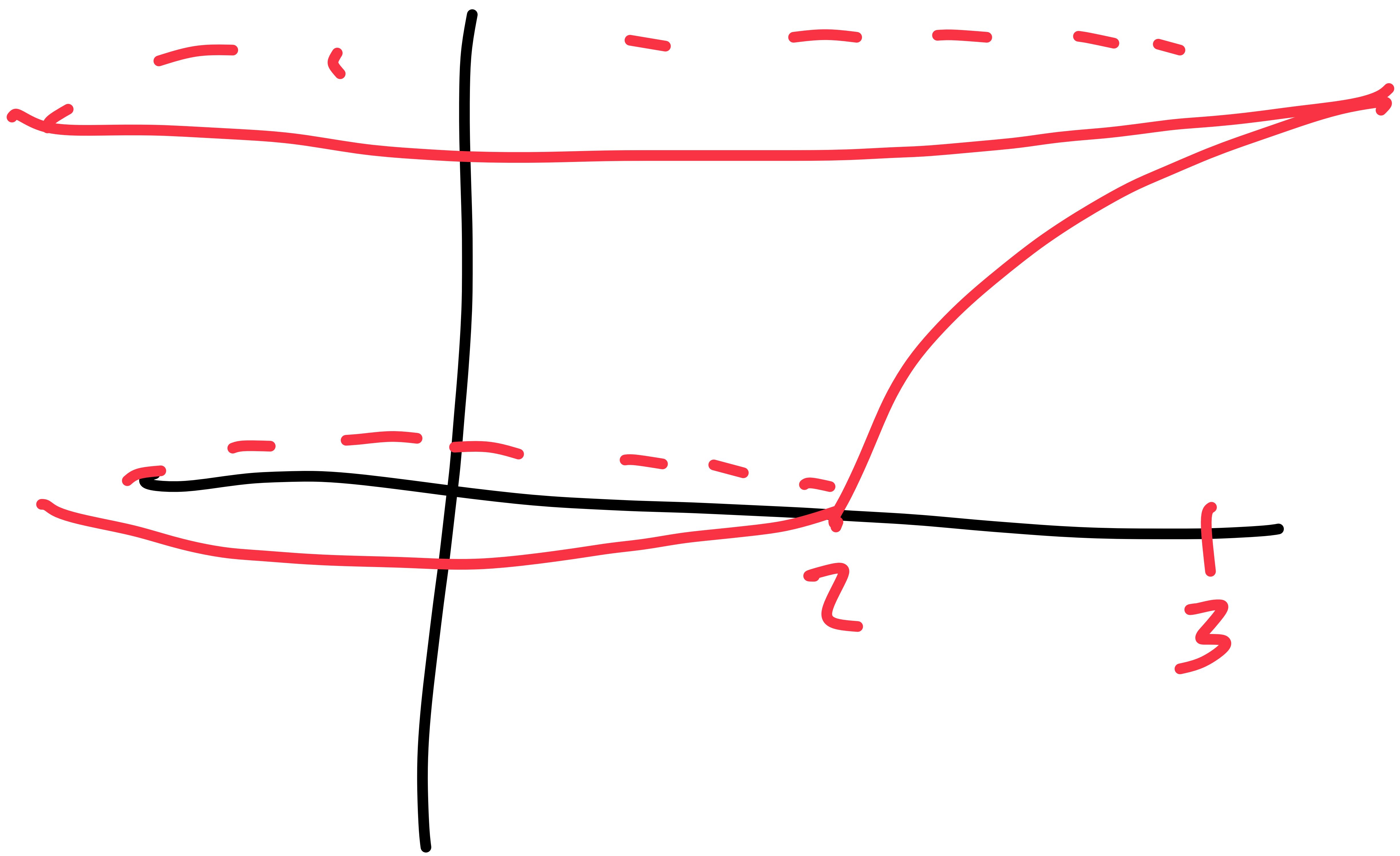
Volume is

$$\int_{y=c}^{d} \pi g(y)^2 \, dy$$

f and g are inverses

$f(g(y)) = y$

$x: 2 \to 3$ so $y: 0 \to 1$



$y = \sqrt{x-2}$

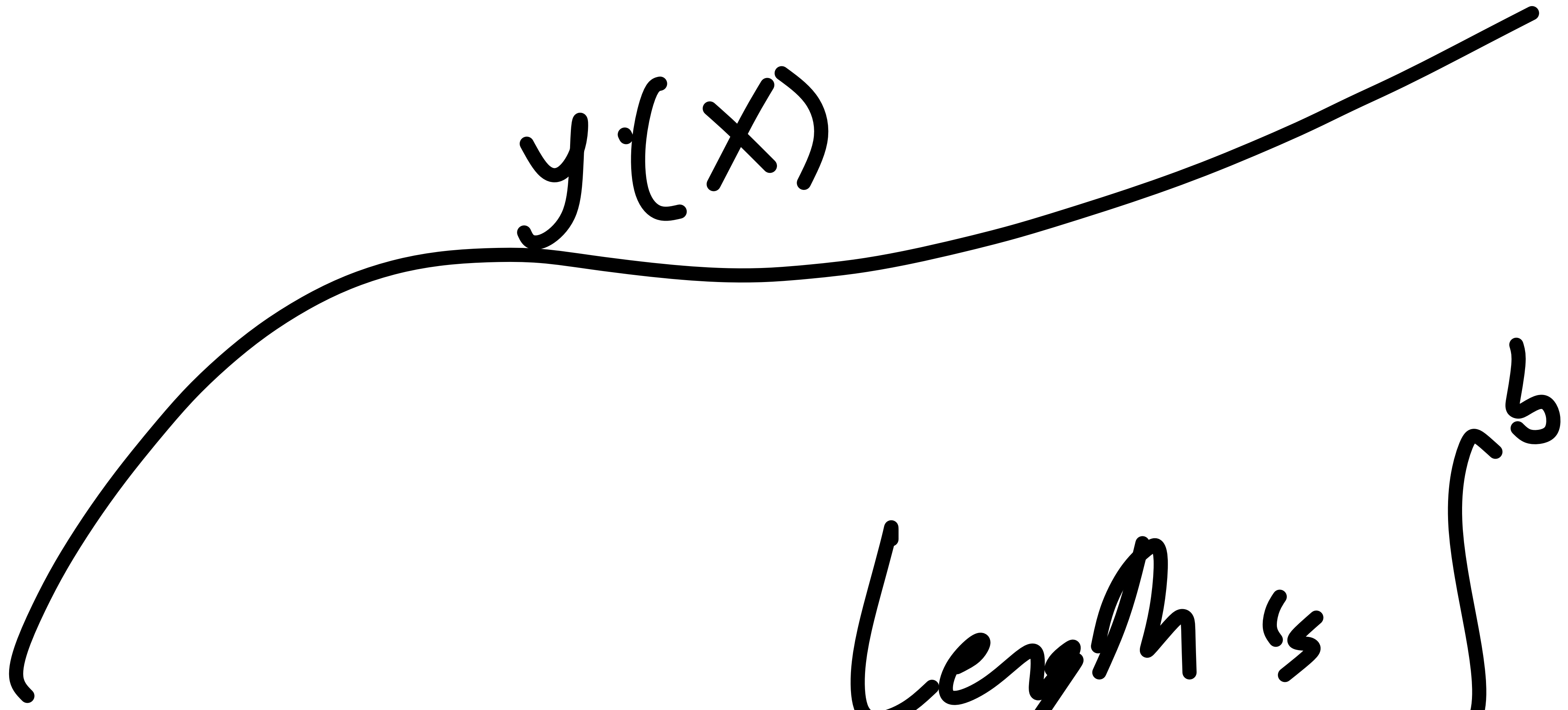
$y = f(x)$

$x = g(y)$

$y^2 = x - 2 \Rightarrow x = y^2 + 2$

so $x = g(y) = y^2 + 2$

Volume $\displaystyle\int_{y=0}^{1} \pi (y^2+2)\,dy = \pi\left[\frac{y^3}{3}\Big|_0^1 + 2y\Big|_0^1 \right] = \frac{7\pi}{3}$ units^3



Length is $\displaystyle\int_{x=a}^{b} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Parametric Form:

$\vec{c}(t) = (x(t), y(t))$

vector

Length is $\displaystyle\int_{t=t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$\langle x(t), y(t)\rangle$

