NOTE: ALL NUMBERING IS TO MARSDEN AND TROMBA

MATH 105: REVIEW TO STOKES

Purpose is to review material of course and conclude with statement of at least one of Green/Gauss/Stokes Thm

↳ This is a massive generalization of the Fund Thm Calc

↳ Briefly, moving derivative from "function" to "region"

$$\iint \cdots \int_C d\omega = \int \cdots \int_{\partial C} \omega$$

where ∂C is boundary of C

and $d\omega$ is derivative of ω

Sec 4.2: Arc Length

$c(t)$: curve in 3-dim space (or n-dim space)

If $c(t) = (x(t), y(t), z(t))$

Then $c'(t) = (x'(t), y'(t), z'(t))$ is speed

and length of path $c(t)$ from t_0 to t_1 is

$$\int_{t_0}^{t_1} \|c'(t)\|\, dt = \int_{t_0}^{t_1} \sqrt{c'(t) \cdot c'(t)}\, dt$$

↳ most of the time these integrals are hard to do

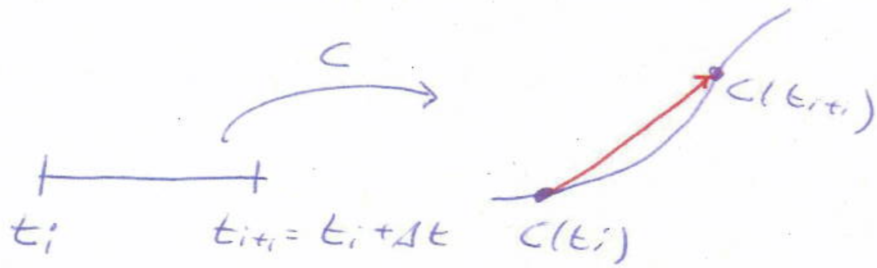
↳ reduces to a Calc-II problem!

↳ comes down to parametrizing a curve

Sec 4.2: Arc Length (cont)

Justification for formula

Polygonal approximation
Chop into many small time steps



length of red line segment is $\| C(t_{i+1}) - C(t_i) \|$

↳ using MUT lots of time, done

x-coord is $x(t_{i+1}) - x(t_i) = x'(s_i)\,\Delta t \approx x'(t_i)\,\Delta t$

$$y(t_{i+1}) - y(t_i) = y'(r_i)\,\Delta t \approx y'(t_i)\,\Delta t$$

$$z(t_{i+1}) - z(t_i) = z'(q_i)\,\Delta t \approx z'(t_i)\,\Delta t$$

yields red line segment has length $\approx \|C'(t_i)\|\,\Delta t$

So Length $\approx \sum \|C'(t_i)\|\,\Delta t \longrightarrow \int_{a}^{b} \|C'(t)\|\,dt$

__What does this review?__

↳ Parametrizing curves
↳ Riemann Sums
↳ MUT

~~HW (§4.2) #1 (see formula on bottom of page)~~
~~Suggested: #12, #18~~

SECTION 7.1: THE PATH INTEGRAL

Defn: The path integral, or integral of f along the path c and denoted $\int_c f\,ds$ is $\int_a^b f(c(t))\,\|c'(t)\|\,dt$ where $c:[a,b] \to \mathbb{R}^3$ is a C^1 map.

↳ idea: If $f=1$ get arc-length
Think of f as a density

↳ See pg 424 for interpretation if $c:[a,b] \to \mathbb{R}^2$
↳ gives area of fence of height $f(t)$ at $c(t)$

Ex: $f(x,y,z) = x+y+z$, $c(t) = (\sin t, \cos t, t)$ $\quad 0 \le t \le 2\pi$

What Does This Review

↳ Parametrizing curves

↳ Derivatives and integrals

~~Homework: #36~~
~~Suggested: #6, #7, #13~~

SECTION 7.2: LINE INTEGRALS

Defn: line integral of vector-field $\vec{F}$ along a C^1 path c,

$c:[a,b] \to \mathbb{R}^3$, is

$$\int_c \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(c(t)) \cdot c'(t)\, dt$$

↳ interpretation: pages 429–430: work done

↳ if take $\vec{F}$ to be $c'(t)/\|c(t)\|$ (unit tangent)

Then get arc length

Notation: $\vec{F} = (F_1, F_2, F_3)$, often write

$$\int_c \vec{F}\cdot d\vec{s} = \int_c F_1\,dx + \int_c F_2\,dy + \int_c F_3\,dz$$

which is $\displaystyle\int_c \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$

$$= \int_c \vec{F}(c(t)) \cdot c'(t)\, dt$$

Example: $c(t) = (t, t^2, 1)$, $\vec{F}(x,y,z) = (x^2, xy, 1)$, $0 \le t \le 1$

So $c'(t) = (1, 2t, 0)$

$F(c(t)) = (t^2, t^3, 1)$

$$\int_c \vec{F}\cdot d\vec{s} = \int_0^1 (t^2\cdot 1 + t^3\cdot 2t + 1\cdot 0)\, dt = \frac{11}{15}$$

$$\boxed{\text{KEY THM: } \int_c \nabla f \cdot d\vec{s} = f(c(b)) - f(c(a))}$$

only depends on the endpoints!

