Math 140: Calculus II: Spring '22 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/

Lecture 5: 2-14-22:
https://youtu.be/osKvsXoCuIM

Linear polynomials: $x+1$, $x+2$, $3x - 7$, …

Quadratic: $x^2 - 4$, $x^2 - 4x + 8$, …

Cubic: $x^3 + 11x - 8$

Quartic: $a x^4 + b x^3 + c x^2 + d x + e$ or $a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$

Factor as $a(x - r_1) (x - r_2) (x - r_3) (x - r_4)$

If want all the roots to be the same: take $r_1 = r_2 = r_3 = r_4$, get $a (x - r)^4$.

Simplest example is x^4.

Two roots the same, other two different from each other and the two that are the same:
Try $r_1 = r_2 = 0$, $r_3 = 1$, $r_4 = -1$: Get $a (x) (x) (x - 1) (x + 1) = a x^2 (x^2 - 1) = a x^4 - a x^2$.

Fundamental Theorem of Algebra: A polynomial of degree n with complex coefficients has n complex roots (and this includes multiplicities).

Plan for the day: Lecture 5: February 14, 2022:

Intermediate Value Theorem

Mean Value Theorem

Fundamental Theorem of Calculus

Simpson's Rule

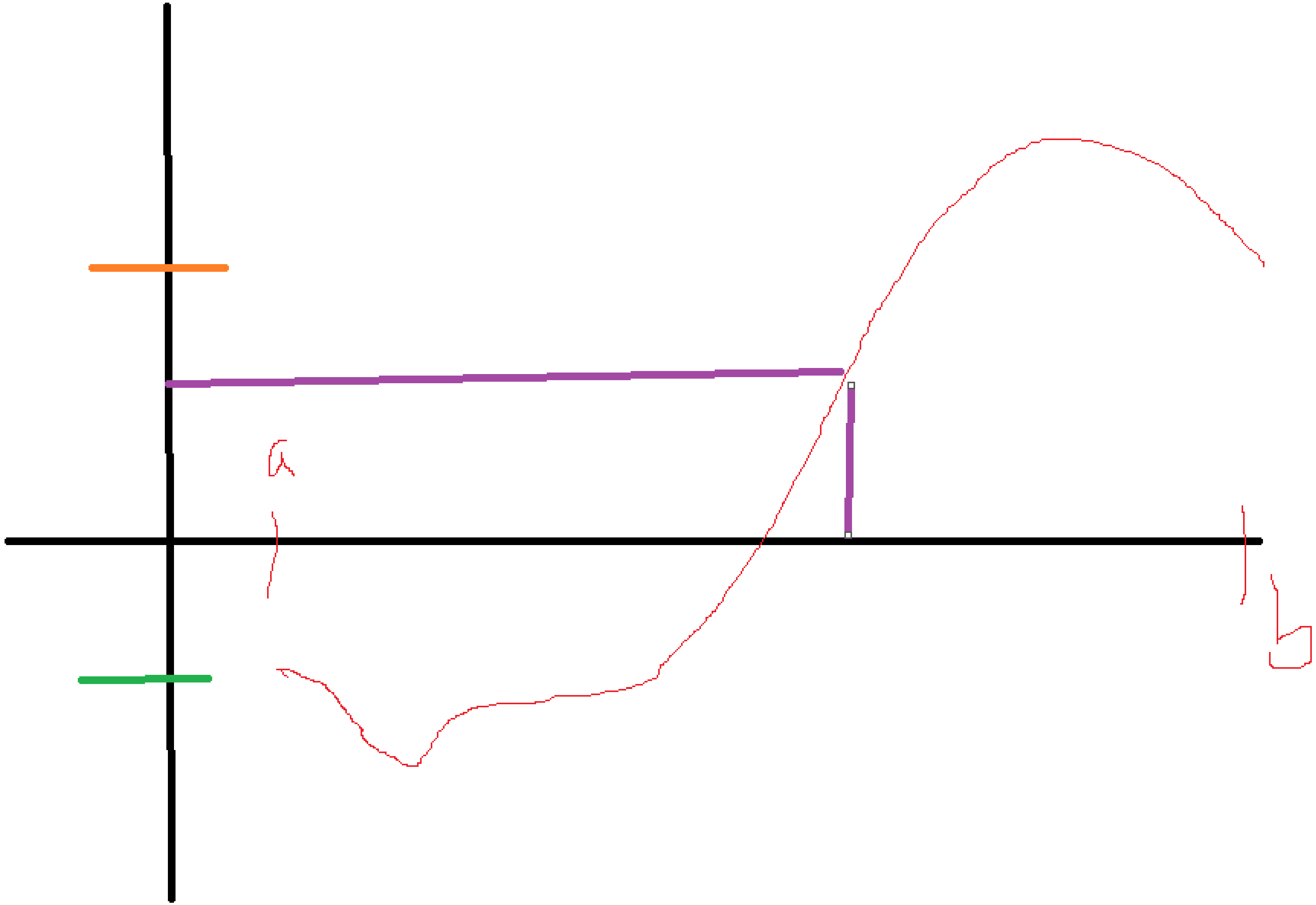
Theorem A.2.1 (Intermediate Value Theorem (IVT)). *Let f be a continuous function on $[a, b]$. For all C between $f(a)$ and $f(b)$ there exists a $c \in [a, b]$ such that $f(c) = C$. In other words, all intermediate values of a continuous function are obtained.*

