MATH 105: PRACTICE PROBLEMS FOR CHAPTER 3: SPRING 2010

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Question 1 : Compute the partial derivatives of order 1 and order 2 for:

(1) $f(x,y,z) = e^{x+y}\cos(x)\sin(y)$.
Solution: We can proceed by brute force, but it helps to notice that our function factors into a function depending only on x and another depending only on y. Thus $f(x,y,z) = A(x)B(y)$, say. We immediately see all partials involving a z vanish. Further, it is clear that all partial derivatives will exist and be continuous (our function f is of class $\mathcal{C}^2$), and thus the order in which we differentiate does not matter. In this case,

$$f(x,y,z) \;=\; [e^x\cos(x)] \;\cdot\; [e^y\sin(y)]\,.$$

We find

$$\frac{\partial f}{\partial x} \;=\; \left(\frac{\partial}{\partial x}[e^x\cos(x)]\right)\cdot[e^y\sin(y)]\,.$$

Thus

$$\frac{\partial f}{\partial x} \;=\; [e^x\cos(x) - e^x\sin(x)]\cdot[e^y\sin(y)]$$

and

$$\frac{\partial f}{\partial y} \;=\; [e^x\cos(x)]\cdot[e^y\sin(y) + e^y\cos(y)]\,.$$

The mixed partials are equal (since f is of class $\mathcal{C}^2$), and thus

$$\frac{\partial^2 f}{\partial x\partial y} \;=\; \frac{\partial^2 f}{\partial y\partial x} \;=\; [e^x\cos(x) - e^x\sin(x)]\cdot[e^y\sin(y) + e^y\cos(y)]\,.$$

We are left with the double derivatives with respect to x and y, which are

$$\frac{\partial^2 f}{\partial x^2} \;=\; -2e^x\sin(x)\,[e^y\sin(y)]$$

and

$$\frac{\partial^2 f}{\partial y^2} \;=\; [e^x\cos(x)]\cdot 2e^y\cos(y)$$

(as terms cancel when we simplify the algebra).

(2) $f(x,y) = \sin(e^x/e^y)$.
Solution: We could proceed by brute force, but again it is best to try and simplify the algebra first. If we Thoreau it, we realize we have $f(x,y) = \sin(e^{x-y})$, and this

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removes a quotient rule (at the cost of a chain rule, but of course this is minor as we already had the chain rule). We now just take lots of derivatives:

$$\frac{\partial f}{\partial x} = e^{x-y}\cos(e^{x-y}) = -\frac{\partial f}{\partial y}.$$

In other words, x and y almost have a symmetric role in f, but there is an extra minus sign in front of y, which surfaces when we take derivatives. Thus

$$\frac{\partial f}{\partial x} = e^{x-y}\cos(e^{x-y}), \quad \frac{\partial f}{\partial y} = -e^{x-y}\cos(e^{x-y}).$$

Similarly, for the second derivatives we have

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= e^{x-y}\cos\left(e^{x-y}\right) - e^{2x-2y}\sin(e^{x-y}) \\ \frac{\partial^2 f}{\partial y^2} &= e^{x-y}\cos\left(e^{x-y}\right) - e^{2x-2y}\sin(e^{x-y}) \\ \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} &= -e^{x-y}\cos\left(e^{x-y}\right) + e^{2x-2y}\sin\left(e^{x-y}\right). \end{aligned}$$

Note that for this problem differentiation with respect to x is and y is *almost* the same, differing by a minus sign. This is why $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = -\frac{\partial^2 f}{\partial x \partial y}$.

Question 2 : Compute the second order Taylor series expansions about $(0,0)$ of

(1) $A(x,y) = (x+y-5)^3$.

