Math 140: Calculus II: Spring '22 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/

Lecture 6: 2-16-22:

https://youtu.be/HOvHoX8UtLw (slides here)

Logarithms: watch https://www.youtube.com/watch?v=-SsbkPaB6j8 (22 minutes)

Plan for the day: Lecture 4: February 11, 2022:

Review Exponential Function and Logarithm

Applications

Compound interest definition, series definition

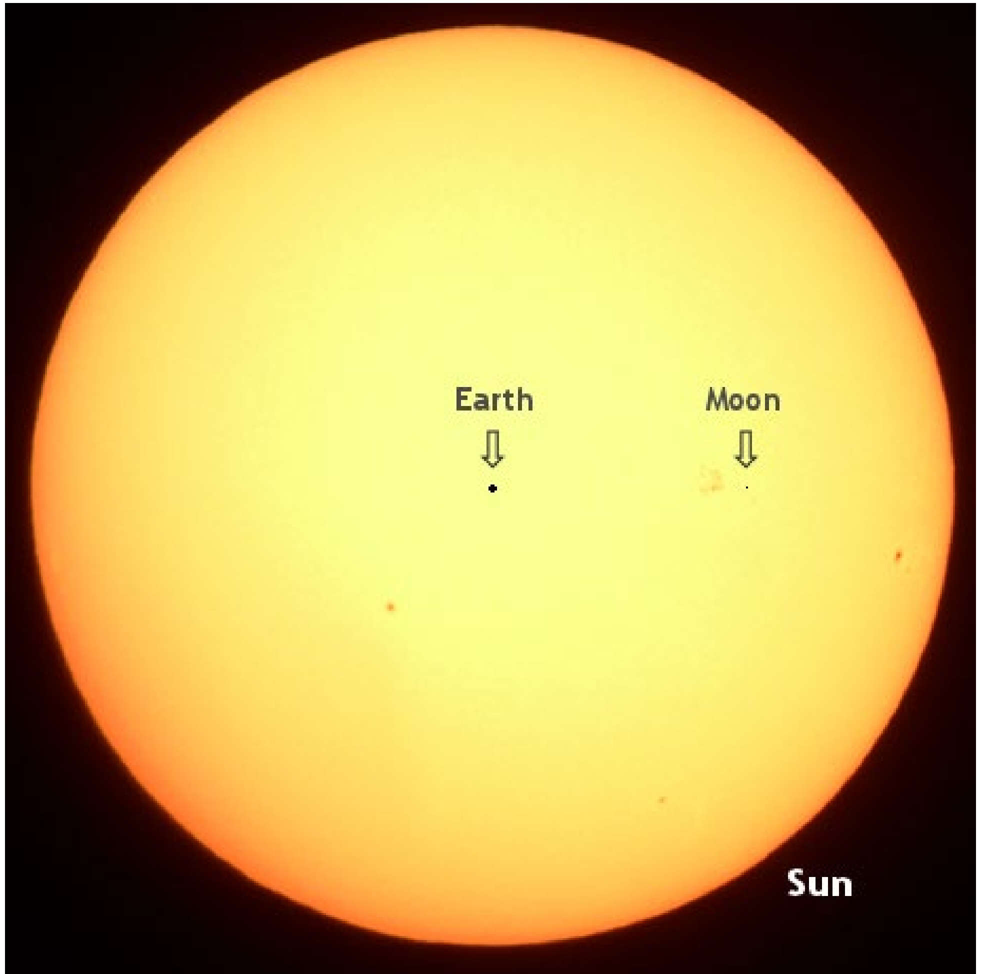
Log Laws

Introduction to Logarithms

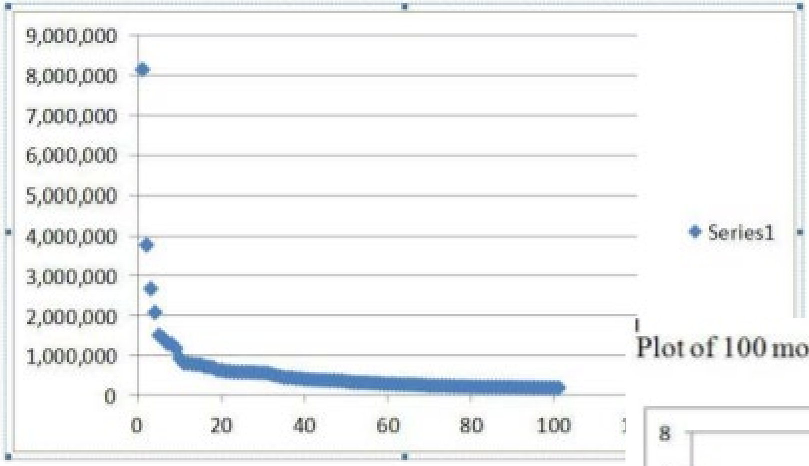
Steven Miller, Williams College
sjm1@Williams.edu