Sketch of the proof. We proceed by **Divide and Conquer**. Without loss of generality, assume $f(a) < C < f(b)$. Let x_1 be the midpoint of $[a, b]$. If $f(x_1) = C$ we are done. If $f(x_1) < C$, we look at the interval $[x_1, b]$. If $f(x_1) > C$ we look at the interval $[a, x_1]$.

In either case, we have a new interval, call it $[a_1, b_1]$, such that $f(a_1) < C < f(b_1)$ and the interval has half the size of $[a, b]$. We continue in this manner, repeatedly taking the midpoint and looking at the appropriate half-interval.

If any of the midpoints satisfy $f(x_n) = C$, we are done. If no midpoint works, we divide infinitely often and obtain a sequence of points x_n in intervals $[a_n, b_n]$. This is where rigorous mathematical analysis is required (see §A.3 for a brief review, and [Rud] for complete details) to show x_n converges to an $x \in (a, b)$.

For each n we have $f(a_n) < C < f(b_n)$, and $\lim_{n\to\infty} |b_n - a_n| = 0$. As f is continuous, this implies $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(x) = C$. □


a
b

Theorem A.2.2 (Mean Value Theorem (MVT)). *Let $f(x)$ be differentiable on $[a,b]$. Then there exists a $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c) \cdot (b - a).$$

We give an interpretation of the Mean Value Theorem. Let $f(x)$ represent the distance from the starting point at time x. The average speed from a to b is the distance traveled, $f(b) - f(a)$, divided by the elapsed time, $b - a$. As $f'(x)$ represents the speed at time x, the Mean Value Theorem says that there is some intermediate time at which we are traveling at the average speed.

To prove the Mean Value Theorem, it suffices to consider the special case when $f(a) = f(b) = 0$; this case is known as Rolle's Theorem:

Theorem A.2.3 (Rolle's Theorem). *Let f be differentiable on $[a,b]$, and assume $f(a) = f(b) = 0$. Then there exists a $c \in (a,b)$ such that $f'(c) = 0$.*

Theorem A.2.3 (Rolle's Theorem). *Let f be differentiable on $[a, b]$, and assume $f(a) = f(b) = 0$. Then there exists a $c \in (a, b)$ such that $f'(c) = 0$.*

Exercise A.2.4. *Show the Mean Value Theorem follows from Rolle's Theorem.* Hint: *Consider*

$$h(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a).$$

Note $h(a) = f(a) - f(a) = 0$ and $h(b) = f(b) - (f(b) - f(a)) - f(a) = 0$. The conditions of Rolle's Theorem are satisfied for $h(x)$, and

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Proof of Rolle's Theorem. Without loss of generality, assume $f'(a)$ and $f'(b)$ are non-zero. If either were zero we would be done. Multiplying $f(x)$ by -1 if needed, we may assume $f'(a) > 0$. *For convenience, we assume $f'(x)$ is continuous.* This assumption simplifies the proof, but is not necessary. In all applications in this book this assumption will be met.

Case 1: $f'(b) < 0$**:** As $f'(a) > 0$ and $f'(b) < 0$, the Intermediate Value Theorem applied to $f'(x)$ asserts that all intermediate values are attained. As $f'(b) < 0 < f'(a)$, this implies the existence of a $c \in (a, b)$ such that $f'(c) = 0$.

Case 2: $f'(b) > 0$**:** $f(a) = f(b) = 0$, and the function f is increasing at a and b. If x is real close to a then $f(x) > 0$ if $x > a$. This follows from the fact that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{A.17}$$

As $f'(a) > 0$, the limit is positive. As the denominator is positive for $x > a$, the numerator must be positive. Thus $f(x)$ must be greater than $f(a)$ for such x. Similarly $f'(b) > 0$ implies $f(x) < f(b) = 0$ for x slightly less than b.