Solution: The standard approach is to recall

$$(\nabla f)(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

and

$$(Hf)(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix},$$

and then use our result that the second order expansion is

$$f(0,0) + (\nabla f)(0,0)\cdot(x,y) + \frac{1}{2}(x,y)(Hf)(0,0)\begin{pmatrix} x \\ y \end{pmatrix}.$$

For our function,

$$f(0,0) = (-5)^3 = -125$$

and

$$(Df)(x,y,z) = (3(x+y-5)^2, 3(x+y-5)^2) = 3(x+y-5)^2(1,1),$$

which implies

$$(Df)(0,0) = (75,75).$$

A slightly more lengthy computation gives

$$Hf = \begin{pmatrix} 6(x+y-5) & 6(x+y-5) \\ 6(x+y-5) & 6(x+y-5) \end{pmatrix},$$

so

$$(Hf)(0,0) \;=\; \begin{pmatrix} -30 & -30 \\ -30 & -30 \end{pmatrix},$$

leading to a Taylor expansion of

$$\begin{aligned}
& -125 + (75,75)\cdot(x,y) + \frac{1}{2}(x,y)\begin{pmatrix} -30 & -30 \\ -30 & -30 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
= \;& -125 + 75x + 75y + \frac{1}{2}(x,y)\begin{pmatrix} -30x + -30y \\ -30x + -30y \end{pmatrix} \\
= \;& -125 + 75(x+y) + \frac{1}{2}(x(-30x + -30y) + y(-30x + -30y)) \\
= \;& -125 + 75(x+y) - 15(x+y)^2,
\end{aligned}$$

where we have done a little algebra simplification. There are of course other ways we could have written the answer, such as

$$-125 + 75x + 75y - 15x^2 - 30xy - 15y^2.$$

The two expressions are equivalent; it's just a question of which do you prefer. I like doing a little factoring, but that is not required.

Alternatively, we could have used our trick. Using the binomial theorem, we find

$$(u-5)^3 \;=\; u^3 + \binom{3}{1}u^2\cdot(-5) + \binom{3}{2}u\cdot(-5)^2 + (-5)^3 \;=\; -125 + 75u - 15u^2 - u^3;$$

as we are interested in just a second order Taylor series expansion, we see that taking $u = x+y$ means we may ignore the u^3 term, and the second order Taylor expansion is just

$$-125 + 75(x+y) - 15(x+y)^2,$$

which matches what we found previously.

(2) $f(x,y,z) = e^{x+y}\cos(x)\sin(y)$.
Solution: As always, we could solve this by brute force, taking all the derivatives (which is done in the first problem). Alternatively, we can think a bit about the algebra, and make our life easier. We expand out each factor and multiply together, keeping only the terms that have at most two x's and y's. We find

$$\left[1 + x + \frac{x^2}{2} + \cdots\right]\;\left[1 + y + \frac{y^2}{2} + \cdots\right]\left[1 - \frac{x^2}{2} + \cdots\right]\;[y - \cdots],$$

and if we only keep terms involving at most two x's and y's then we have

$$y + xy + y^2.$$

(3) $g(x,y) = \sin\left(e^{x-y} - 1\right)$.
Solution: Note

$$e^{x-y} - 1 \;=\; 1 + (x-y) + \frac{(x-y)^2}{2!} + \frac{(x-y)^3}{3!} + \cdots - 1,$$

or
$$x - y + \frac{(x-y)^2}{2}$$
plus higher order terms. As
$$\sin u \ = \ u - u^3/3! + \cdots ,$$
since we are just interested in the second order Taylor expansion the only term that survives is u, and
$$u \ = \ x - y + \frac{(x-y)^2}{2} + \cdots ;$$
the higher order terms (those involving $(x-y)^3$ or higher powers) do not matter; thus the only part of u that is important for us is the first two terms, the $x-y$ piece and the $(x-y)^2/2!$ piece. Thus the second order Taylor series expansion is just
$$x - y + \frac{(x-y)^2}{2!}.$$
This can also be found by brute force differentiation and then invoking our standard expansion.

Question 3 : Find the critical points of

(1) $A(x,y) = (x+y-5)^3$.
Solution: We have
$$(DA)(x,y) \ = \ (3(x+y-5)^2, 3(x+y-5)^2) \ = \ 3(x+y-5)^2(1,1).$$
Thus for $(DA)(x,y) = (0,0)$ we need $x+y-5=0$, or $y = -x+5$. Thus the critical points are a line with slope -1 and y-intercept 5.

(2) $B(x,y,z) = xy^2 + x^2y$.
Solution: We have
$$(DB)(x,y) \ = \ (y^2 + 2xy, 2xy + x^2).$$
To have $(DB)(x,y) = (0,0)$, we need $y^2 + 2xy = 0$ and $2xy + x^2 = 0$. This implies $x^2 = y^2$ as both equal $-2xy$.