Ellipse $\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$

$x(t)=2\cos t \qquad y(t)=3\sin t$

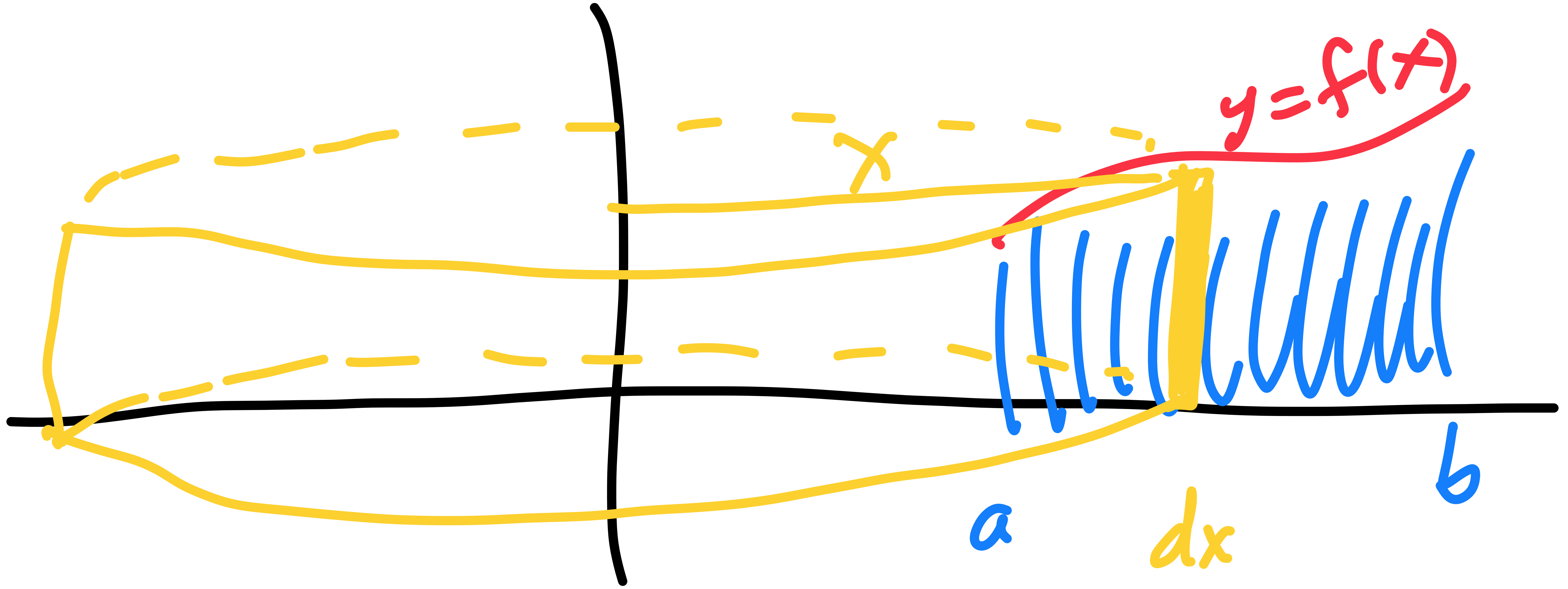
Note: $\left(\frac{x(t)}{2}\right)^2+\left(\frac{y(t)}{3}\right)^2=\cos^2 t+\sin^2 t=1$ ✓

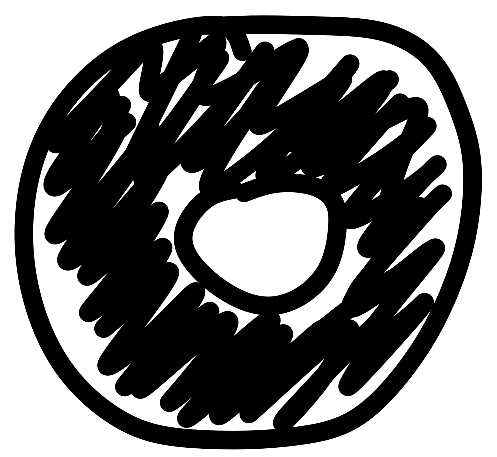
$x'(t)=\frac{dx}{dt}=-2\sin t \qquad y'(t)=\frac{dy}{dt}=3\cos t$

