1. Homework Problems

1.1. **Class 01:** Introduction: Motivation and Mechanics: https://youtu.be/RyLma1yOQEg

1.2. **Class 02:** Email me what you want to get out of the course, what majors you are considering, and comments on Michael Uslan's graduation speech: https://tinyurl.com/468hj5v9. Lecture: From Pythagoras to π: https://youtu.be/v3j0oiTrj7w

1.3. **Class 03:** From Pascal to Calculus: https://youtu.be/v3j0oiTrj7w

1.4. **Class 04:** Homework: You do not need to turn in, but make sure you can do all the differentiation problems from the calculus problem review sheet, and make sure you are comfortable with the material from Calculus I: https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/handouts/105CalcReviewProblems.pdf (note the solution to the problems are later in the link). Review day / application day: no HW due, lecture topics chosen by class.

1.5. **Class 05:** Page 198: (22), (24), (46), (53), (54). Lecture: Fundamental Theorem of Calculus: watch https://www.youtube.com/watch?v=Q1TQtH6POyI (50 minutes)
Page 198: #22: Evalute the antiderivative $\int \frac{1+x^2}{\sqrt{x}}dx$.
Page 198: #24: Evalute the antiderivative $\int \cos 3x dx$.
Page 198: #46: Evalute the antiderivative $\int \frac{dx}{(x-1)^3}$ with quick formula 1.
Page 198: #53: Evalute the antiderivative $\int (x-1)^2 x dx$.
Page 198: #54: Evalute the antiderivative $\int (x^2-x)^4(2x-1)dx$.

1.6. **Class 06:** Page 217: (33a), (38), (39a) and also compare with (38). Lecture: Logarithms: watch https://www.youtube.com/watch?v=-SsbkPaB6j8 (22 minutes)
Page 217: #33a: Compute the average value of $f(x) = x^{\frac{1}{5}}$ on $[0, 1]$.
Page 217: #38: In an approximating sum (23.1) $\sum_{k=1}^{n} f(x_k^*)\Delta_k x$,if we seelct x_k^* to be the midpoint of the kth subinterval, then the sum is said to be obtained by the *midpoint rule*. Apply the midpoint rule to approximate $\int_0^1 x^2 dx$, using a division into five equal subintervals, and comapre with the exact result of $\frac{1}{3}$.
Page 217: #39a: Apply Simson's rule to approximate $\int_0^1 x^2 dx$ and compare with the results from the answer obtained by the fundamental theorem.

1.7. **Class 07:** Review day / application day: no HW due, lecture topics chosen by class.

1.8. **Class 08:** Page 224: (8a,h), (9a), (11), (13). Lecture: Integration by Parts: https://youtu.be/ZL4U8sB0Vj4 (26 minutes) slides: https://web.williams.edu/Mathematics/sjmiller/public_html/115Fa21/Math115Int09.pdf
Page 224: #8a: Find the derivative of $y = \ln(x+3)^2 = 2\ln(x+3)$.
Page 224: #8h: Find the derivative of $y = x\ln x - x$.
Page 225: #9a: Find the antiderivative of $\int \frac{1}{7x}dx$.
Page 226: #11: Express in terms of $\ln 2$ and $\ln 3$: (a) $\ln(3^7)$; (b) $\ln \frac{2}{27}$.
Page 226: #13: Find the area under the curve $y = \frac{1}{x}$ and above the x-axis, between $x = 2$ and $x = 4$.

1.9. **Class 09:** Page 280: (14), (18), (22). Lecture: Areas between curves, volumes, and trig substitution: https://youtu.be/IbPM8WJ1x10 (39 minutes). slides: https://web.williams.edu/Mathematics/sjmiller/public_html/115Fa21/Math115Int07.pdf
Page 280: #14: Use integration by parts to verify $\int x\cos x dx = x\sin x + \cos x + C$.
Page 280: #18: Use integration by parts to verify $\int x^2 e^{-3x}dx = -\frac{1}{3}e^{-3x}(x^2 + \frac{2}{3}x + \frac{2}{9}) + C$.
Page 281: #22: Show that $\int_0^{2\pi} x\sin nx dx = -\frac{2\pi}{n}$ for any positive interger n.

1.10. **Class 10:** Review day / application day: no HW due, lecture topics chosen by class.

1.11. **Class 11:** Page 293: (29), (35), (40). Lecture: u-substitution: https://youtu.be/KMPQ85_2ESY (54 minutes) slides: https://web.williams.edu/Mathematics/sjmiller/public_html/115Fa21/Math115Int04.pdf
Page 292: #29: $\int \cos^2 x dx$.
Page 293: #35: $\int \sin^2 x\cos^5 x dx$.
Page 293: #40: $\int \cos 3x\cos 2x dx$.

1.12. **Class 12:** Page 310: (2), (31), (35); Lecture: Partial Fractions: https://youtu.be/S5QAJ15b4JM (part I) slides: https://web.williams.edu/Mathematics/sjmiller/public_html/115Fa21/Math115Int10.pdf.
Page 307: #2: Find $\int \frac{dx}{(x-2)\sqrt{x+2}}$.
Page 311: #31: Evaluate $\int \sin\sqrt{x}dx$.
Page 311: #35: Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$.

1.13. **Class 13:** Page 304: (7), (8), (14), (23). Review day / application day: lecture topics chosen by class.
Page 304: #7: Evaluate $\int \frac{dx}{x^2-9}$.
Page 304: #8: Evaluate $\int \frac{xdx}{x^2-3x-4}$.
Page 305: #14: Evaluate $\int \frac{dx}{x^3+x}$.
Page 305: #23: Evaluate $\int \frac{dx}{e^{2x}-3e^x}$.

1.14. **Class 14:** Partial Fractions II: https://youtu.be/fmdWptrelmE (part II) slides: https://web.williams.edu/Mathematics/sjmiller/public_html/115Fa21/Math115Int11.pdf.

1.15. **Class 15:** No class, take-home midterm

1.16. **Class 16:**

1.17. **Class 17:**

1.18. **Class 18:**

1.19. **Class 19:**

1.20. **Class 20:** Sequences and Series; read handout https://people.math.gatech.edu/~cain/notes/cal10.pdf: Homework due: Problem 1. Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ that diverges. Problem 2. Give an example of a sequence of distinct terms a_n such that the sequence $\{a_n\}_{n=1}^{\infty}$ converges. Problem 3. Give an example of a sequence of distinct terms a_n such that $|a_n| < 2013$ and the sequence $\{a_n\}_{n=1}^{\infty}$ does not converge. Problem 10-4 (Cain-Herod): Find the limit of the sequence $a_n = 3/n^2$, or explain why it does not converge. Problem 10-5 (Cain-Herod https://people.math.gatech.edu/~cain/notes/cal10.pdf): Find the limit of the sequence $a_n = \frac{3n^2+2n-7}{n^2}$.

1.21. **Class 21:** Topic TBD Homework: (1) Cain-Herod https://people.math.gatech.edu/~cain/notes/cal10.pdf : Find the limit of the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$. (2) Cain-Herod: Find a value of n that will insure that $1+1/2+1/3+\cdots+1/n > 10^6$. Prove your value works. (3) Cain-Herod: Question 14: Determine if the series $\sum_{k=0}^{\infty} \frac{1}{2e^k+k}$ converges or diverges. (4) Cain-Herod: Question 15: Determine if the series $\sum_{k=0}^{\infty} \frac{1}{2k+1}$ converges or diverges. (5) Let $f(x) = \cos x$, and compute the first eight derivatives of $f(x)$ at $x = 0$, and determine the n-th derivative.

1.22. **Class 22:** Topic TBD Homework: (1) Cain-Herod https://people.math.gatech.edu/~cain/notes/cal10.pdf 10-18: Is the series $\left(\sum_{k=0}^{n} \frac{10^k}{k!}\right)$ convergent or divergent? (2) Cain-Herod 10-21: Is the following series convergent or divergent? $\sum_{k=1}^{n} \frac{3^k}{5^k(k^4+k+1)}$. (3) Let $a_n = \frac{1}{(n \ln n)}$ (one divided by n times the natural log of n). Prove that this series diverges. *Hint: what is the derivative of the natural log of x? Use u-substitution.* (4) Let $a_n = \frac{1}{(n \ln^2 n)}$ (one divided by n times the square of the natural log of n). Prove that this series converges. *Hint: use the same method as the previous problem. (5) Give an example of a sequence or series that you have seen in another class, in something you've read, in something you've observed in the world, ...*

1.23. **Class 25:** Topic TBD

Recall that the **Taylor series** of degree n for a function f at a point x_0 is given by

$$f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

where $f^{(k)}$ denotes the k^{th} derivative of f. We can write this more compactly with summation notation as

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k,$$

where $f^{(0)}$ is just f. In many cases the point x_0 is 0, and the formulas simplify a bit to

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k \;=\; f(0) + f'(0)x + \frac{f''(0)}{2!}\, x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\, x^n.$$

The reason Taylor series are so useful is that they allow us to understand the behavior of a complicated function near a point by understanding the behavior of a related polynomial near that point; the higher the degree of our approximating polynomial, the smaller the error in our approximation. Fortunately, for many applications a first order Taylor series (ie, just using the first derivative) does a very good job. This is also called the **tangent line** method, as we are replacing a complicated function with its tangent line.

One thing which can be a little confusing is that there are $n+1$ terms in a Taylor series of degree n; the problem is we start with the zeroth term, the value of the function at the point of interest. You should never be impressed if someone tells you the Taylor series at x_0 agrees with the function at x_0 – this is forced to hold from the definition! The reason is all the $(x-x_0)^k$ terms vanish, and we are left with $f(x_0)$, so of course the two will agree. Taylor series are only useful when they are close to the original function for x close to x_0.

http://www.williams.edu/go/math/sjmiller/public_html/105/handouts/MVT_TaylorSeries.pdf

Question 1.1. *Find the first five terms of the Taylor series for* $f(x) = x^8 + x^4 + 3$ *at* $x = 0$.

Question 1.2. *Find the first three terms of the Taylor series for* $f(x) = x^8 + x^4 + 3$ *at* $x = 1$.

Question 1.3. *Find the first three terms of the Taylor series for* $f(x) = \cos(5x)$ *at* $x = 0$.

Question 1.4. *Find the first five terms of the Taylor series for* $f(x) = \cos^3(5x)$ *at* $x = 0$.

Question 1.5. *Find the first two terms of the Taylor series for* $f(x) = e^x$ *at* $x = 0$.