If $x = y$ the only solution is $(0,0)$, as we must also satisfy $y^2 + 2xy = 0$; since $x = y$ this equation becomes $x^2 + 2x^3 = 0$ or $3x^2 = 0$, and clearly the only solution is $x = 0$.

If, however, it is $x = -y$ that holds instead of $x = y$, then the first equation $y^2 + 2xy = 0$ becomes $y^2 - 2y^2 = 0$, or $-y^2 - 0$. If $-y^2 - 0$ then we must have $y = 0$, and again the only solution is $(0,0)$.

Question 4 : Find the candidates for the extrema of the functions below, subject to the given constraints.

(1) $f(x,y,z) = xy^2z^3$ subject to $g(x,y,z) = x^2 + y^2 + z^2 = 1$.
Solution: We have
$$Df \ = \ \nabla f \ = \ (y^2z^3, 2xyz^3, 3xy^2z^2)$$

and

$$Dg \;=\; \nabla g \;=\; (2x, 2y, 2z) \;=\; 2(x, y, z).$$

We thus need $(\nabla f)(x, y, z) = \lambda(\nabla g)(x, y, z)$ for some λ and $g(x, y, z) = 1$. We have

$$y^2z^3 \;=\; 2\lambda x, \quad 2xyz^3 \;=\; 2\lambda y, \quad 3xy^2z^2 \;=\; 2\lambda z, \qquad x^2 + y^2 + z^2 \;=\; 1.$$

There are two cases: $\lambda = 0$ or $\lambda \neq 0$.

If $\lambda = 0$ then at least one of x, y and z must vanish; equivalently, $\lambda = 0$ means $xyz = 0$. If $xyz = 0$ then of course $f(x, y, z) = xy^2z^3 = 0$. Clearly this cannot be a maximum or minimum for the function; if x, y, z are all positive that will give a larger value, while if x, y are positive and z is negative that will give a smaller value. Thus the maximum / minimum cannot be when $\lambda = 0$.

We can find the solution by taking ratios of equations. From our analysis above, we may assume $\lambda xyz \neq 0$. The first equation divided by the second equation yields

$$\frac{y^2z^3}{2xyz^3} \;=\; \frac{2\lambda x}{2\lambda y},$$

or

$$\frac{y}{2x} \;=\; \frac{x}{y},$$

which gives $y^2 = 2x^2$. If we look at the first equation divided by the second equation we get $\frac{y^2z^3}{3xy^2z^2} = \frac{x}{z}$, or $\frac{z}{3x} = \frac{x}{z}$ which implies $z^2 = 3x^2$. We substitute into the constraint $g(x, y, z) = 1$ and find

$$x^2 + y^2 + z^2 \;=\; x^2 + 2x^2 + 3x^2 \;=\; 1;$$

thus $6x^2 = 1$ or $x = \pm\sqrt{6}/6$. As $y^2 = 2x^2$ we have $y = \pm\sqrt{2}x$, so $y = \pm\sqrt{3}/3$; a similar analysis gives $z = \pm\sqrt{2}/2$. There are thus *eight* candidates for the maxima and minima:

$$\left(\pm\frac{\sqrt{6}}{6}, \pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{2}}{2}\right).$$

We need to evaluate f at these eight points. Note that as $f(x, y, z) = xy^2z^3$ it doesn't matter if we take $y = \sqrt{3}/3$ or $y = -\sqrt{3}/3$; both lead to the same value. If $x = z$ then we get a maximum, while if $x = -z$ we have a minimum.

We now present another method to solve the equations. We have

$$Df \;=\; \nabla f \;=\; (y^2z^3, 2xyz^3, 3xy^2z^2)$$

and

$$Dg \;=\; \nabla g \;=\; (2x, 2y, 2z) \;=\; 2(x, y, z).$$

We thus need $(\nabla f)(x, y, z) = \lambda(\nabla g)(x, y, z)$ for some λ. We have

$$y^2z^3 \;=\; 2\lambda x, \quad 2xyz^3 \;=\; 2\lambda y, \quad 3xy^2z^2 \;=\; 2\lambda z.$$