Why do we care about Logarithms

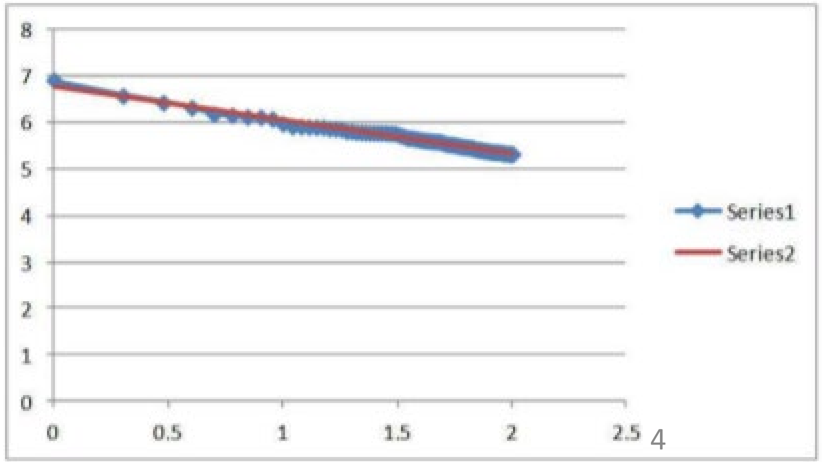
- Discuss objects across many orders of magnitude.
- Linearize many non-linear functions (calculus becomes available).