Question 1.6. *Find the first six terms of the Taylor series for* $f(x) = e^{x^8} = \exp(x^8)$ *at* $x = 0$.

Question 1.7. *Find the first four terms of the Taylor series for* $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \exp(-x^2/2)/\sqrt{2\pi}$ *at* $x = 0$.

Question 1.8. *Find the first three terms of the Taylor series for* $f(x) = \sqrt{x}$ *at* $x = \frac{1}{3}$.

Question 1.9. *Find the first three terms of the Taylor series for* $f(x) = (1 + x)^{1/3}$ *at* $x = \frac{1}{2}$.

Question 1.10. *Find the first three terms of the Taylor series for* $f(x) = x \log x$ *at* $x = 1$.

Question 1.11. *Find the first three terms of the Taylor series for* $f(x) = \log(1 + x)$ *at* $x = 0$.

Question 1.12. *Find the first three terms of the Taylor series for* $f(x) = \log(1 - x)$ *at* $x = 1$.

Question 1.13. *Find the first two terms of the Taylor series for* $f(x) = \log((1 - x) \cdot e^x) = \log((1 - x) \cdot \exp(x))$ *at* $x = 0$.

Question 1.14. *Find the first three terms of the Taylor series for* $f(x) = \cos(x) \log(1 + x)$ *at* $x = 0$.

Question 1.15. *Find the first two terms of the Taylor series for* $f(x) = \log(1 + 2x)$ *at* $x = 0$.

1.24. **Class 28:** Radial Integration, Introduction to Multivariable Calculus

Question 1.16. *Find the volume of a sphere of radius R (without loss of generality you can argue you can do a sphere of radius 1 and multiply by* R^3*).*

Question 1.17. *Find the volume of region formed by rotating* $y = \sin(x)$ *about the x axis, with x ranging from 0 to* 2π.

Question 1.18. *Find the volume of the region formed by rotating the region between* $y = \sin(x)$ *and y = 0 about the y-axis for x ranging from 0 to* π. *You may use a program such as Mathematica to evaluate the integral.*

1.25. **Class 29:** Review / Application Day

1.26. **Class 30:** Library Trip (no written HW)

1.27. **Class 31:** Watch: Double Plus Ungood: https://www.youtube.com/watch?v=Esa2TYwDmwA&t=309s

Question 1.19. *Calculate, to at least 40 decimal places,* $\frac{100}{9801}$. *Do you notice a pattern? Do you think it will continue forever - why or why not?*

Question 1.20. *Calculate, to at least 40 decimal places,* $\frac{1000}{998999}$. *Do you notice a pattern? Do you think it will continue forever - why or why not?*

1.28. **Class 32:** Differential Equations and Trafalgar

Question 1.21. *Solve the difference equation* $a(n + 1) = 7a(n) - 12a(n - 1)$ *with initial conditions* $a(0) = 3$ *and* $a(1) = 10$.

Question 1.22. *Consider the whale problem from class, but now assume that on every two pairs of 1 year old whales give birth to one new pair of whales, and every four pairs of 2 year old whales give birth to one new pair. Prove or disprove: eventually the whales dies out.*

1.29. **Class 33:** Review Day

1.30. **Class 34:** Application: Mathematical Modeling I: German Tank Problem: https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/GermanTankProblem_Talk_PennState2020.pdf

1.31. **Class 35:** Application: Mathematical Modeling II: German Tank Problem: https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/GermanTankProblem_Talk_PennState2020.pdf

1.32. **Class 36:** Library Trip (no written HW)

2. HOMEWORK SOLUTIONS

2.1. Problems: HW Class 05.

Page 198: #22: Evalute the antiderivative $\int \frac{1+x^2}{\sqrt{x}} dx$.

We use the sum rule and then the power rule:

$$\begin{aligned}\int \frac{1+x^2}{\sqrt{x}} dx &= \int \frac{1}{\sqrt{x}} dx + \int \frac{x^2}{\sqrt{x}} dx \\ &= \int x^{-1/2} dx + \int x^{3/2} dx \\ &= 2x^{1/2} + \frac{2}{5} x^{5/2} + C \\ &= 2x^{1/2}\left(1 + \frac{1}{5}x^2\right) + C.\end{aligned}$$

Page 198: #24: Evalute the antiderivative $\int \cos 3x dx$.

Using the chain rule (or the multiple rule):

$$\int \cos 3x dx = \frac{\sin 3x}{3} + C.$$

Page 198: #46: Evalute the antiderivative $\int \frac{dx}{(x-1)^3}$ with quick formula 1.

Using quick formula 1 we set $g(x) = x - 1$ and $r = -3$:

$$\begin{aligned}\int \frac{dx}{(x-1)^3} &= \int (x-1)^{-3}(1) dx \\ &= \frac{1}{-3+1}(x-1)^{-3+1} + C \\ &= -\frac{1}{2(x-1)^2} + C.\end{aligned}$$

Page 198: #53: Evalute the antiderivative $\int (x-1)^2 x dx$.

Distributing and then using the power rule gives:

$$\begin{aligned}\int (x-1)^2 x dx &= \int (x^2 - 2x + 1) x dx \\ &= \int x^3 - 2x^2 + x dx \\ &= \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 + C.\end{aligned}$$

NOTE: For this problem, it is worth expanding $(x-1)^2$ and then multiplying by x, using the sum rule, and integrating term by term. It would be a very different story if we had say $(x-1)^{140}x$, as we would have to use the binomial theorem and we would have a lot of terms. There is a nice trick, though, in this case. We can write x as $x - 1 + 1$ and thus

$$(x-1)^{140}x = (x-1)^{140}(x-1+1) = (x-1)^{141} + (x-1)^{140}.$$

Now we can just integrate each piece by the power rule. Note this gives a significantly cleaner answer, with just two terms. This is a terrific example of the powe of looking at the problem the right way – we did nothing, as we added zero, but we did it in such a way that we obtained a significantly easier integral to study.

Page 198: #54: Evalute the antiderivative $\int (x^2 - x)^4 (2x - 1) dx$.

Let $f(x) = x^2 - x$, then observe that $f'(x) = 2x - 1$. Thus using the chain rule:

$$\int (x^2 - x)^4 (2x - 1) dx = \frac{1}{5}(x^2 - x)^5 + C.$$

2.2. **Problems: HW Class 06.**

Page 217: #33a: Compute the average value of $f(x) = x^{\frac{1}{5}}$ on $[0, 1]$.

Recall the average value formula:

$$\text{Average of } f \text{ on } [a, b] \text{ is } \frac{1}{b-a}\int_a^b f(x)dx.$$

For $a = 0, b = 1, f(x) = x^{\frac{1}{5}}$, we have that

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)dx &= \frac{1}{1}\int_0^1 x^{1/5}dx \\
&= \frac{5}{6}x^{6/5}\Big|_0^1 \\
&= \frac{5}{6}\cdot 1 - \frac{5}{6}\cdot 0 \\
&= \frac{5}{6}.
\end{aligned}$$

Page 217: #38: In an approximating sum $\sum_{k=1}^n f(x_k^*)\Delta_k x$,if we select x_k^* to be the midpoint of the kth subinterval, then the sum is said to be obtained by the *midpoint rule*. Apply the midpoint rule to approximate $\int_0^1 x^2 dx$, using a division into five equal subintervals, and comapre with the exact result of $\frac{1}{3}$.

For our 5 equal subintervals, we have $[0, 1/5], [1/5, 2/5], [2/5, 3/5], [3/5, 4/5], [4/51]$, with midpoints $1/10, 3/10, 5/10, 7/10$ and $9/10$, respectively, and $\Delta x = 1/5$.

Now, find the values of the function f(x) at the midpoints

$$\begin{aligned}
f(1/10) &= 1/100 \\
f(3/10) &= 9/100 \\
f(5/10) &= 25/100 \\
f(7/10) &= 49/100 \\
f(9/10) &= 81/100,
\end{aligned}$$

and plug into the Midpoint Rule formula:

$$\sum_{k=1}^n f(x_k^*)\Delta_x = \frac{1+9+25+49+81}{100}\frac{1}{5} = \frac{165}{500} = \frac{330}{1000},$$

Which is less than the exact answer of $1/3$ by about $3/1000$.

Page 217: #39a: Apply Simson's rule to approximate $\int_0^1 x^2 dx$ and compare with the results from the answer obtained by the fundamental theorem.

Simpson's Rule:

$$\int_a^b (x)dx \approx \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$$

for even n.

For $n = 4, a = 0, b = 1$, we get subintervals $[0, 1/4], [1/4, 2/4], [2/4, 3/4], [3/4, 1]$.Thus,

$$\begin{aligned}
\int_0^1 x^2 dx &\approx \frac{1-0}{3\cdot 4}[f(0) + 4f(1/4) + 2f(2/4) + 4f(3/4) + f(1)] \\
&= \frac{1}{12}\left[0 + \frac{1}{4} + \frac{1}{2} + \frac{9}{4} + 1\right] \\
&= \frac{1}{12}\left[\frac{1}{4} + \frac{2}{4} + \frac{9}{4} + \frac{4}{4}\right] \\
&= \frac{1}{12}\left[\frac{16}{4}\right] \\
&= \frac{1}{3},
\end{aligned}$$

which is equal to the exact answer.

Remark 2.1. *The formula for Simpson's rule requires n to be an EVEN number. We have the points $x_0, x_1, \ldots, x_n$, and then we evaluate and have the sum $f(x_0) + 4f(x_1) + 2f(x_2) + \cdots$, alternating between 4 and 2 until the end, which is $f(x_n)$.*

Why does n need to be even? We are using parabolas to approximate the area, not rectangles or trapezoids. How many points determine a line? Two. How many determine a parabola? Three. (There WAS a reason we discussed this in class!). Thus to get a parabola we need three points, so we use the interval $[x_0, x_1]$ UNION $[x_1, x_2]$ as the first interval, and that gives us three points: x_0, x_1, x_2. We need to have an even number of intervals $[x_k, x_{k+1}]$ as we take two at a time, so n must be even.

2.3. **Problems: HW Class 08.**

Page 224: #8a: Find the derivative of $y = \ln(x+3)^2 = 2\ln(x+3)$.

Using chain rule we get:

$$\begin{aligned}\frac{d}{dx}(2\ln(x+3)) &= \frac{2}{x+3} * \frac{d}{dx}(x+3) \\ &= \frac{2}{x+3} * 1 \\ &= \frac{2}{x+3}.\end{aligned}$$

Page 224: #8h: Find the derivative of $y = x\ln x - x$.