Let's multiply the first equation by $6x$, the second by $3y$ and the third by $2z$; the point of this is that after such multiplication, the left hand side of the three equations will all be the same, namely $6xy^2z^3$. This means the three right hand sides equal this and each other, or

$$6xy^2z^3 \;=\; 12\lambda x^2 \;=\; 6\lambda y^2 \;=\; 4\lambda z^2.$$

Arguing as before, we see that we don't have a maximum or minimum when $\lambda = 0$. Assume now $\lambda \neq 0$. Then $6x^2 = 3y^2 = 2z^2$ and thus we can express z^2 and y^2 in terms of x^2 (explicitly, $y^2 = 2x^2$ and $z^2 = 3x^2$). As $x^2 + y^2 + z^2 = 1$, at the extremum we have $x^2 + 2x^2 + 3x^2 = 1$, or $6x^2 = 1$ which implies $x = \pm\sqrt{6}/6$. We can now find the values of y and z.

(2) $f(x, y) = (x - 4)^2 + y^2$ subject to $g(x, y) = (x/5)^2 + (y/3)^2 = 1$.
Solution: We have

$$\nabla f \ = \ (2x - 8, 2y), \quad \nabla g \ = \ (2x/25, 2y/9).$$

Setting $\nabla f = \lambda \nabla g$ yields

$$(2x - 8, 2y) \ = \ \lambda(2x/25, 2y/9).$$

Divide both sides by 2 and multiply by 225 (you don't need to do this; this just clears the denominators and makes all the numbers in site integers), yielding

$$(225x - 900, 225y) \ = \ \lambda(9x, 25y).$$

The first equation is $225x - 900 = 9\lambda x$, while the second equation is $225y = 25\lambda y$.

As $225y = 25\lambda y$, there are two possibilities: either $y = 0$ or if $y \neq 0$ then $\lambda = \frac{225}{25} = 9$. Let's consider first the case $y \neq 0$, so $\lambda = 9$. Returning to $225x - 900 = 9\lambda x$, we see $225x - 900 = 81x$, or $144x - 900 = 0$ so $x = 900/144$. Recall, however, that x and y must also satisfy our constraint $g(x, y) = 1$, and the point $x = 900/144$ and $y = 0$ does not. Thus, when $y \neq 0$ there are no candidates.

We are thus left with analyzing the case $y = 0$. We must satisfy $225x - 900 = 9\lambda x$. This is one equation with two unknowns, which is not comforting. Fortunately we have the other equation, namely the constraint $(x/5)^2 + (y/3)^2 = 1$. As $y = 0$ the solutions are $x = \pm 5$, and then for each choice of x we can (if we so desire) find the corresponding value of λ. We see that in the case when $y = 0$ that the candidates are $x = \pm 5$. Substituting into our function we see the maximum is when $x = -5$ and the minimum is when $x = 5$.

There is a geometrical interpretation to this problem. The function f is the distance from the point (x, y) on an ellipse to one of the foci, namely the one at $(4, 0)$. Not surprisingly, the distance is smallest at $(5, 0)$ and furthest at $(-5, 0)$.

Question 5 : Use the fact that the derivative of a sum is the sum of the derivatives to prove that the derivative of a sum of three terms is the sum of the three derivatives.
Solution: The idea to solve this problem is quite useful in mathematics. We know that for any two functions $f(x)$ and $g(x)$ that $\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{df}{dx} + \frac{dg}{dx}$. We now use this result to

show a similar claim holds for the sum of three functions. We have

$$\begin{aligned}
A(x) &= f(x)+g(x)+h(x) \\
\frac{dA}{dx} &= \frac{d}{dx}\left(f(x)+g(x)+h(x)\right) \\
&= \frac{d}{dx}\left(B(x)+h(x)\right), \quad B(x) \ = \ f(x)+g(x) \\
&= \frac{dB}{dx}+\frac{dh}{dx} \\
&= \left(\frac{df}{dx}+\frac{dg}{dx}\right)+\frac{dh}{dx} \\
&= \frac{df}{dx}+\frac{dg}{dx}+\frac{dh}{dx},
\end{aligned}$$

where we constantly used the fact that the derivative of a sum of two functions is the sum of the two derivatives.

A similar argument (formally, using induction) shows the derivative of any finite sum is the finite sum of the derivatives). It is not clear that this result holds for derivatives of infinite sums (it does in some cases, while it fails in others).