Therefore the function $f(x)$ is positive for x slightly greater than a and negative for x slightly less than b. If the first derivative were always positive then $f(x)$ could never be negative as it starts at 0 at a. This can be seen by again using the limit definition of the first derivative to show that if $f'(x) > 0$ then the function is increasing near x. Thus the first derivative cannot always be positive. Either there must be some point $y \in (a, b)$ such that $f'(y) = 0$ (and we are then done) or $f'(y) < 0$. By the Intermediate Value Theorem, as 0 is between $f'(a)$ (which is positive) and $f'(y)$ (which is negative), there is some $c \in (a, y) \subset [a, b]$ such that $f'(c) = 0$. □

Assume average speed is 50mph

If start or end at 50mph, easy to find a time when going the average speed!

What if speed is always greater than 50mph? Impossible!

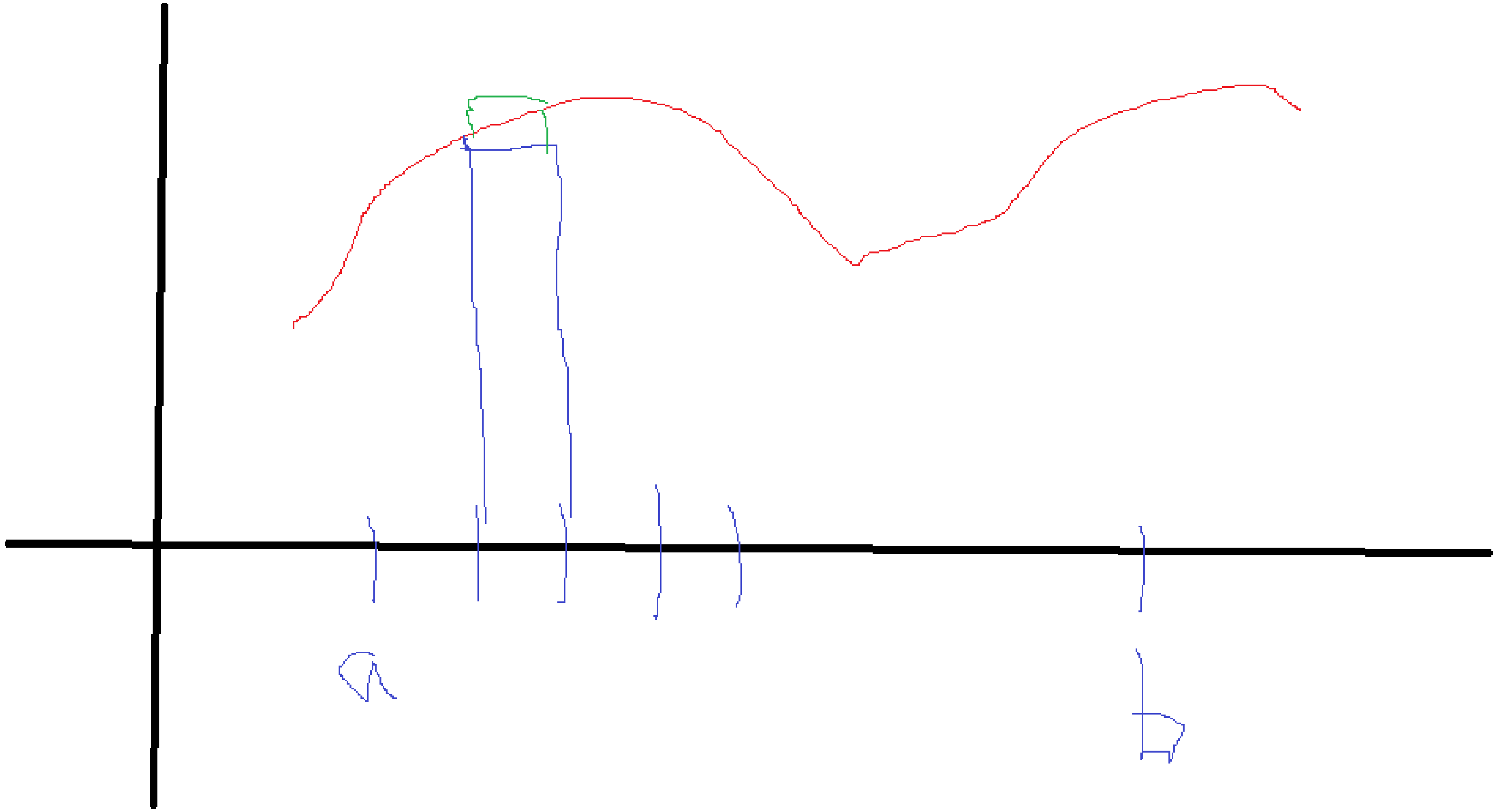
What if speed is always less than 50mph? Impossible!