Using the product rule and then summing the two pieces together we get:

$$\begin{aligned}\frac{d}{dx}(x\ln x - x) &= \frac{d}{dx}(x\ln x) - \frac{d}{dx}(x) \\ &= x\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}(x) - 1 \\ &= x\frac{1}{x} + \ln x * 1 - 1 \\ &= 1 + \ln x - 1 \\ &= \ln x.\end{aligned}$$

Page 225: #9a: Find the antiderivative of $\int \frac{1}{7x}dx$.

Multiplication of constants can be separated to get:

$$\begin{aligned}\int \frac{1}{7x}dx &= \frac{1}{7}\int \frac{1}{x}dx \\ &= \frac{1}{7}(\ln|x| + C) \\ &= \frac{\ln|x|}{7} + C.\end{aligned}$$

Page 226: #11: Express in terms of $\ln 2$ and $\ln 3$: (a) $\ln(3^7)$; (b) $\ln\frac{2}{27}$.

For a, we can use the logarithm property that $\ln(x^a) = a * ln(x)$. Therefore, $\ln(3^7) = 7 * \ln(3)$.
For b, we can the logarithm property that $\ln(\frac{x}{y}) = \ln(x) - \ln(y)$ and $\ln(x^a) = a * ln(x)$. Therefore, $\ln\frac{2}{27} = \ln(2) - \ln(27) = \ln(2) - \ln(3^3) = \ln(2) - 3\ln(3)$.

Page 226: #13: Find the area under the curve $y = \frac{1}{x}$ and above the x-axis, between $x = 2$ and $x = 4$.

This are is equivalent to:

$$\begin{aligned}\int_2^4 \frac{1}{x}dx &= \ln x|_{x=2}^{x=4} \\ &= \ln 4 - \ln 2 \\ &= \ln 2^2 - \ln 2 \\ &= 2\ln 2 - \ln 2 \\ &= \ln 2.\end{aligned}$$

2.4. **Problems: HW Class 09.**

Recall the general formula for integration by parts:

$$\int u dv \;=\; uv - \int v du.$$

Page 280: #14: Use integration by parts to verify $\int x \cos x dx = x \sin x + \cos x + C$.
Let $u = x$ and $dv = \cos x dx$.
Then $du = dx$ and $v = \sin x$.
So

$$\begin{aligned}\int x \cos x dx &= x \sin x - \int \sin x dx \\ &= x \sin x - (-\cos x) + C \\ &= x \sin x + \cos x + C\end{aligned}$$

Page 280: #18: Use integration by parts to verify $\int x^2 e^{-3x} dx = -\frac{1}{3} e^{-3x}(x^2 + \frac{2}{3}x + \frac{2}{9}) + C$.
Let $u = x^2$ and $dv = e^{-3x}$.
Then $du = 2x$ and $v = -\frac{1}{3}e^{-3x}$. So

$$\begin{aligned}\int x^2 e^{-3x} dx &= -\frac{1}{3}x^2 e^{-3x} - \int \left(-\frac{1}{3}e^{-3x}\right)(2x)dx \\ &= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\int e^{-3x}(x)dx \\ &= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} - \int -\frac{1}{3}e^{-3x}dx\right] \\ &= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx\right] \\ &= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} + \frac{1}{3}(-\frac{1}{3}e^{-3x})\right] + C \\ &= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C \\ &= -\frac{1}{3}e^{-3x}(x^2 + \frac{2}{3}x + \frac{2}{9}) + C. \qquad (2.1)\end{aligned}$$

Note we integrated by parts twice, we could use new variables, say $\tilde{u}$ and $\tilde{v}$, but the convention is often to somewhat confusingly over-ride and re-use; we avoid that issue by not showing what we are declaring the variables to be!

Page 281: #22: Show that $\int_0^{2\pi} x \sin nx dx = -\frac{2\pi}{n}$ for any positive interger n.
Let $u = x$ and $dv = \sin nx dx$.
Then $du = dx$ and $v = -\frac{1}{n}\cos nx$. So

$$\begin{aligned}\int_0^{2\pi} x \sin nx dx &= \left[-\frac{x}{n}\cos nx\right]_0^{2\pi} - \int_0^{2\pi} -\frac{1}{n}\cos nx dx \\ &= \left[-\frac{x}{n}\cos nx\right]_0^{2\pi} + \frac{1}{n}\int_0^{2\pi} \cos nx dx \\ &= \left[-\frac{x}{n}\cos nx\right]_0^{2\pi} + \frac{1}{n}\left[\frac{1}{n}\sin nx\right]_0^{2\pi} \\ &= -\frac{2\pi}{n}(1) - 0 + \frac{1}{n}[0 - 0] \\ &= -\frac{2\pi}{n}\end{aligned}$$

2.5. **Problems: HW Class 11.**

Page 292: #29: $\int \cos^2 xdx$.
Integrating $\cos^2 x$ immediately with chain rule or by parts is not easy. Therefore we will use the identity $\cos^2 x = \frac{1+\cos 2x}{2}$.

$$\begin{aligned}\int \cos^2 xdx &= \int \frac{1+\cos 2x}{2}dx \\ &= \int \frac{1}{2}dx + \int \frac{\cos 2x}{2}dx \\ &= \frac{x}{2} + \frac{\sin 2x}{4} + C.\end{aligned}$$

Page 293: #35: $\int \sin^2 x \cos^5 xdx$.
Notice that $\cos x$ has an odd power while $\sin x$ has an even power. This will help us to factor out a $\cos x$ and replace $\sin^2 x$. Let $u = \sin x$ then $du = \cos x$.

$$\begin{aligned}\int \sin^2 x \cos^5 xdx &= \int \sin^2 x \cos^4 x \cos xdx \\ &= \int \sin^2 x(1-\sin^2 x)^2 \cos xdx \\ &= \int u^2(1-u^2)^2 du \\ &= \int u^2(1-2u^2+u^4)du \\ &= \int u^2 - 2u^4 + u^6 du \\ &= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C \\ &= \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C.\end{aligned}$$

There are other ways to do this problem. You can do by integrating by parts and doing the Bring It Over method; you can write that up and submit for extra credit.
Page 293: #40: $\int \cos 3x \cos 2xdx$.
We will use the identity $\cos Ax \cos Bx = \frac{1}{2}(\cos(A-B)x + \cos(A+B)x)$ to break up the expression in the integral into two functions we can integrate more easily.

$$\begin{aligned}\int \cos 3x \cos 2xdx &= \int \frac{1}{2}(\cos(3-2)x + \cos(3+2)x)dx \\ &= \frac{1}{2}\int \cos x + \cos 5xdx \\ &= \frac{1}{2}(\sin x + \frac{\sin 5x}{5} + C) \\ &= \frac{\sin x}{2} + \frac{\sin 5x}{10} + C.\end{aligned}$$

2.6. **Problems: HW Class 12.**

Page 310: #2: $\int \frac{1}{(x-2)\sqrt{x+2}}dx$.
Our problem here is the square root term, so let's set $u = \sqrt{x+2}$.

$$\begin{aligned} u &= \sqrt{x+2} \\ u^2 &= x+2 \\ x &= u^2-2 \\ dx &= 2udu \\ \int \frac{dx}{(x-2)\sqrt{x+2}} &= \int \frac{2udu}{(u^2-4)*u} \\ &= 2\int \frac{du}{u^2-4} \end{aligned}$$

To evaluate this integral, notice that the denominator factors: $u^2-4 = (u-2)(u+2)$. We can thus solve by partial fractions. Let

$$\frac{1}{u^2-4} = \frac{A}{u+2} + \frac{B}{u-2}.$$

Multiply both sides by $(u+2)(u-2)$ to get the equation

$$A(u-2) + B(u+2) = 1.$$

To solve for A and B, set $u = -2$ and $u = 2$, respectively.

$$\begin{aligned} A((-2)-2) + B((-2)+2) &= 1 \\ -4A + 0B &= 1 \\ A &= -\frac{1}{4} \\ A((2)-2) + B((2)+2) &= 1 \\ 0A + 4B &= 1 \\ B &= \frac{1}{4} \end{aligned}$$

So we know

$$\frac{1}{u^2-4} = -\frac{1}{4(u+2)} + \frac{1}{4(u-2)}.$$

And we can integrate this normally:

$$\begin{aligned} 2\int \frac{du}{u^2-4} &= 2\int (-\frac{1}{4(u+2)} + \frac{1}{4(u-2)})du \\ &= \frac{1}{2}\int (\frac{1}{u-2} - \frac{1}{u+2})du \\ &= \frac{1}{2}(\ln|u-2| - \ln|u+2|) + C \\ &= \frac{1}{2}\ln|\frac{u-2}{u+2}| + C \end{aligned}$$

Substitute back in $u = \sqrt{x+2}$, and we get the final answer of

$$\frac{1}{2}\ln|\frac{\sqrt{x+2}-2}{\sqrt{x+2}+2}| + C$$

Page 310: #31: $\int \sin\sqrt{x}dx$ Again, we can't integrate this directly because of the square root term, so let's let $u = \sqrt{x}$. This gives us:

$$\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}dx \\ dx &= 2udu \end{aligned}$$

When we plug this into our original integral, we get

$$\begin{aligned} \int \sin\sqrt{x}dx &= 2\int u\sin udu \\ &= 2(-u\cos u + \int \cos udu) \\ &= 2\sin u - 2u\cos u + C \end{aligned}$$

by integrating by parts. Substituting back in $u = \sqrt{x}$, we get our final answer of

$$2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + C$$

Page 310: #35: $\int \frac{1}{x^2\sqrt{4-x^2}}dx$ It's tricky to figure out the right choice of u here, but we know we want the derivative of the stuff on the inside of the square root to be in the numerator of the integrand. Seeing the x^2 term in the denominator, we might be able to cancel some terms out if we let $u = \frac{1}{x}$ (since this makes $du = -\frac{1}{x^2}$.) It turns out that $u = \frac{2}{x}$ gives a more elegant solution, so we'll use that instead.

$$\begin{aligned} u &= \frac{2}{x} \\ du &= -\frac{2}{x^2}dx \\ x &= \frac{2}{u} \\ dx &= -\frac{2}{u^2}du \end{aligned}$$

Now, we can plug this into our original integral and cancel out terms:

$$\begin{aligned} \int \frac{dx}{x^2\sqrt{4-x^2}} &= \int \frac{-\frac{2}{u^2}du}{(\frac{2}{u})^2\sqrt{4-(\frac{2}{u})^2}} \\ &= -2\int \frac{du}{u^2\frac{4}{u^2}\sqrt{4-\frac{4}{u^2}}} \\ &= -\frac{1}{2}\int \frac{du}{\sqrt{4-\frac{4}{u^2}}} \\ &= -\frac{1}{4}\int \frac{du}{\sqrt{1-\frac{1}{u^2}}} \\ &= -\frac{1}{4}\int \frac{udu}{\sqrt{u^2-1}}. \end{aligned}$$