Plot of 100 most populous cities



Plot of 100 most populous cities: log-log plot



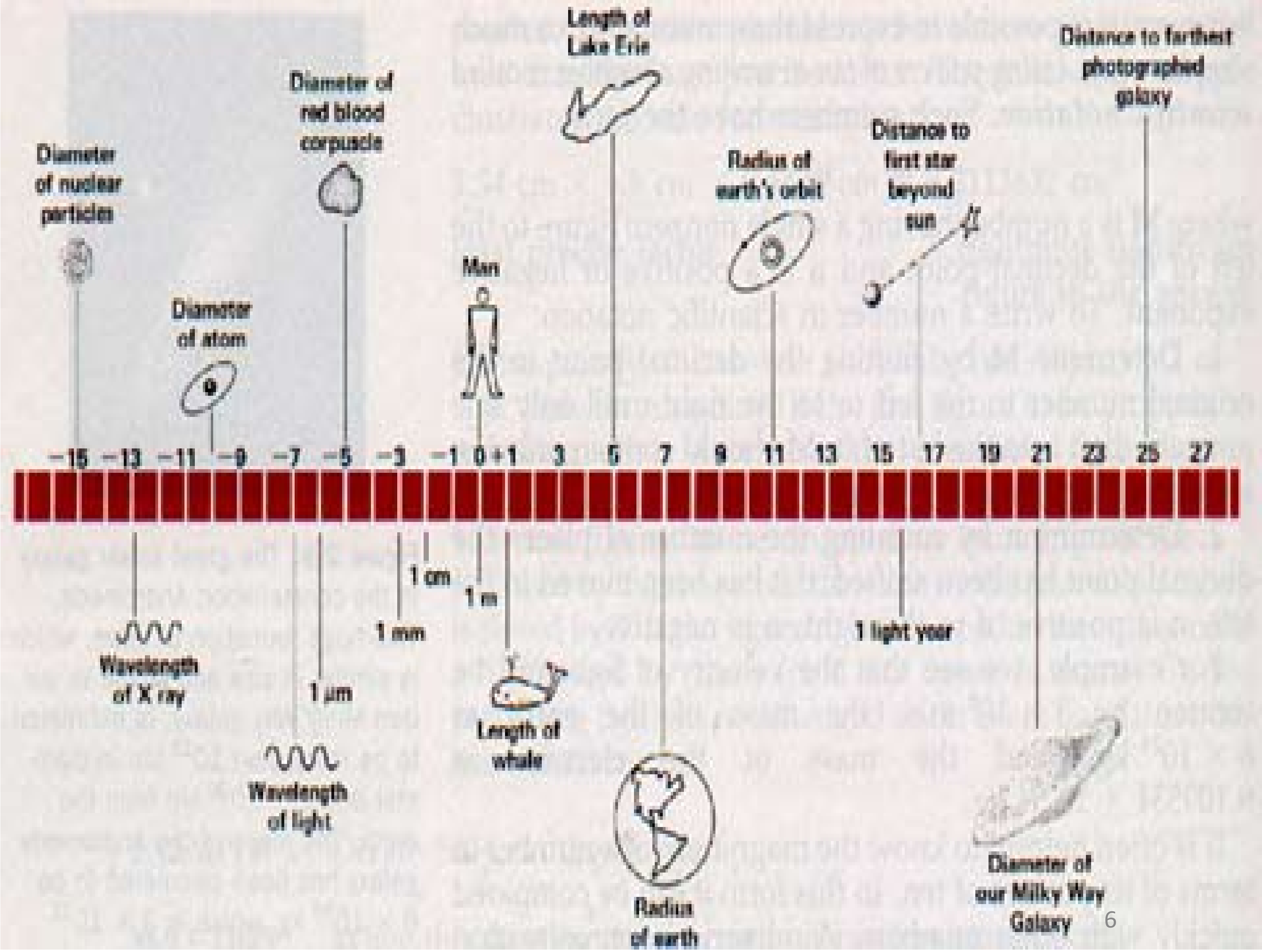
Definition of Logarithms

- **If $x = b^y$ then $\log_b x = y$.**
- Read as the logarithm of x base b is y.
- Often use base 10, and some authors suppress the subscript 10.
- Other popular bases are 2 for computers, and e for calculus; many sources write $\ln x$ for the natural logarithm of x, which is its logarithm base e (e is approximately 2.71828).

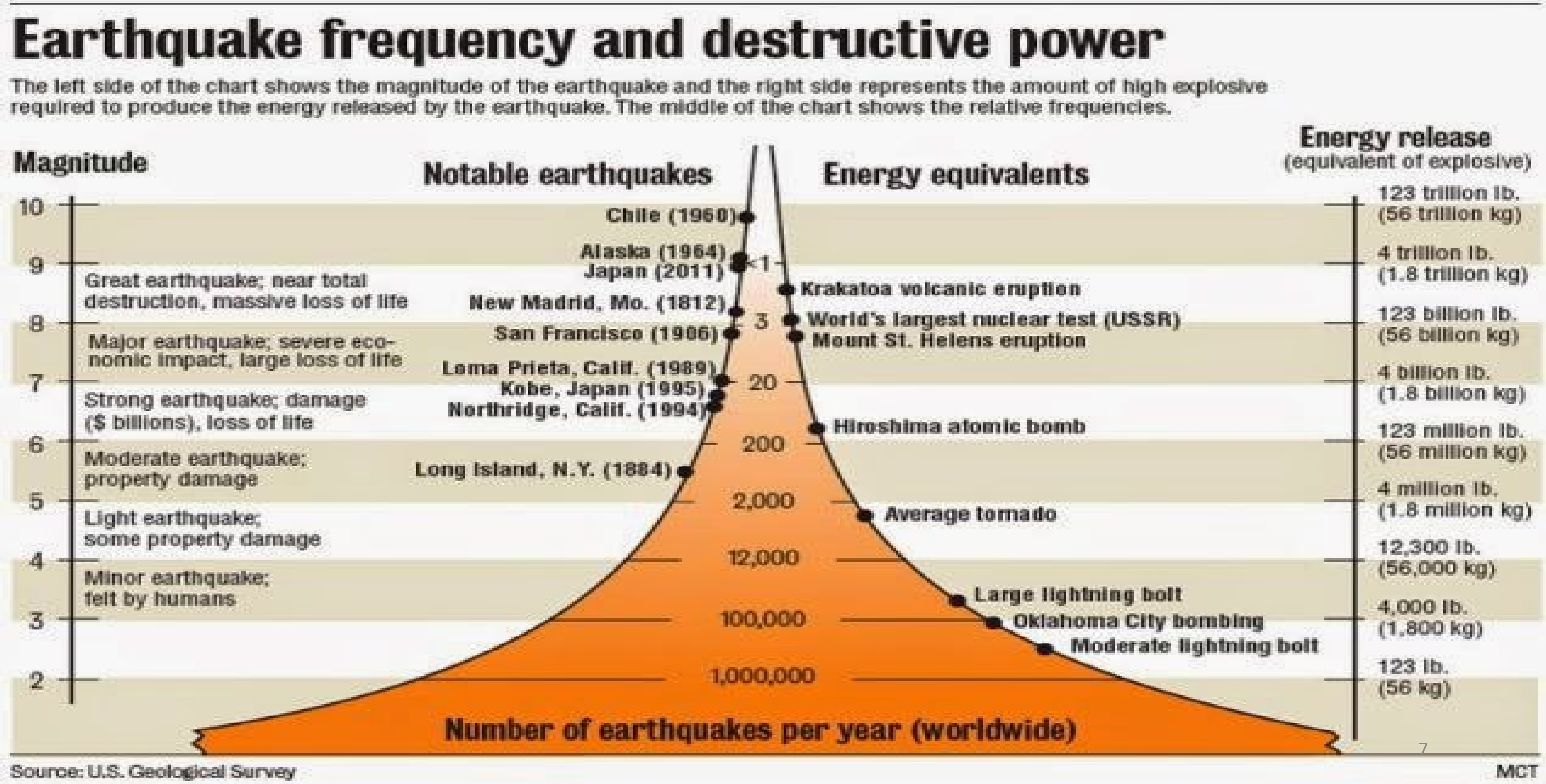
- **Examples: $\log_b x = y$ means we need y powers of b to get x.**
  - $100 = 10^2$ becomes $\log_{10} 100 = 2$. In base e it is about 4.6.
  - $1 = 10^0$ becomes $\log_{10} 1 = 0$. In base e it is still 0.
  - $.001 = 10^{-3}$ becomes $\log_{10} .001 = -3$. In base e it is about -6.9.