Case 1: At some point below 50mph and at another point above 50mph.

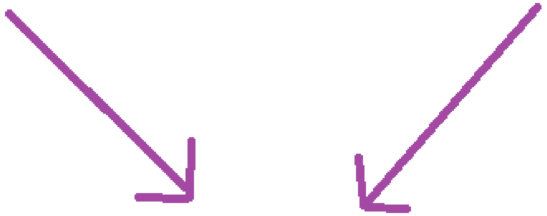
By the IVT, at some point must travel at exactly 50mph.

Case 2: Driving at a constant speed of 50mph.

We leave this to the reader....



L(n) <= True area <= U(n)



If the upper and lower sums converge to a common value as n goes to infinity, that must be the true area.

Fundamental Theorem of Calculus: If f is a nice function and F' = f, then the area under y = f(x) from x=a to x=b is equal to F(b) - F(a). Note: If f is continuous and differentiable and has bounded derivative, f is nice.

Simpson's Rule

https://i.ytimg.com/vi/7EqRRuh-5Lk/maxresdefault.jpg

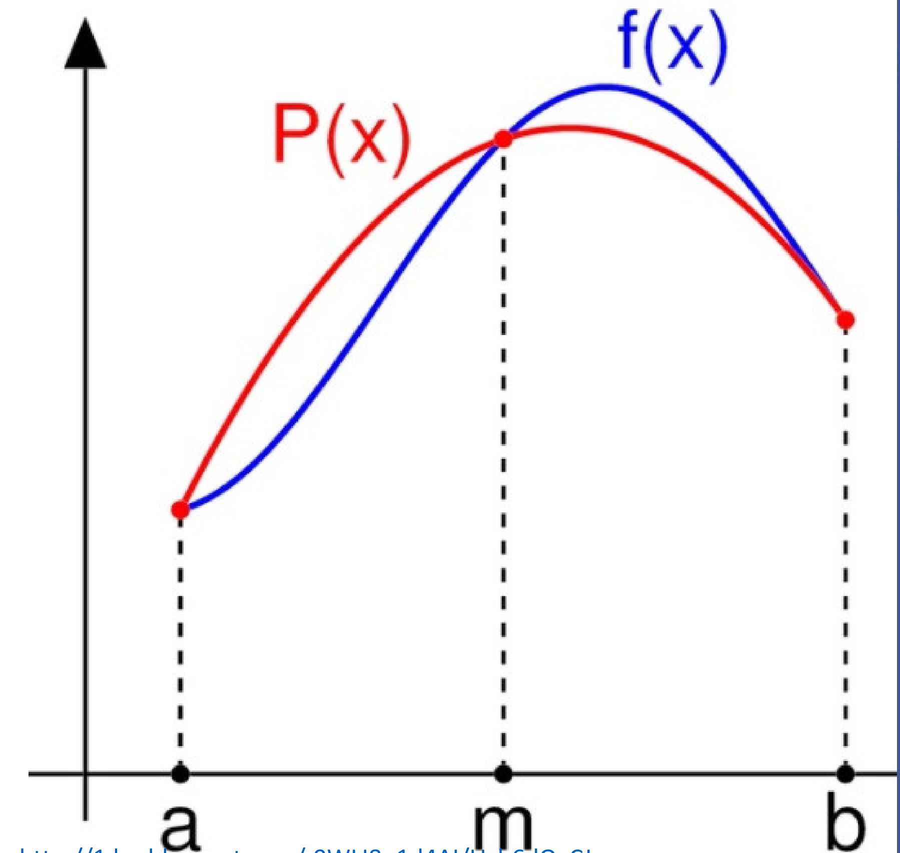
$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots f(x_n)\right]$$

$$\int_a^b f(x)\,dx \approx S_n$$

$$\Delta x = \frac{b-a}{n}$$

Steve Miller's

Math Riddles



http://1.bp.blogspot.com/-9WLI8o1d4AI/Uch6dQxCJ-I/AAAAAAAABHo/Jhp3ATK9chU/s1600/Simpson's+Rule+Illustration.jpg