This is something we're more used to integrating. We'll integrate by substitution again, this time with variable z.

$$\begin{aligned}
z &= u^2 - 1 \\
dz &= 2udu \\
u &= \sqrt{z+1} \\
du &= \frac{dz}{2\sqrt{z+1}} \\
udu &= \frac{1}{2} \\
-\frac{1}{4}\int \frac{udu}{\sqrt{u^2-1}} &= -\frac{1}{4}\int \frac{dz}{2\sqrt{z}} \\
&= -\frac{1}{4}\sqrt{z} + C.
\end{aligned}$$

We can finally substitute back into the world of x:

$$\begin{aligned}
-\frac{1}{4}\sqrt{z} + C &= -\frac{1}{4}\sqrt{u^2-1} + C \\
&= -\frac{1}{4}\sqrt{\frac{4}{x^2} - 1} + C \\
&= -\frac{\sqrt{4-x^2}}{4x} + C.
\end{aligned}$$

2.7. **Problems: HW Class 13.**

Page 304: #7: Evaluate $\int \frac{dx}{x^2-9}$. We'll use partial fractions to convert this integral into two parts that we know how to integrate. Let

$$\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}.$$

Then multiplying both sides by $(x-3)(x+3)$ gives

$$1 = A(x+3) + B(x-3).$$

Next substitute in $x = -3$

$$\begin{aligned} 1 &= A(-3+3) + B(-3-3) \\ 1 &= A(0) + B(-6) \\ \frac{-1}{6} &= B. \end{aligned}$$

Then substitute in $x = 3$

$$\begin{aligned} 1 &= A(3+3) + B(3-3) \\ 1 &= A(6) + B(0) \\ \frac{1}{6} &= A. \end{aligned}$$

Therefore we now know that

$$\frac{1}{x^2-9} = \frac{1}{6}\frac{1}{x-3} - \frac{1}{6}\frac{1}{x+3}.$$

Returning to the original integral we now have that

$$\begin{aligned} \int \frac{dx}{x^2-9} &= \int \frac{1}{6}\frac{1}{x-3} - \frac{1}{6}\frac{1}{x+3} dx \\ &= \frac{1}{6}\int \frac{1}{x-3} - \frac{1}{x+3} dx \\ &= \frac{1}{6}(\ln|x-3| - \ln|x+3| + C) \\ &= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C. \end{aligned}$$

Page 304: #8: Evaluate $\int \frac{xdx}{x^2-3x-4}$. We'll use partial fractions to convert this integral into two parts that we know how to integrate. Let

$$\frac{x}{x^2-3x-4} = \frac{A}{x-4} + \frac{B}{x+1}.$$

Then multiplying both sides by $(x-4)(x+1)$ gives

$$x = A(x+1) + B(x-4).$$

Next substitute in $x = 4$

$$\begin{aligned} x &= A(x+1) + B(x-4) \\ 4 &= A(5) + B(0) \\ \frac{4}{5} &= A. \end{aligned}$$

Then substitute in $x = -1$

$$\begin{aligned} -1 &= A(-1+1) + B(-1-4) \\ -1 &= A(0) + B(-5) \\ \frac{1}{5} &= B. \end{aligned}$$

Therefore we now know that

$$\frac{x}{x^2-3x-4} = \frac{4}{5}\frac{1}{x-4} + \frac{1}{5}\frac{1}{x+1}.$$

Returning to the original integral we now have that

$$\begin{aligned}
\int \frac{xdx}{x^2-3x-4} &= \int \frac{4}{5}\frac{1}{x-4} + \frac{1}{5}\frac{1}{x+1}dx \\
&= \frac{4}{5}\int \frac{1}{x-4}dx - \frac{1}{5}\int \frac{1}{x+1}dx \\
&= \frac{4}{5}\ln|x-4| + \frac{1}{5}\ln|x+1| + C \\
&= \frac{1}{5}\ln|(x-4)^4| + \frac{1}{5}\ln|x+1| + C \\
&= \frac{1}{5}(\ln|(x-4)^4| + \ln|x+1|) + C \\
&= \frac{1}{5}\ln|(x+1)(x-4)^4| + C.
\end{aligned}$$

Page 305: #14: Evaluate $\int \frac{dx}{x^3+x}$. We'll use partial fractions to convert this integral into two parts that we know how to integrate. Let

$$\frac{1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Then multiplying both sides by $x(x^2+1)$ gives

$$\begin{aligned}
1 &= A(x^2+1) + (Bx+C)(x) \\
1 &= Ax^2 + A + Bx^2 + Cx \\
1 &= (A+B)x^2 + Cx + A.
\end{aligned}$$

From this we know $A = 1$ and $C = 0$. Thus, since $A + B = 0$ and $A = 1$, then $B = -1$. Now we can conclude that

$$\frac{1}{x^3+x} = \frac{1}{x} - \frac{x}{x^2+1}.$$

Returning to the original integral we now have that

$$\begin{aligned}
\int \frac{dx}{x^3+x} &= \int \frac{1}{x} - \frac{x}{x^2+1}dx \\
&= \ln|x| - \frac{1}{2}\ln|x^2+1| + C \\
&= \ln|x| - \ln|\sqrt{x^2+1}| + C \\
&= \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C.
\end{aligned}$$

Page 305: #23: Evaluate $\int \frac{dx}{e^{2x}-3e^x}$. Let $u = e^x$, then $du = e^x dx$ so $dx = \frac{du}{e^x}$. Now we have that

$$\int \frac{dx}{e^{2x}-3e^x} = \int \frac{du}{u(u^2-3u)} = \int \frac{du}{u^3-3u^2}$$

We'll use partial fractions to convert this integral into two parts that we know how to integrate. Let

$$\frac{1}{u^3-3u^2} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-3}.$$

Then multiplying both sides by $u^2(u-3)$ gives

$$\begin{aligned}
1 &= A(u(u-3)) + B(u-3) + C(u^2) \\
1 &= A(u^2-3u) + B(u-3) + C(u^2) \\
1 &= (A+C)u^2 + (-3A+B)u + (-3B)
\end{aligned}$$

From this, we know $-3B = 1$ so $B = \frac{-1}{3}$. We also know that $(-3A + B) = 0$ and $B = \frac{-1}{3}$ so $A = \frac{B}{3} = \frac{-1}{9}$. Lastly, we know that $A + C = 0$ and $A = \frac{-1}{9}$ so $C = \frac{1}{9}$. Therefore we now know that

$$\frac{1}{u^3-3u^2} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-3} = \frac{-1}{9u} + \frac{-1}{3u^2} + \frac{1}{9(u-3)}.$$

Returning to the integral with respect to u we now have that

$$\begin{aligned}
\int \frac{du}{u^3 - 3u^2} &= \int \frac{-1}{9u} + \frac{-1}{3u^2} + \frac{1}{9(u-3)} du \\
&= \frac{-1}{9} \int \frac{1}{u} du + \frac{-1}{3} \int \frac{1}{u^2} du + \frac{1}{9} \int \frac{1}{u-3} du \\
&= \frac{-1}{9} \ln|u| + \frac{-1}{3}(-u^{-1}) + \frac{1}{9} \ln|u-3| + C \\
&= \frac{1}{3u} + \frac{1}{9} \ln|u-3| - \frac{1}{9} \ln|u| + C \\
&= \frac{1}{3u} + \frac{1}{9}(\ln|u-3| - \ln|u|) + C. \\
&= \frac{1}{3u} + \frac{1}{9} \ln\left|\frac{u-3}{u}\right| + C.
\end{aligned}$$

Lastly we substitute $e^x = u$ back in

$$\frac{1}{3u} + \frac{1}{9} \ln\left|\frac{u-3}{u}\right| + C = \frac{1}{3e^x} + \frac{1}{9} \ln\left|\frac{e^x - 3}{e^x}\right| + C.$$

2.8. **Class 20:** Sequences and Series; read handout https://people.math.gatech.edu/~cain/notes/cal10.pdf: Homework due: Problem 1. Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ that diverges. Problem 2. Give an example of a sequence of distinct terms a_n such that the sequence $\{a_n\}_{n=1}^{\infty}$ converges. Problem 3. Give an example of a sequence of distinct terms a_n such that $|a_n| < 2013$ and the sequence $\{a_n\}_{n=1}^{\infty}$ does not converge. Problem 10-4 (Cain-Herod): Find the limit of the sequence $a_n = 3/n^2$, or explain why it does not converge. Problem 10-5 (Cain-Herod https://people.math.gatech.edu/~cain/notes/cal10.pdf): Find the limit of the sequence $a_n = \frac{3n^2+2n-7}{n^2}$.
Problem 1:. Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ that diverges.
Solution: There are two ways for a sequence not to converge. It can either get too big (diverge to infinity), or it can bounce around forever and never settle down. For instance, the sequence given by $a_n = n$ for all $n \in \mathbb{N}$ will diverge to infinity, since given any real number $r \in \mathbb{R}$, $a_n > r$ for all $n > r$. A sequence that fails to converge because it bounces is $a_n = (-1)^n$, or more interestingly $a_n = (-1)^n + (-1)^n/n$.

Problem 2:. Give an example of a sequence of distinct terms a_n such that the sequence $\{a_n\}_{n=1}^{\infty}$ converges.
Solution: For a sequence to converge to a limit L, it must eventually get and stay arbitrarily close to L. Consider the sequence $a_n = 1/n$. We claim this converges to 0. To prove this, we need to show that given any $\epsilon > 0$, we can find an N such that $|a_n - 0| < \epsilon$ for all $n > N$. Let N be any integer exceeding $2/\epsilon$. Then for $n > N$, $a_n < \epsilon/2$, so $|a_n - 0| < \epsilon/2 < \epsilon$, so a_n does indeed converge to 0. Arguing more informally, we would say $\lim_{n\to\infty} |a_n - 0| = \lim_{n\to\infty} 1/n$, and this limit is zero, thus proving that 0 is indeed the limit of the sequence. For a more interesting example, consider the sequence $a_n = 3 + 1/n$, which converges to 3.