$\sqrt{x'(t)^2+y'(t)^2}=\sqrt{4\sin^2 t+9\cos^2 t}$

hard integral: $\int_{t=0}^{2\pi}\sqrt{4\sin^2 t+9\cos^2 t}\,dt$

Washer method





Cylinder Volume: Base * height $= \pi x^2 \cdot f(x)$

Move Δx units: Base * height $= \pi(x+\Delta x)^2 f(x+\Delta x)$

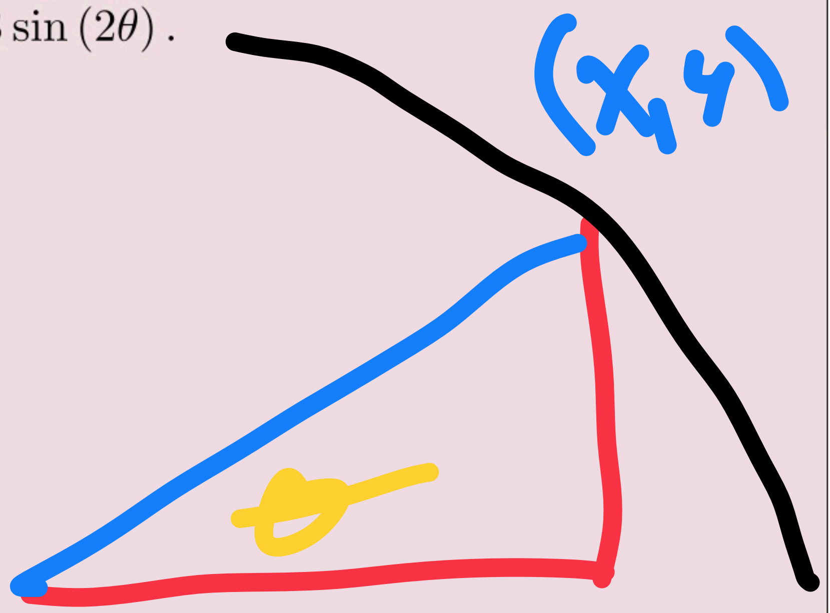
$\pi(x^2 + 2\pi x \Delta x + (\Delta x)^2) f(x)$

difference $2\pi x \Delta x f(x) \Rightarrow Vol = \int_{x=a}^{b} 2\pi x f(x) dx$

Suppose f is continuous and nonnegative on the interval $\alpha \le \theta \le \beta$ with $0 < \beta - \alpha \le 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is
$A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta.$

Finding an Area of a Polar Region

Find the area of one petal of the rose defined by the equation $r = 3\sin(2\theta)$.



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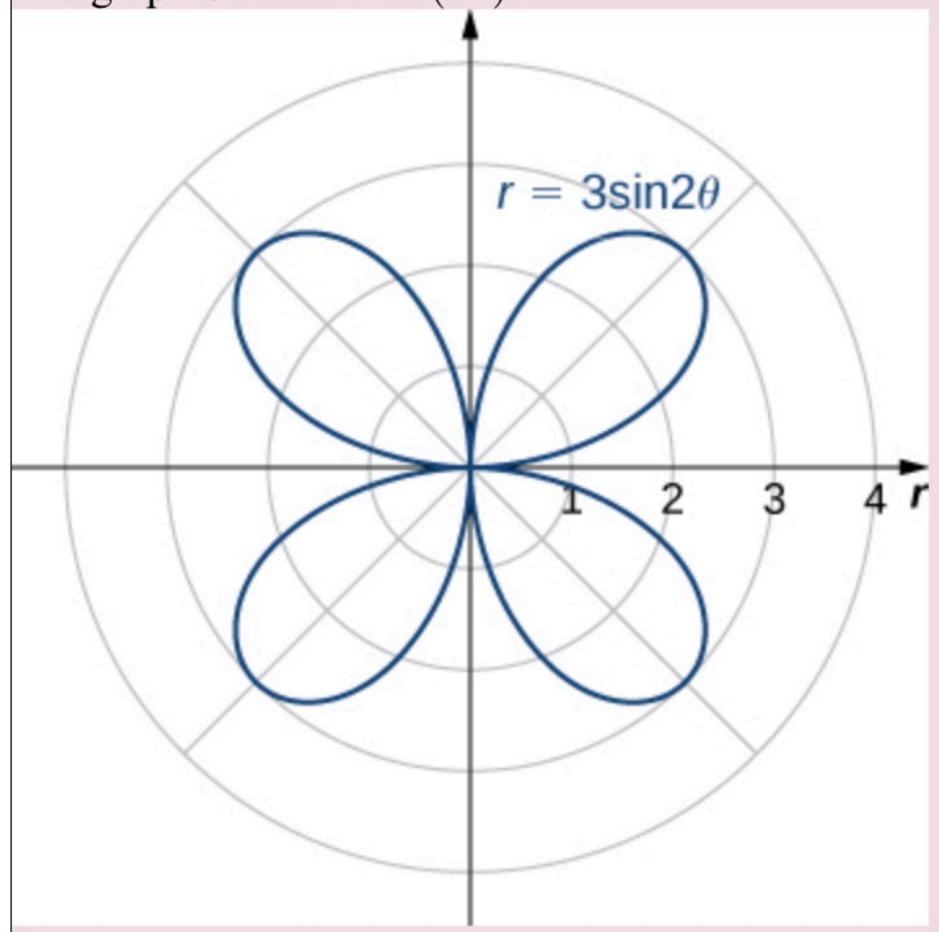