PROOF: If $F(t) = f(c(t))$ then $F'(t) = (\nabla f)(c(t)) \cdot c'(t)$

By FTC in 1-var: $\int_a^b F'(t)\,dt = F(b) - F(a)$ ∎

What We Reviewed: Chain Rule, Derivatives, FTC, paths

Section 8.1: Green's Thm (AKA, Stoke's in the Plane)

Green's Thm: Let D be a simple region with boundary C. Suppose $P, Q : D \to \mathbb{R}$ are C^1 functions. Then we have

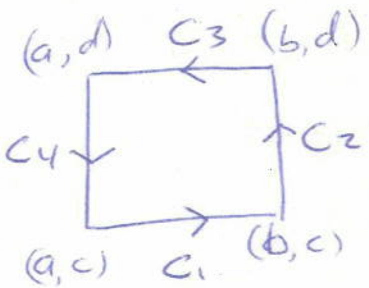
$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where C^+ is the oriented boundary (st the region D is on left as you travel).

Let $\vec{F} = (P, Q, 0)$, let ∂D be boundary of D.

Rewrite: $$\int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D (\nabla \times \vec{F}) \cdot \hat{k}\, dx\,dy$$

Proof: First do for a rectangle



Note on C_1, C_3 have $y'(t) = 0$

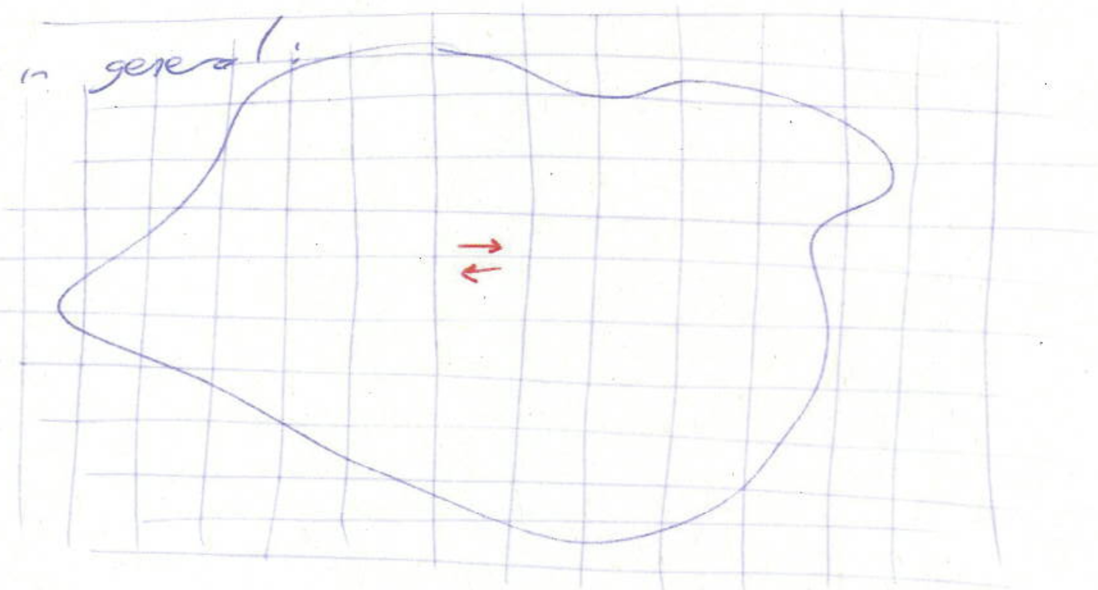
Note on C_2, C_4 have $x'(t) = 0$

$$\begin{aligned}
\int_{C^+} P\,dx &= \int_{C_1} P\,dx + \int_{C_3} P\,dx \\
&= \int_a^b P(x,c)\,dx + \int_b^a P(x,d)\,dx \\
&= \int_a^b \left[P(x,c) - P(x,d)\right] dx \\
&= \int_a^b \int_c^d -\frac{\partial P}{\partial y}\, dy\, dx
\end{aligned}$$

similarly get $\displaystyle\int_{C^+} Q\,dy = \int_c^d \int_a^b \frac{\partial Q}{\partial x}\, dx\,dy$

Section 8.1: Green's Thm (cont)

Proof in general:



Interior line integrals cancel in pairs
left with line integral over boundary
↳ control errors

What Did We Review?

↳ Double Integrals

↳ Derivatives

↳ Cross Product / Partial Derivatives

Application: $\text{Area}(D) = \frac{1}{2} \iint_{\partial D} x\,dy - y\,dx$

↳ See example 2, page 524

See Thm 4, page 527

~~Homework: #3a~~

~~Suggested: #8~~