Problem 3:. Give an example of a sequence of distinct terms a_n such that $|a_n| < 2018$ and the sequence $\{a_n\}_{n=1}^{\infty}$ does not converge.
Solution: Here we are looking for a bounded sequence that does not converge. Since the sequence cannot diverge to infinity, it must continually bounce around. Consider the sequence

$$\{a_n\}_{n=1}^{\infty} \;=\; \left\{1,\ \frac{2}{3},\ \frac{1}{3},\ \frac{4}{5},\ \frac{1}{5},\ \frac{6}{7},\ \frac{1}{7},\ \frac{8}{9},\ \cdots,\ \right\}$$

where the odd terms are given by $a_{2k+1} = 1/(2k+1)$, and the even terms are given by $a_{2k} = 2k/(2k+1)$. Notice that this sequence is bounded since every term is less than or equal to 1, and cannot converge because the odd terms converge to 0 while the even terms converge to 1.
Problem 10-4 (Cain-Herod): Find the limit of the sequence $a_n = 3/n^2$, or explain why it does not converge.
Solution: We can use the limit of a quotient is the quotient of the limit as the limit of the denominator is not zero and we do not have ∞/∞. We see that the numerator is always 3 while the denominator increases and approaches infinity. Thus we know that $\lim_{n\to\infty} 3/n^2 = 0$.

Problem 10-5 (Cain-Herod): Find the limit of the sequence $a_n = \frac{3n^2+2n-7}{n^2}$
Solution: We cannot use the limit of a quotient is the quotient of the limits as we have ∞/∞. One approach is to use L'Hopital's rule and take derivatives of the numerator and the denominator. We have

$$\lim_{n\to\infty} \frac{3n^2+2n-7}{n^2} \;=\; \lim_{n\to\infty} \frac{6n+2}{2n} \;=\; \lim_{n\to\infty} \frac{6}{2} \;=\; \lim_{n\to\infty} 3 \;=\; 3.$$

Another approach is to pull out the highest power of n in the numerator and denominator:

$$\lim_{n\to\infty} \frac{3n^2+2n-7}{n^2} \;=\; \lim_{n\to\infty} \frac{n^2(3+2/n-7/n^2)}{n^2\cdot 1} \;=\; \lim_{n\to\infty} \frac{3+2/n-7/n^2}{1} \;=\; \lim_{n\to\infty} \left(3+\frac{2}{n}-\frac{7}{n^2}\right) \;=\; 3.$$

The analysis is easier than some of the other problems as the denominator was just n to a power.

2.9. **Class 21:** Topic Sequences and Series: Ratio and Root Test: Homework: (1) Cain-Herod https://people.math.gatech.edu/~cain/notes/cal10.pdf : Find the limit of the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$. (2) Cain-Herod: Find a value of n that will insure that $1 + 1/2 + 1/3 + \cdots + 1/n > 10^6$. Prove your value works. (3) Cain-Herod: Question 14: Determine if the series $\sum_{k=0}^{\infty} \frac{1}{2e^k+k}$ converges or diverges. (4) Cain-Herod: Question 15: Determine if the series $\sum_{k=0}^{\infty} \frac{1}{2k+1}$ converges or diverges. (5) Let $f(x) = \cos x$, and compute the first eight derivatives of $f(x)$ at $x = 0$, and determine the n-th derivative.
Problem 10-8 (Cain-Herod): Find the limit of the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$.
Solution: This is the same as finding the sum of the infinite geometric sequence $1 + 1/3 + (1/3)^2 + (1/3)^3 + \cdots$ and then subtracting off 1, as we want to start the sum at $n = 1$ and not $n = 0$. We can use the formula that the sum of infinite geometric sequence with ratio r starting at $n = 0$ is $\frac{1}{1-r}$, provided of course that $|r| < 1$. For us $r = 1/3$, and thus the sum, starting from $n = 0$, is $1/(1 - \frac{1}{3}) = 1/(2/3) = 3/2$; however, we want the sum to start with the $n = 1$ term and not the $n = 0$ term, so we must subtract the $n = 0$ term, which is 1. Thus the answer is $3/2 - 1 = 1/2$.

Problem 10-10: Find a value of n that will insure that $1 + 1/2 + 1/3 + \cdots + 1/n > 10^6$. Prove your value works.
Solution: By a result stated in class, we know that for N large

$$\sum_{n=1}^{N} \frac{1}{N} \approx \ln N.$$

So we must solve $\ln N = 10^6$; the solution is $N = \exp(10^6)$, which is about $9.8 \cdot 10^{434294}$.

It is possible to solve this without using the asymptotic relation for the sum. We showed in class that if we group the terms $1/3$ and $1/4$ we get at least $1/2$, and if we group terms $1/5, 1/6, 1/7, 1/8$ we get at least $1/2$, and so on. If we go up to the term $n = 2^2$ we have at least 1/2 two times, if we go up to $n = 2^3$ we have 1/2 at least 3 times, and in general if we go up to n^k then we have 1/2 at least k times. If we want to have the sum at least 10^6, we just need to take $k = 2 \cdot 10^6$, which means $n = 2^{2 \cdot 10^6} = 4^{10^6}$, which is approximately $3.0 \cdot 10^{602059}$. Note how much larger this is than the answer we get from using the sum of the first N terms is about $\ln N$.
Page 10-8: Question 14: Determine if the series $\sum_{k=0}^{\infty} \frac{1}{2e^k+k}$ converges or diverges.
Solution: We will use the comparison test to determine if this series converges or diverges. The series $\sum_{k=0}^{\infty} \frac{1}{2e^k+k}$ is less than the series $\sum_{k=0}^{\infty} \frac{1}{2e^k}$, which is less than the convergent series $\sum_{k=0}^{\infty} \frac{1}{e^k} = \sum_{k=0}^{\infty} (1/e)^k$. This last series is a geometric series with ratio $r = 1/e$, as $|r| < 1$, the geometric series converges. Thus, by the comparison test, the original sequence converges because $|\frac{1}{2e^k+k}| \le \frac{1}{e}$.

Page 10-8: Question 15: Determine if the series $\sum_{k=0}^{\infty} \frac{1}{2k+1}$ converges or diverges.
Solution: We will use the comparison test to determine if this series converges or diverges. We want to compare this to a multiple of the harmonic series; we know the harmonic series diverges, and multiplying each term by a constant won't change if it converges or diverges. We have $4k \ge 2k + 1$ for all $k \ge 1$. This implies $\frac{1}{2k+1} \ge \frac{1}{4k} = \frac{1}{4}\frac{1}{k}$. Thus our series is greater, term by term, than the harmonic series (multiplied by 1/4). As the harmonic series diverges, so too does our series.

Another proof is to note that the sum over the odd indexed terms (which are just the odd terms) in the harmonic series is at least as large as the sum over the even terms, and since the total sum diverges so too must the sum over just the odd indexed terms.

Page 10-8: Question 16: Determine if the series $\sum_{k=2}^{\infty} \frac{1}{\log k}$ converges or diverges.
Solution: We will use the comparison test to determine if this series converges or diverges. The growth of a log function is slower than a linear function: $\log k \le k$; taking the reciprocal reverses the relation, so $\frac{1}{\log k} \ge \frac{1}{k}$. Thus our series is greater, term by term, than the harmonic series. As the harmonic series diverges, so too does our series.

Additional question: Let $f(x) = \cos x$, and compute the first eight derivatives of $f(x)$ at $x = 0$, and determine the n^{th} derivative.
Solution: We will begin by computing the first eight derivatives.

$$\begin{aligned}
f'(x) &= -\sin x \\
f''(x) &= -\cos x \\
f'''(x) &= \sin x \\
f^{(iv)}(x) &= \cos x \\
f^{(v)}(x) &= -\sin x \\
f^{(vi)}(x) &= -\cos x \\
f^{(vii)}(x) &= \sin x \\
f^{(viii)}(x) &= \cos x.
\end{aligned}$$

Now compute the derivatives at $f(0)$.

$$\begin{aligned}
f'(0) &= -\sin 0 = 0, \quad f''(0) = -\cos 0 = -1 \\
f'''(0) &= \sin 0 = 0, \quad f^{(iv)}(0) = \cos 0 = 1 \\
f^{(v)}(0) &= -\sin 0 = 0, \quad f^{(vi)}(0) = -\cos 0 = -1 \\
f^{(vii)}(0) &= \sin 0 = 0, \quad f^{(viii)}(0) = \cos 0 = 1.
\end{aligned}$$

We see the pattern: 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1 and so on. Specifically, the even derivatives vanish, and if $f(x) = \cos x$ then $f^{(4k+1)}(0) = -1$ while $f^{(4k+3)}(0) = 1$.

2.10. **Class 22:** Topic TBD Homework: (1) Cain-Herod https://people.math.gatech.edu/~cain/notes/cal10.pdf 10-18: Is the series $\left(\sum_{k=0}^{n} \frac{10^k}{k!}\right)$ convergent or divergent? (2) Cain-Herod 10-21: Is the following series convergent or divergent? $\sum_{k=1}^{n} \frac{3^k}{5^k(k^4+k+1)}$. (3) Let $a_n = \frac{1}{(n \ln n)}$ (one divided by n times the natural log of n). Prove that this series diverges. *Hint: what is the derivative of the natural log of x? Use u-substitution.* (4) Let $a_n = \frac{1}{(n \ln^2 n)}$ (one divided by n times the square of the natural log of n). Prove that this series converges. *Hint: use the same method as the previous problem. (5) Give an example of a sequence or series that you have seen in another class, in something you've read, in something you've observed in the world, ...*

Problem 10-18: Is the series $\left(\sum_{k=0}^{n} \frac{10^k}{k!}\right)$ convergent or divergent?

Solution: We use the ratio test:

$$\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{10^{k+1}}{(k+1)!}\right| \left|\frac{k!}{10^k}\right| = \lim_{k\to\infty} \left|\frac{10}{k+1}\right| = 0 < 1,$$

so the series converges as the ratio ρ is less than 1.

Problem 10-21: Is the following series convergent or divergent?

$$\sum_{k=1}^{n} \frac{3^k}{5^k(k^4+k+1)}$$

Solution: We use the Comparison Test:

$$\sum_{k=1}^{n} \left(\frac{3}{5}\right)^k \frac{1}{(k^4+k+1)} < \sum_{k=1}^{n} \frac{1}{(k^4+k+1)} < \sum_{k=1}^{n} \frac{1}{k^4},$$

which converges (it is a p-series with $p = 4$), and thus the original series also converges. Alternatively, we have $a_k \leq (3/5)^k$, and we obtain convergence by comparing with a geometric series with ratio $3/5$.