When $\theta = 0$ we have $r = 3\sin(2(0)) = 0$. The next value for which $r = 0$ is $\theta = \pi/2$. This can be seen by solving the equation $3\sin(2\theta) = 0$ for θ. Therefore the values $\theta = 0$ to $\theta = \pi/2$ trace out the first petal of the rose. To find the area inside this petal, use (Figure) with $f(\theta) = 3\sin(2\theta)$, $\alpha = 0$, and $\beta = \pi/2$:

$$\begin{aligned} A &= \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta \\ &= \frac{1}{2}\int_{0}^{\pi/2}[3\sin(2\theta)]^2 d\theta \\ &= \frac{1}{2}\int_{0}^{\pi/2} 9\sin^2(2\theta)\, d\theta. \end{aligned}$$

To evaluate this integral, use the formula $\sin^2\alpha = (1 - \cos(2\alpha))/2$ with $\alpha = 2\theta$:

$$\begin{aligned} A &= \frac{1}{2}\int_{0}^{\pi/2} 9\sin^2(2\theta)\, d\theta \\ &= \frac{9}{2}\int_{0}^{\pi/2}\frac{(1-\cos(4\theta))}{2} d\theta \\ &= \frac{9}{4}\left(\int_{0}^{\pi/2} 1 - \cos(4\theta)\, d\theta\right) \\ &= \frac{9}{4}\left(\theta - \frac{\sin(4\theta)}{4}\right)\Big|_{0}^{\pi/2} \\ &= \frac{9}{4}\left(\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right) - \frac{9}{4}\left(0 - \frac{\sin 4(0)}{4}\right) \\ &= \frac{9\pi}{8}. \end{aligned}$$

Find the area inside the cardioid defined by the equation $r = 1 - \cos\theta$.

Polar Coordinates:

$x = r\cos\theta$ $y = r\sin\theta$

$r = \sqrt{x^2+y^2}$ $\theta = \arctan\left(\frac{y}{x}\right)$

$r = 3\sin(2\theta)$

Study $r = 3\sin(2\theta)$

$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

$= (x^2+y^2)^{1/2}$

$r = 3 \cdot 2\sin(\theta)\cos(\theta)$

$r = 6 \cdot \frac{y}{r} \frac{x}{r}$

$r^3 = 6xy$

$(x^2+y^2)^{3/2} = 6xy$

or: $(x^2+y^2)^3 = 36x^2y^2$

$x^6+3x^4y^2+3x^2y^4+y^6 = 36x^2y^2$

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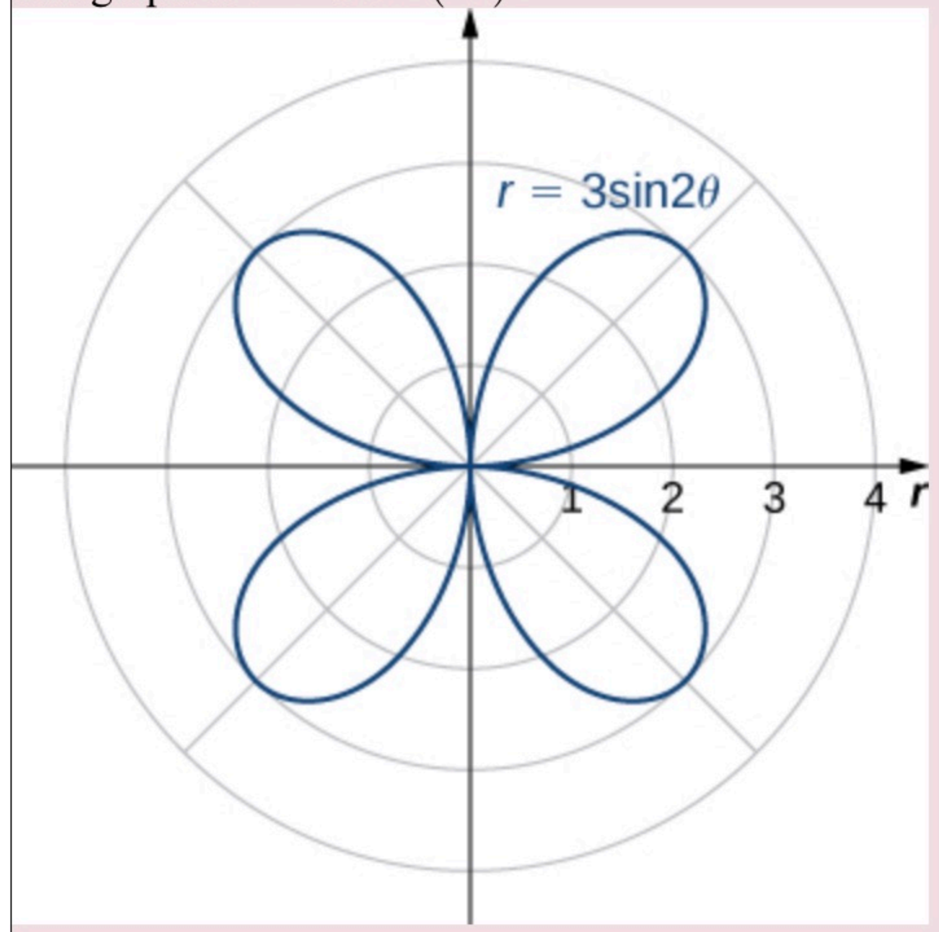

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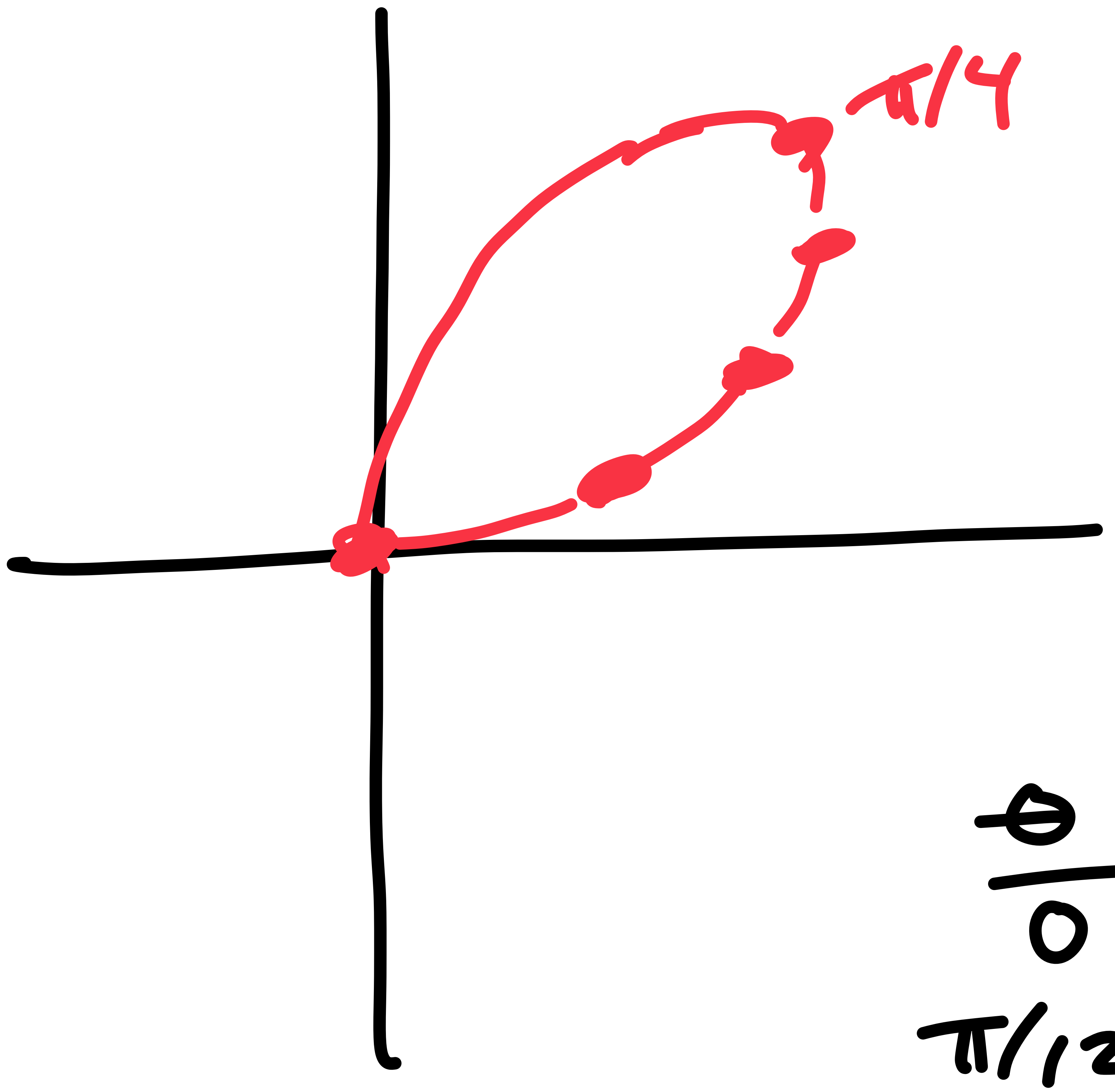
Plot of $r = 3\sin 2\theta$