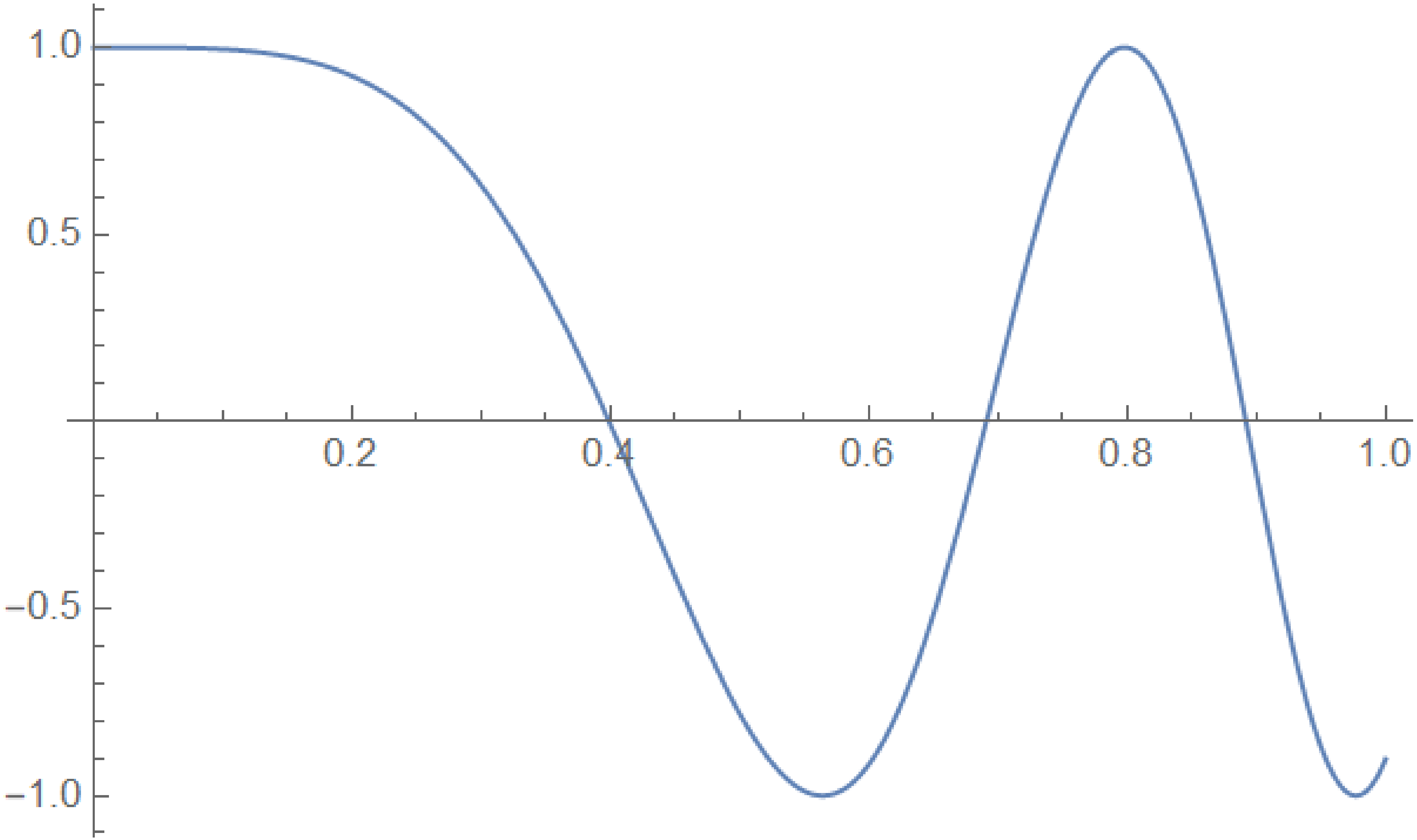
General Code

Consider an army with 10 generals. One wants a security system such that any three of them can determine the code to launch nuclear missiles, but no two of them can. It is possible to devise such a system by using a quadratic polynomial, such as a x^2 + bx + c; to launch the missiles, one must input (a,b,c). One cannot just tell each general one of a, b, or c (as then it is possible that some subset of three generals won't know a, b and c); however, if a general knows two of (a,b,c), then a set of two generals can launch the missiles! What information should be given to the generals so that any three can find (a,b,c) but no two can? What about the general situation with N generals and any M can launch (but no set of M-1) can?