Problem 3: Let $a_n = \frac{1}{(n \ln n)}$ (one divided by n times the natural log of n). Prove that this series diverges. *Hint: what is the derivative of the natural log of x? Use u-substitution.*

Solution: We use the integral test. We start the series with $n = 2$ as $\ln 1 = 0$ and we cannot divide by zero. Set $f(x) = \frac{1}{x \ln x}$; note $f(n) = a_n$. The convergence / divergence of the series is equivalent to the convergence or divergence of the integral $\int_2^\infty \frac{1}{x \ln x} dx$. Through substitution by parts, we have $u = \ln x$, $du = \frac{dx}{x}$, and $x : 2 \to \infty$ becomes $u : \ln 2 \to \infty$. Then

$$\int_2^\infty \frac{1}{\ln x} \frac{dx}{x} = \int_{\ln 2}^\infty \frac{1}{u} du = [\ln u]_{\ln 2}^\infty.$$

As this clearly diverges, the original series diverges as well.

Problem 4: Let $a_n = \frac{1}{(n \ln^2 n)}$ (one divided by n times the square of the natural log of n). Prove that this series converges. *Hint: use the same method as the previous problem.*

Solution: We integrate $\int_2^\infty \frac{1}{x \ln^2 x} dx$, where we cannot have $n = 1$ (see previous problem). Through u-substitution, we have $u = \ln x$, $du = \frac{dx}{x}$, and $x : 2 \to \infty$ becomes $u : \ln 2 \to \infty$. Then

$$\int_2^\infty \frac{1}{\ln^2 x} \frac{dx}{x} = \int_{\ln 2}^\infty \frac{1}{u^2} du = \left[-\frac{1}{u}\right]_{\ln 2}^\infty = \frac{1}{\ln 2}.$$

As this converges, the original series converges as well.

Problem 5: Give an example of a sequence or series that you have seen in another class, in something you've read, in something you've observed in the world, ...

2.11. **Class 25:** Topic TBD

Recall that the **Taylor series** of degree n for a function f at a point x_0 is given by

$$\begin{aligned} f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \end{aligned}$$

where $f^{(k)}$ denotes the k^{th} derivative of f. We can write this more compactly with summation notation as

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

where $f^{(0)}$ is just f. In many cases the point x_0 is 0, and the formulas simplify a bit to

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k \;=\; f(0) + f'(0)x + \frac{f''(0)}{2!}\, x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\, x^n.$$

The reason Taylor series are so useful is that they allow us to understand the behavior of a complicated function near a point by understanding the behavior of a related polynomial near that point; the higher the degree of our approximating polynomial, the smaller the error in our approximation. Fortunately, for many applications a first order Taylor series (ie, just using the first derivative) does a very good job. This is also called the **tangent line** method, as we are replacing a complicated function with its tangent line.

One thing which can be a little confusing is that there are $n + 1$ terms in a Taylor series of degree n; the problem is we start with the zeroth term, the value of the function at the point of interest. You should never be impressed if someone tells you the Taylor series at x_0 agrees with the function at x_0 – this is forced to hold from the definition! The reason is all the $(x - x_0)^k$ terms vanish, and we are left with $f(x_0)$, so of course the two will agree. Taylor series are only useful when they are close to the original function for x close to x_0.

http://www.williams.edu/go/math/sjmiller/public_html/105/handouts/MVT_TaylorSeries.pdf

(Not sure why the numbering is off by one...)

Question 2.2. *Find the first five terms of the Taylor series for $f(x) = x^8 + x^4 + 3$ at $x = 0$.*

Solution: To find the first five terms requires evaluating the function and its first four derivatives:

$$\begin{aligned} f(0) &= 3 \\ f'(x) &= 8x^7 + 4x^3 \quad\Rightarrow\quad f'(0) \;=\; 0 \\ f''(x) &= 56x^6 + 12x^2 \quad\Rightarrow\quad f''(0) \;=\; 0 \\ f'''(x) &= 336x^5 + 24x \quad\Rightarrow\quad f'''(0) \;=\; 0 \\ f^{(4)}(x) &= 1680x^4 + 24 \quad\Rightarrow\quad f^{(4)} \;=\; 24. \end{aligned}$$

Therefore the first five terms of the Taylor series are

$$f(0) + f'(0)x + \cdots + \frac{f^{(4)}(0)}{4!}\, x^4 \;=\; 3 + \frac{24}{4!}\, x^4 \;=\; 3 + x^4.$$

This answer shouldn't be surprising as we can view our function as $f(x) = 3 + x^4 + x^8$; thus our function is presented in such a way that it's easy to see its Taylor series about 0. If we wanted the first six terms of its Taylor series expansion about 0, the answer would be the same. We won't see anything new until we look at the degree 8 Taylor series (ie, the first nine terms), at which point the x^8 term appears. □

Question 2.3. *Find the first three terms of the Taylor series for $f(x) = x^8 + x^4 + 3$ at $x = 1$.*

Solution: We can find the expansion by taking the derivatives and evaluating at 1 and not 0. We have

$$\begin{aligned} f(x) &= x^8 + x^4 + 3 \quad\Rightarrow\quad f(1) \;=\; 5 \\ f'(x) &= 8x^7 + 4x^3 \quad\Rightarrow\quad f'(1) \;=\; 12 \\ f''(x) &= 56x^6 + 12x^2 \quad\Rightarrow\quad f''(1) \;=\; 68. \end{aligned}$$

Therefore the first three terms gives

$$f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 \;=\; 5 + 12(x - 1) + 34(x - 1)^2.$$

□

Important Note: Another way to do this problem is one of my favorite tricks, namely converting a Taylor expansion about one point to another. We write x as $(x-1)+1$; we have just added zero, which is one of the most powerful tricks in mathematics. We then have

$$x^8 + x^4 + 3 \;=\; ((x-1)+1)^8 + ((x-1)+1)^4 + 3;$$

we can expand each term by using the Binomial Theorem, and after some algebra we'll find the same answer as before. For example, $((x-1)+1)^4$ equals

$$\binom{4}{0}(x-1)^4 1^0 + \binom{4}{1}(x-1)^3 1^1 + \binom{4}{2}(x-1)^2 1^2 + \binom{4}{3}(x-1)^1 1^3 + \binom{4}{4}(x-1)^0 1^5.$$

In this instance, it is not a good idea to use this trick, as this makes the problem more complicated rather than easier; however, there are situations where this trick does make life easier, and thus it is worth seeing. We'll see another trick in the next problem (and this time it *will* simplify things).

Question 2.4. *Find the first three terms of the Taylor series for $f(x) = \cos(5x)$ at $x = 0$.*

Solution: The standard way to solve this is to take derivatives and evaluate. We have

$$\begin{aligned} f(x) &= \cos(5x) &\Rightarrow\; f(0) &= 1 \\ f'(x) &= -5\sin(5x) &\Rightarrow\; f'(0) &= 0 \\ f''(x) &= -25\cos(5x) &\Rightarrow\; f''(0) &= -25. \end{aligned}$$

Thus the answer is

$$f(0) + f'(0)x + \frac{f''(0)}{2}\,x^2 \;=\; 1 - \frac{25}{2}x^2.$$

□

Important Note: We discuss a faster way of doing this problem. This method assumes we know the Taylor series expansion of a related function, $g(u) = \cos(u)$. This is one of the three standard Taylor series expansions one sees in calculus (the others being the expansions for $\sin(u)$ and $\exp(u)$; a good course also does $\log(1 \pm u)$). Recall

$$\cos(u) \;=\; 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k}}{(2k)!}.$$

If we replace u with $5x$, we get the Taylor series expansion for $\cos(5x)$:

$$\cos(5x) \;=\; 1 - \frac{(5x)^2}{2!} + \frac{(5x)^4}{4!} - \frac{(5x)^6}{6!} + \cdots.$$

As we only want the first three terms, we stop at the x^2 term, and find it is $1 - 25x^2/2$. The answer is the same as before, but this seems much faster. Is it? At first it seems like we avoided having to take derivatives. We haven't; the point is we took the derivatives years ago in Calculus when we found the Taylor series expansion for $\cos(u)$. We now use that. We see the advantage of being able to recall previous results – we can frequently modify them (with very little effort) to cover a new situation; however, we can of course only do this if we remember the old results!

Question 2.5. *Find the first five terms of the Taylor series for $f(x) = \cos^3(5x)$ at $x = 0$.*

Solution: Doing (a lot of!) differentiation and algebra leads to

$$1 - \frac{75}{2}\,x^2 + \frac{4375}{8}\,x^4 - \frac{190625}{48}\,x^6;$$

we calculated more terms than needed because of the comment below. Note that $f'(x) = -15\cos^2(5x)\sin(5x)$. To calculate $f''(x)$ involves a product and a power rule, and we can see that it gets worse and worse the higher derivative we need! It is worth doing all these derivatives to appreciate the alternate approach given below. □

Important Note: There is a faster way to do this problem. From the previous exercise, we know

$$\cos(5x) \;=\; 1 - \frac{25}{2}\,x^2 + \text{terms of size } x^3 \text{ or higher}.$$

Thus to find the first five terms is equivalent to just finding the coefficients up to x^4. Unfortunately our expansion is just a tad too crude; we only kept up to x^2, and we need to have up to x^4. So, let's spend a little more time and compute the Taylor series of $\cos(5x)$ of degree 4: that is

$$1 - \frac{25}{2}\,x^2 + \frac{625}{24}\,x^4.$$

If we cube this, we'll get the first six terms in the Taylor series of $\cos^3(5x)$. In other words, we'll have the degree 5 expansion, and all our terms will be correct up to the x^6 term. The reason is when we cube, the only way we can get a term of degree 5 or less is covered. Thus we need to compute

$$\left(1 - \frac{25}{2}\,x^2 + \frac{625}{24}\,x^4\right)^3;$$

however, as we only care about the terms of x^5 or lower, we can drop a lot of terms in the product. For instance, one of the factors is the x^4 term; if it hits another x^4 term or an x^2 it will give an x^6 or higher term, which we don't care about. Thus, taking the cube but only keeping terms like x^5 or lower degree, we get

$$1 + \binom{3}{1} 1^2 \left(-\frac{25}{2}\,x^2\right) + \binom{3}{2} 1 \left(-\frac{25}{2}\,x^2\right)^2 + \binom{3}{1} 1^2 \left(\frac{625}{24}\,x^4\right).$$

After doing a little algebra, we find the same answer as before.