θ	r
0	0
$\pi/12$	$3/2$
$\pi/8$	$3\sqrt{2}/2$
$\pi/6$	$3\sqrt{3}/2$
$\pi/4$	3

Suppose f is continuous and nonnegative on the interval $\alpha \le \theta \le \beta$ with $0 < \beta - \alpha \le 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is
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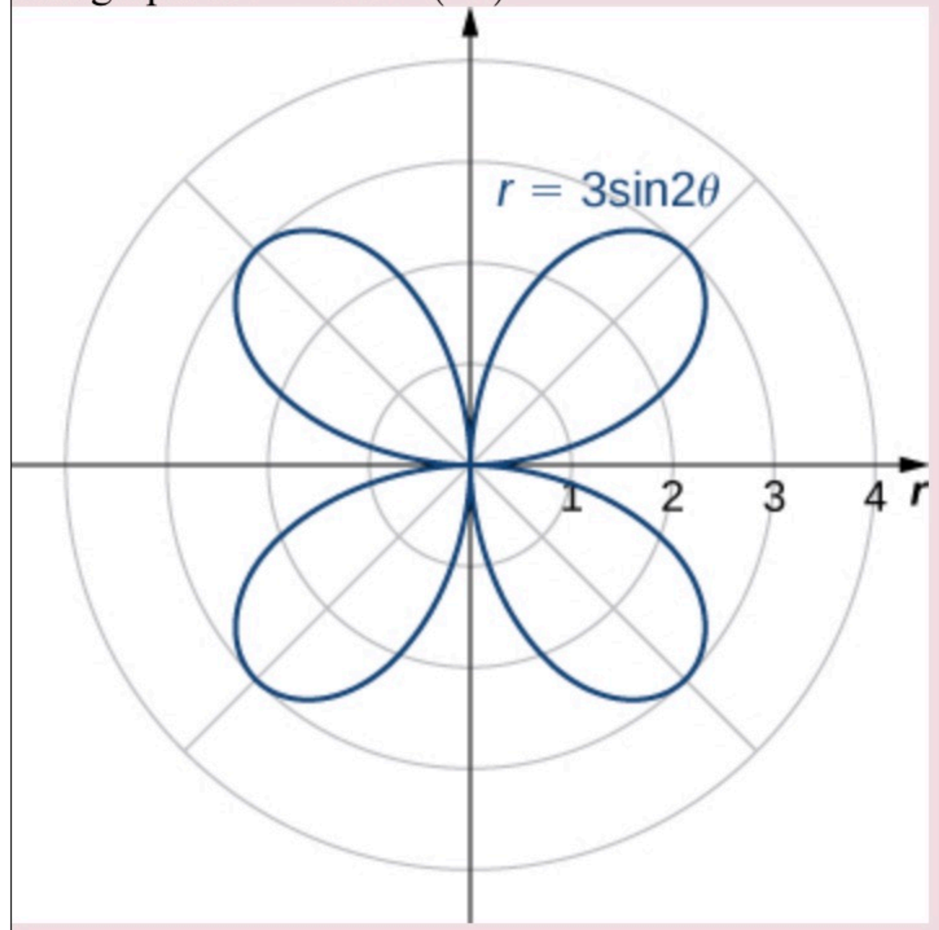