Free textbooks available here: https://aimath.org/textbooks/approved-textbooks/

```
integratecompare[a_, b_, num_, ff_] := Module[{},
  (* need num even for simpson's rule *)
  f[x_] := ff[x];
  currf = f[a];
  left = 0;
  average = 0;
  right = 0;
  simpson = 0;
  deltax = (b - a) / num;
  simpson = simpson + currf * deltax * 1 / 3;
  (* use current and new to minimize computations *)
  (* don't compute same value for each method, do once and add! *)
  For[n = 1, n ≤ num, n++,
   {
    newf = f[a + n deltax];
    left = left + currf * deltax;
    average = average + deltax * (currf + newf) / 2;
    right = right + newf * deltax;
    simpson = simpson + If[Mod[n, 2] == 0, 2, 4] * newf * deltax / 3;
    currf = newf;
   }]; (* end of n loop *);
  simpson = simpson - currf * deltax / 3; (* need to fix last term *)
  answer = SetAccuracy[Integrate[f[x], {x, a, b}], 15];
  Print["Function is ", ff[x]];
  Print["Going from ", a, " to ", b, " with ", num, " divisions."];
  Print["Left      = ", SetAccuracy[left, 15]];
  Print["Average   = ", SetAccuracy[average, 15]];
  Print["Right     = ", SetAccuracy[right, 15]];
  Print["Simpson   = ", SetAccuracy[simpson, 15]];
  Print["Answer is  ", SetAccuracy[answer, 15]];
  Print[" "];
  Print["Below are the errors."];
  Print["Left:      = ", SetAccuracy[left - answer, 15]];
  Print["Average:   = ", SetAccuracy[average - answer, 15]];
  Print["Right:     = ", SetAccuracy[right - answer, 15]];
  Print["Simpson:   = ", SetAccuracy[simpson - answer, 15]];
  Print["(Delta x)^1: ", SetAccuracy[deltax, 15]];
  Print["(Delta x)^2: ", SetAccuracy[deltax^2, 15]];
  Print["(Delta x)^3: ", SetAccuracy[deltax^3, 15]];
  Print["(Delta x)^4: ", SetAccuracy[deltax^4, 15]];
  Print["(Delta x)^5: ", SetAccuracy[deltax^5, 15]];
 ]
```

```
Plot[Cos[(Pi x)^2], {x, 0, 1}]
```



Function is $\mathrm{Cos}[\pi^2 x^2]$

Going from 0 to 1 with 10 divisions.

Left = 0.2825793013048

Average = 0.1874450332081

Right = 0.0923107651115

Simpson = 0.1801730919687

Answer is 0.180065837931052

Below are the errors.

Left: = 0.10251346337373

Average: = 0.00737919527708

Right: = −0.08775507281957

Simpson: = 0.00010725403762

(Delta x)^1: 0.10000000000000

(Delta x)^2: 0.01000000000000

(Delta x)^3: 0.00100000000000

(Delta x)^4: 0.00010000000000

(Delta x)^5: 0.00001000000000

Function is $\mathrm{Cos}[\pi^2 x^2]$

Going from 0 to 1 with 100 divisions.

Left = 0.1896500777597

Average = 0.180136650950

Right = 0.1706232241403

Simpson = 0.1800657127816

Answer is 0.180065837931052

Below are the errors.

Left: = 0.00958423982863

Average: = 0.00007081301896

Right: = −0.00944261379070

Simpson: = $-1.2514948 \times 10^{-7}$

(Delta x)^1: 0.01000000000000

(Delta x)^2: 0.00010000000000

(Delta x)^3: $1.00000000 \times 10^{-6}$

(Delta x)^4: 1.000000×10^{-8}

(Delta x)^5: 1.0000×10^{-10}

Function is $\mathrm{Cos}[\pi^2 x^2]$

Going from 0 to 1 with 1000 divisions.

Left = 0.181017888432

Average = 0.18006654575

Right = 0.179115203070

Simpson = 0.180065837919

Answer is 0.180065837931052

Below are the errors.

Left: = 0.00095205050123

Average: = 7.0782026×10^{-7}

Right: = −0.00095063486070

Simpson: = -1.252×10^{-11}

(Delta x)^1: 0.00100000000000

(Delta x)^2: $1.00000000 \times 10^{-6}$

(Delta x)^3: 1.00000×10^{-9}

(Delta x)^4: 1.00×10^{-12}

(Delta x)^5: $0. \times 10^{-15}$