So, was it worth it? To each his own, but again the advantage of this method is we reduce much our problem to something we've already done. If we wanted to do the first seven terms of the Taylor series, we would just have to keep a bit more, and expand the original function $\cos(5x)$ a bit further. As mentioned above, to truly appreciate the power of this method you should do the problem the long way (ie, the standard way).

Question 2.6. *Find the first two terms of the Taylor series for $f(x) = e^x$ at $x = 0$.*

Solution: This is merely the first two terms of one of the most important Taylor series of all, the Taylor series of e^x. As $f'(x) = e^x$, we see $f^{(n)}(x) = e^x$ for all n. Thus the answer is

$$f(0) + f'(0)x \;=\; 1 + x.$$

More generally, the full Taylor series is

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

□

Question 2.7. *Find the first six terms of the Taylor series for $f(x) = e^{x^8} = \exp(x^8)$ at $x = 0$.*

Solution: The first way to solve this is to keep taking derivatives using the chain rule. Very quickly we see how tedious this is, as $f'(x) = 8x^7\exp(x^8)$, $f''(x) = 64x^{14}\exp(x^8) + 56x^6\exp(x^8)$, and of course the higher derivatives become even more complicated. We use the faster idea mentioned above. We know

$$e^u \;=\; 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots \;=\; \sum_{n=0}^{\infty} \frac{u^n}{n!},$$

so replacing u with x^8 gives

$$e^{x^8} \;=\; 1 + x^8 + \frac{(x^8)^2}{2!} + \cdots.$$

As we only want the first six terms, the highest term is x^5. Thus the answer is just 1 – we would only have the x^8 term if we wanted at least the first nine terms! For this problem, we see how much better this approach is; knowing the first two terms of the Taylor series expansion of e^u suffice to get the first six terms of e^{x^8}. This is magnitudes easier than calculating all those derivatives. Again, we see the advantage of being able to recall previous results.

Question 2.8. *Find the first four terms of the Taylor series for $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} = \exp(-x^2/2)/\sqrt{2\pi}$ at $x = 0$.*

Solution: The answer is

$$\frac{1}{2\pi} - \frac{x^2}{4\pi}.$$

We can do this by the standard method of differentiating, or we can take the Taylor series expansion of e^u and replace u with $-x^2/2$. □

Question 2.9. *Find the first three terms of the Taylor series for $f(x) = \sqrt{x}$ at $x = \frac{1}{3}$.*

Solution: If $f(x) = x^{1/2}$, $f'(x) = \frac{1}{2}x^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2}$. Evaluating at 1/3 gives

$$\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{2}\left(x - \frac{1}{3}\right) - \frac{3\sqrt{3}}{8}\left(x - \frac{1}{3}\right)^2.$$

□

Question 2.10. *Find the first three terms of the Taylor series for $f(x) = (1+x)^{1/3}$ at $x = \frac{1}{2}$.*

Solution: Doing a lot of differentiation and algebra yields

$$\left(\frac{3}{2}\right)^{1/3}+\frac{1}{3}\left(\frac{2}{3}\right)^{2/3}\left(x-\frac{1}{2}\right)-\frac{2}{27}\left(\frac{2}{3}\right)^{2/3}\left(x-\frac{1}{2}\right)^{2}.$$

□

Question 2.11. *Find the first three terms of the Taylor series for $f(x) = x\log x$ at $x = 1$.*

Solution: One way is to take derivatives in the standard manner and evaluate; this gives

$$(x-1)+\frac{(x-1)^2}{2}.$$

□

Important Note: Another way to do this problem involves two tricks we've mentioned before. The first is we need to know the series expansion of $\log(x)$ about $x = 1$. One of the the most important Taylor series expansions, which is often done in a Calculus class, is

$$\log(1+u) \;=\; u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}+\cdots \;=\; \sum_{n=1}^{\infty}\frac{(-1)^{k+1}u^k}{k}.$$

We then write

$$x\log x \;=\; ((x-1)+1)\cdot\log\left(1+(x-1)\right);$$

we can now grab the Taylor series from

$$((x-1)+1)\cdot\left((x-1)-\frac{(x-1)^2}{2}\right) \;=\; (x-1)+\frac{(x-1)^2}{2}+\cdots.$$

Question 2.12. *Find the first three terms of the Taylor series for $f(x) = \log(1 + x)$ at $x = 0$.*

Question 2.13. *Find the first three terms of the Taylor series for $f(x) = \log(1 - x)$ at $x = 1$.*

Solution: The expansion for $\log(1-x)$ is often covered in a Calculus class; equivalently, it can be found from $\log(1+u)$ by replacing u with $-x$. We find

$$\log(1-x) \;=\; -\left(x+\frac{x}{2}+\frac{x}{3}+\cdots\right) \;=\; -\sum_{n=1}^{\infty}\frac{x^n}{n}.$$

For this problem, we get $x + \frac{x^2}{2}$. □

Question 2.14. *Find the first two terms of the Taylor series for $f(x) = \log((1 - x)\cdot e^x) =$ $\log((1 - x)\cdot \exp(x))$ at $x = 0$.*

Solution: Taking derivatives and doing the algebra, we see the answer is just zero! The first term that has a non-zero coefficient is the x^2 term, which comes in as $-x^2/2$. A better way of doing this is to simplify the expression before taking the derivative. As the logarithm of a product is the sum of the logarithms, we have $\log((1 - x)\cdot e^x)$ equals $\log(1 - x) + \log e^x$. But $\log e^x = x$, and $\log(1 - x) = -x - x^2/2 - \cdots$. Adding the two expansions gives $-x^2/2 - \cdots$, which means that the first two terms of the Taylor series vanish. □

Question 2.15. *Find the first three terms of the Taylor series for $f(x) = \cos(x)\log(1 + x)$ at $x = 0$.*

Solution: Taking derivatives and doing the algebra gives $x - x^2/2$. □

Important Note: A better way of doing this is to take the Taylor series expansions of each piece and then multiply them together. We need only take enough terms of each piece so that we are sure that we get the terms of order x^2 and lower correct. Thus

$$\cos(x)\log(1+x) \;=\; \left(1-\frac{x^2}{2}+\cdots\right)\cdot\left(x-\frac{x^2}{2}+\cdots\right) \;=\; x-\frac{x^2}{2}+\cdots.$$

Question 2.16. *Find the first two terms of the Taylor series for $f(x) = \log(1 + 2x)$ at $x = 0$.*

Solution: The fastest way to do this is to take the Taylor series of $\log(1 + u)$ and replace u with $2x$, giving $2x$. □

2.12. **Class 27:** Topic Volumes of Revolution Homework: (1) Solve $y'(x) = x^2$. (2) Solve $\frac{dy}{dx} = y$. (3) Solve $\frac{d^2y}{dx^2} = y$. (4) Solve $\frac{d^2y}{dx^2} = -y$. (5) Solve $\frac{dy}{dx} = x + xy$. (6) A player hits a fastball 5 ft above the ground. If the ball leaves his bat at 100mph, what angle should he hit it so that the horizontal distance is maximized, and what is the maximum distance? Note: You can use a computer to estimate/approximate some quantities. (7) Imagine the Earth has a uniform density of 1 kg per cubic meter. If there is a massive explosion and all the Earth more than half the radius from the center shoots off into space, what is the ratio of the new acceleration due to gravity at the Earth's reduced surface relative to the original acceleration at the surface of the full Earth?
Problem 1: Solve $y'(x) = x^2$.
Solution: We can integrate this as is.

$$\int x^2 dx = \frac{1}{3}x^3 + C$$

Problem 2: Solve $\frac{dy}{dx} = y$.
Solution: Solve by separation of variables:

$$\begin{aligned} \frac{dy}{dx} &= y \\ \frac{dy}{y} &= dx \\ \int \frac{dy}{y} &= \int dx \\ \ln y &= x + C \\ y &= e^{x+C} \\ &= ke^x \end{aligned}$$

Problem 3: Solve $\frac{d^2y}{dx^2} = y$.
Solution: We are looking for functions that are equal to their own second derivative. Exponential and trig functions have derivatives dependent on the original function. If we take ke^x, all of its derivatives will be equal to itself, so $y = ke^x$.

Problem 4: Solve $\frac{d^2y}{dx^2} = -y$.
Solution: Again, we want to consider exponential and trig functions. The derivatives of ke^x will always be ke^x, but what about ke^{-x}? Here, we alternate between $\frac{dy}{dx} = y$ and $\frac{dy}{dx} = -y$, but $\frac{dy}{dx} = -y$ only for odd numbered derivatives. The trig functions $\sin x$ and $\cos x$, though, cycle in the way we want.

$$\begin{aligned} \frac{d}{dx}(\sin x) &= \cos x \\ \frac{d}{dx}(\cos x) &= -\sin x \\ \frac{d}{dx}(-\sin x &= -\cos x \\ \frac{d}{dx}(-\cos x) &= \sin x \end{aligned}$$

So sin, cos, scalar multiples of these functions, and addition of these all solve this equation.

Problem 5: Solve $\frac{dy}{dx} = x + xy$.
Solution: Again, we can separate the variables and integrate:

$$\begin{aligned}
\frac{dy}{dx} &= x(1+y) \\
\frac{dy}{1+y} &= xdx \\
\int \frac{dy}{1+y} &= \int xdx \\
\ln 1+y &= \frac{1}{2}x^2 + C \\
1+y &= e^{\frac{1}{2}x^2+C} \\
y &= ke^{\frac{1}{2}x^2} - 1
\end{aligned}$$

Problem 6: A player hits a fastball 5 ft above the ground. If the ball leaves his bat at 100mph, what angle should he hit it so that the horizontal distance is maximized, and what is the maximum distance? Note: You can use a computer to estimate/approximate some quantities.
Solution: For a projectile in motion acted on by only the force of gravity, we can model its x and y positions with the equations

$$\begin{aligned}
x &= x_0 + v_{x0}t \\
y &= y_0 + v_{y0}t - \frac{1}{2}gt^2
\end{aligned}$$

Where g is the force of gravity on Earth, approximately 9.8 meters per second per second. We know the initial x position is 0 and the intial y position is 5 feet, and we know the initial velocity of the ball is 100mph. *Note: I'm going to convert imperial units to metric - you could do it in imperial, just preference*. In metric units, 5 feet is approximately 1.52 meters and 100mph is approximately 44.7 meters per second. Now, the initial velocities in the x and y directions are modeled by

$$\begin{aligned}
v_{x0} &= v_0 \cos\theta \\
v_{y0} &= v_0 \sin\theta
\end{aligned}$$