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Area is

$$\frac{1}{2}\int_{\theta=0}^{2\pi} r(\theta)^2 d\theta$$

$r(\theta) = 3\sin 2\theta$

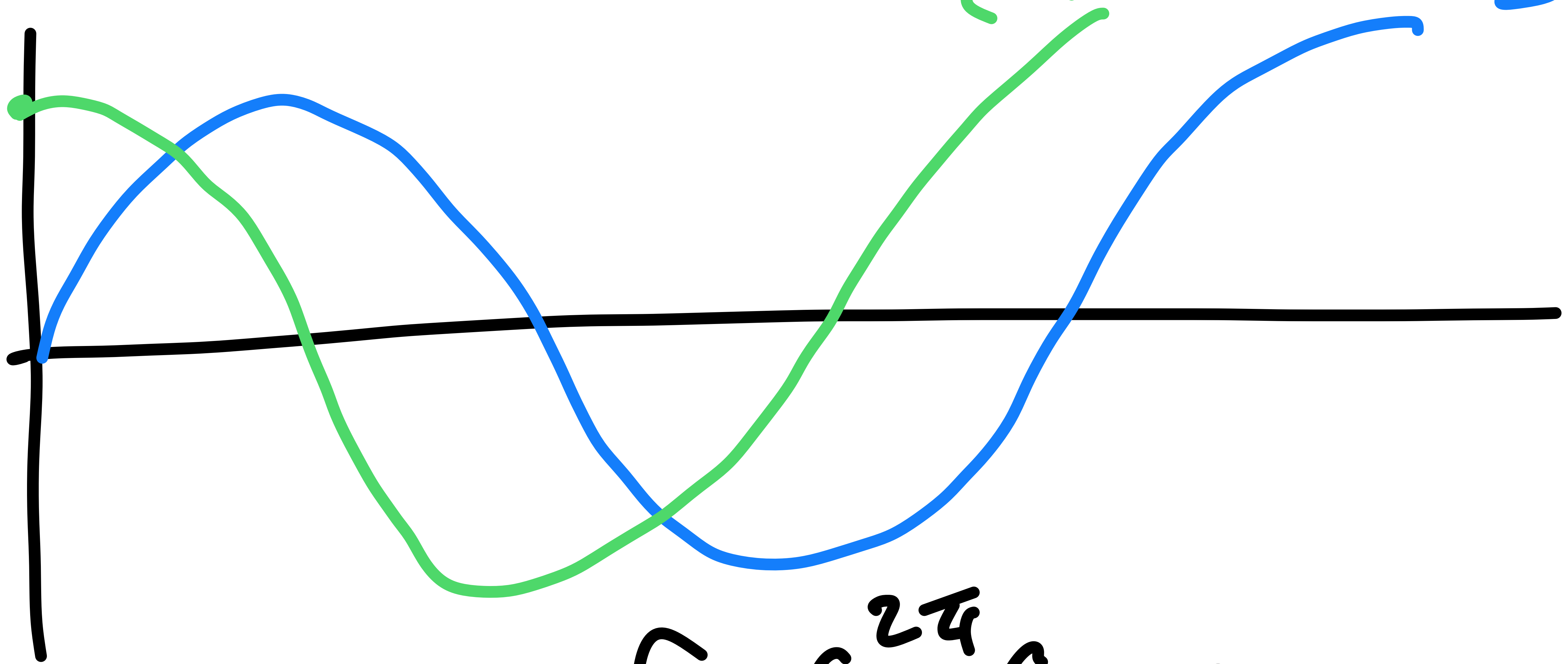
$r(\theta)^2 = 9\sin^2(2\theta)$

$$\frac{1}{2}\int_{\theta=0}^{2\pi} 9\sin^2(2\theta)\,d\theta$$

$$= \frac{9}{2}\int_0^{2\pi} \underbrace{\sin^2(2\theta)\,d\theta}_{\text{integral is } \pi}$$

answer is $\frac{9}{2}\pi$

$$\int_0^{2\pi} \sin^2(2\theta)\,d\theta = \int_0^{2\pi} \cos^2(2\theta)\,d\theta$$



$$\int_0^{2\pi} \sin^2(2\theta)\,d\theta = \frac{1}{2}\left[\int_0^{2\pi} \left[\sin^2(2\theta) + \cos^2(2\theta)\right] d\theta\right]$$

$$= \frac{1}{2}\int_0^{2\pi} 1\,d\theta = \frac{1}{2}\theta\Big|_0^{2\pi} = \frac{2\pi}{2} = \pi$$