With this, we can compute that

$$y = 1.52 + 44.7\sin\theta t - 4.9t^2$$

We want to know how long the baseball is in the air - a function of t in terms of θ - which we can solve for by looking at the case where $y = 0$, when the ball hits the ground.

$$\begin{aligned}
0 &= 1.52 + 44.7\sin\theta t - 4.9t^2 \\
4.9t^2 - 44.7\sin\theta - 1.52 &= 0 \\
t &= \frac{44.7\sin\theta + \sqrt{1998.1(\sin\theta)^2 + 29.8}}{9.8}
\end{aligned}$$

Which we obtain with the quadratic formula. Note that there is a value for t in which the square root is subtracted in the numerator, but this t is negative, and the baseball cannot go back in time after it is hit, so we don't care about it. Now, finally, we can substitute this value of t into our equation for the x position of the ball to get a function of x only dependent on θ.

$$x = 44.7\cos\theta(\frac{44.7\sin\theta + \sqrt{1998.1(\sin\theta)^2 + 29.8}}{9.8})$$

Graphing this function, we see that x has a maximum at 45 degrees (and $45 + 360n$ degrees for integer values of n), so the batter should hit the ball at an angle of 45 degrees. This gives a maximum distance of

$$\begin{aligned}
x &= 44.7\cos 45(\frac{44.7\sin 45 + \sqrt{1998.1(\sin 45)^2 + 29.8}}{9.8}) \\
&= 44.7\frac{\sqrt{2}}{2}(\frac{44.7\frac{\sqrt{2}}{2} + \sqrt{1998.1(\frac{\sqrt{2}}{2})^2 + 29.8}}{9.8}) \\
&= 205.4m
\end{aligned}$$

Problem 7: Imagine the Earth has a uniform density of 1 kg per cubic meter. If there is a massive explosion and all the Earth more than half the radius from the center shoots off into space, what is the ratio of the new acceleration due to gravity at the Earth's reduced

surface relative to the original acceleration at the surface of the full Earth?
Solution: The formula for the force of gravity between two objects is given by

$$F = G\frac{m_1 m_2}{r^2}$$

where m_1 is the mass of the first object, m_2 is the mass of the second, r is the distance between them, and G is a constant. The volume of a sphere is given by $\frac{4}{3}\pi * r^3$, so the volume of a sphere that has had half of its radius exploded off is

$$\frac{4}{3}\pi * (\frac{r}{2})^3 = \frac{1}{8}(\frac{4}{3}\pi * r^3)$$

or $\frac{1}{8}$ its original volume. Since the Earth has uniform density, this means that its mass becomes $\frac{1}{8}$ its original mass. Now, we can compare the force of gravity on Earth's surface before and after the explosion:

$$\begin{aligned} F_0 &= G\frac{m_1 m_2}{r^2} \\ F_1 &= G\frac{\frac{1}{8}m_1 m_2}{(\frac{r}{2})^2} \\ &= G\frac{\frac{1}{8}m_1 m_2}{\frac{1}{4}r^2} \\ &= \frac{1}{2}G\frac{m_1 m_2}{r^2} \\ &= \frac{1}{2}F_0 \end{aligned}$$

2.13. **Class 28:** Radial Integration, Introduction to Multivariable Calculus

Problem 1: Find the volume of a sphere of radius R (without loss of generality you can argue you can do a sphere of radius 1 and multiply by R^3).

Solution: Let's consider a sphere of radius R centered at the origin. Cutting it across the yz-plane leaves a circle in the yz-plane with radius R. If we just consider the top semi-circle of these circle, then we can revolve it arond the x-axis to get a full sphere. This semi-circle is traced by the function $f(x) = \sqrt{R^2 - x^2}$. For each $x \in [-R, R]$, $f(x)$ gives us the radius of a circular disc we can rotate about the x-axis. Therefore, for a given x, the area of this circular disc is $\pi f(x)^2$. Each disc is infinitely thin and we need to sum them up from $-R$ to R to get the entire sphere. Putting this all together we get

$$\begin{aligned}
\int_{-R}^{R} \pi f(x)^2 dx &= \int_{-R}^{R} \pi(\sqrt{R^2 - x^2})^2 dx \\
&= \pi \int_{-R}^{R} R^2 - x^2 dx \\
&= \pi \left(R^2 x - \frac{x^3}{3} \right) \Big|_{-R}^{R} \\
&= \pi \left(R^2 * R - \frac{R^3}{3} \right) - \pi \left(R^2 * (-R) - \frac{(-R)^3}{3} \right) \\
&= \pi \left(R^2 * R - \frac{R^3}{3} - R^2 * (-R) + \frac{(-R)^3}{3} \right) \\
&= \pi \left(R^3 - \frac{R^3}{3} + R^3 - \frac{R^3}{3} \right) \\
&= \frac{4}{3} \pi R^3
\end{aligned}$$

Thus, we get that the volume for a sphere of radius R is $\frac{4}{3}\pi R^3$ as we'd expect.

Problem 2: Find the volume of region formed by rotating $y = \sin(x)$ about the x axis, with x ranging from 0 to 2π.

Solution: Our function $f(x) = \sin(x)$ gives us the height of each disc at a particular x value. We take this radius, square it and multiply by π to get the area of each disc and then sum up these areas from 0 to 2π

$$\begin{aligned}
\int_{0}^{2\pi} \pi f(x)^2 dx &= \int_{0}^{2\pi} \pi \sin^2(x) dx \\
&= \pi \int_{0}^{2\pi} \frac{1}{2} - \frac{1}{2}\cos(2x) dx \\
&= \pi \left(\frac{x}{2} - \frac{1}{4}\sin(2x) \right) \Big|_{0}^{2\pi} \\
&= \pi \left(\frac{2\pi}{2} - \frac{1}{4}\sin(2 * 2\pi) \right) - \pi \left(\frac{0}{2} - \frac{1}{4}\sin(0) \right) \\
&= \pi \left(\pi - \frac{1}{4}\sin(4\pi) \right) \\
&= \pi^2
\end{aligned}$$

Problem 3: Find the volume of the region formed by rotating the region between y = sin(x) and y = 0 about the y-axis for x ranging from 0 to π. You may use a program such as Mathematica to evaluate the integral.

Solution: This problem is nearly identical to the last except now our bound is from 0 to π instead of to 2π. Notice that $\sin(x)$ is identical once rotated about the x-axis from 0 to π and from π to 2π. Therefore, our answer will be half of the answer to the last problem or $\frac{\pi^2}{2}$.

Alternatively, we can integrate like last probnlem:

$$\begin{aligned}
\int_0^{\pi} \pi f(x)^2 dx &= \int_0^{2\pi} \pi \sin^2(x) dx \\
&= \pi \int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos(2x) dx \\
&= \pi \left(\frac{x}{2} - \frac{1}{4}\sin(2x) \right) \Big|_0^{\pi} \\
&= \pi \left(\frac{2\pi}{2} - \frac{1}{4}\sin(2 * 2\pi) \right) - \pi \left(\frac{\pi}{2} - \frac{1}{4}\sin(2\pi) \right) \\
&= \pi \left(\frac{2\pi}{2} - \frac{1}{4}\sin(2 * 2\pi) \right) - \pi \left(\frac{\pi}{2} - \frac{1}{4}\sin(2\pi) \right) \\
&= \pi^2 - \frac{\pi^2}{2} \\
&= \frac{\pi^2}{2}
\end{aligned}$$

2.14. **Class 31:** Watch: Double Plus Ungood: https://www.youtube.com/watch?v=Esa2TYwDmwA&t=309s

Question 2.17. *Calculate, to at least 40 decimal places, $\frac{100}{9801}$. Do you notice a pattern? Do you think it will continue forever - why or why not?*

Solution:

Below is code to solve in Mathematica.

```
In[1]:= f[x_] := x / (1 - x - x^2)
In[2]:= f[1/1000]
SetAccuracy[f[1/1000], 40]

Out[3]= 1000/998999
Out[4]= 0.0010010020030050080130210340550891442334
```

Question 2.18. *Calculate, to at least 40 decimal places, $\frac{1000}{998999}$. Do you notice a pattern? Do you think it will continue forever - why or why not?*

Solution:

Below is code to solve in Mathematica.

```
In[1]:= x D[1/(1 - x), x]
Out[2]= x/(1 - x)^2

In[3]:= a[x_] := x/(1 - x)^2
a[1/100]
SetAccuracy[a[1/100], 40]

Out[4]= 100/9801
Out[5]= 0.0102030405060708091011121314151617181920
```

2.15. **Class 32:** Differential Equations and Trafalgar

Question 2.19. *Solve the difference equation* $a(n+1) = 7a(n) - 12a(n-1)$ *with initial conditions* $a(0) = 3$ *and* $a(1) = 10$.

Solution:

Below is code to solve in Mathematica.

```
In[1]:= RSolve[{a[n + 1] == 7 a[n] - 12 a[n - 1], a[0] == 3, a[1] == 10}, a[n], n]

Out[2]= {{a[n] -> 2 3^n + 4^n}}
```

Arguing theoretically, we try $a_n = r^n$, which leads to the characteristic equation

$$r^{n+1} \ = \ 7r^n - 12r^{n-1}.$$

One solution of course is $r = 0$, which is trivial; the others come from dividing by r^{n-1} which yields

$$r^2 - 7r + 12 \ = \ 0 \quad \text{or} \quad (r-3)(r-4) \ = \ 0.$$

Thus the two roots are 3 and 4, and the general solution is

$$a_n \ = \ c_1 3^n + c_2 4^n.$$

All that is left is finding c_1 and c_2. We have the initial conditions $a(0) = 3$ and $a(1) = 10$. Thus

$$c_1 + c_2 \ = \ 3 \quad \text{and} \quad c_1 3 + c_2 4 \ = \ 10.$$

The first gives us $c_2 = 3 - c_1$, substituting into the second yields

$$3c_1 + 4(3 - c_1) \ = \ 10 \quad \text{or} \quad 12 - c_1 \ = \ 10.$$

Thus $c_1 = 2$ which implies $c_2 = 1$, and the solution is

$$a_n \ = \ 2 \cdot 3^n + 4^n.$$

Question 2.20. *Consider the whale problem from class, but now assume that on every two pairs of 1 year old whales give birth to one new pair of whales, and every four pairs of 2 year old whales give birth to one new pair. Prove or disprove: eventually the whales dies out.*

Solution: The whale population dies out exponentially fast. If we start with four whale pairs, after one year they yield two new pairs, and after two years they give a third; thus four pairs give rise to three, which means the population decreases by 25%. Thus as we march forward in time, the whale population decreases by a fixed percentage and will die off....