Examples of Logarithms

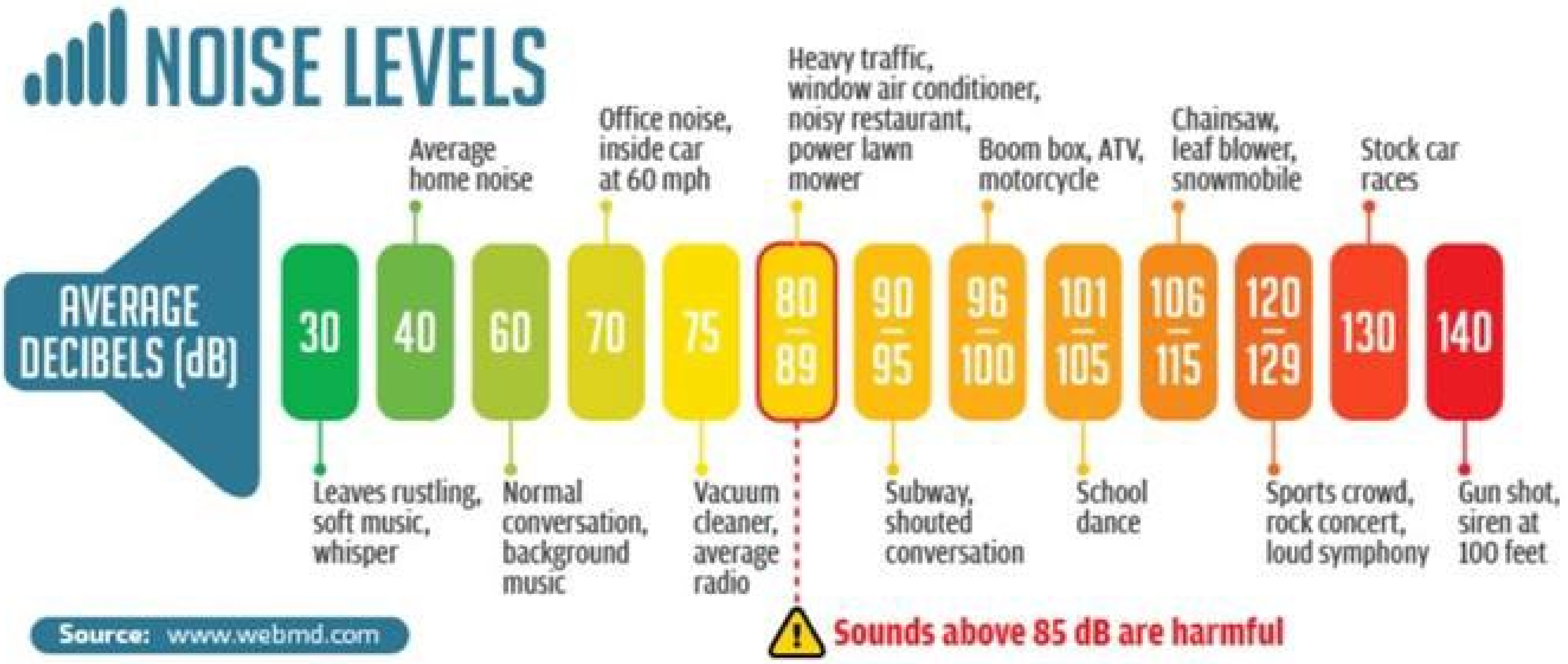
Order of Magnitude of some Lengths	
LENGTH	**meters**
radius of proton	10^{-15}
radius of atom	10^{-10}
radius of virus	10^{-7}
radius of amoeba	10^{-4}
height of human being	10^{0}
radius of earth	10^{7}
radius of sun	10^{9}
earth-sun distance	10^{11}
radius of solar system	10^{13}
distance of sun to nearest star	10^{16}
radius of milky way galaxy	10^{21}
radius of visible Universe	10^{26}



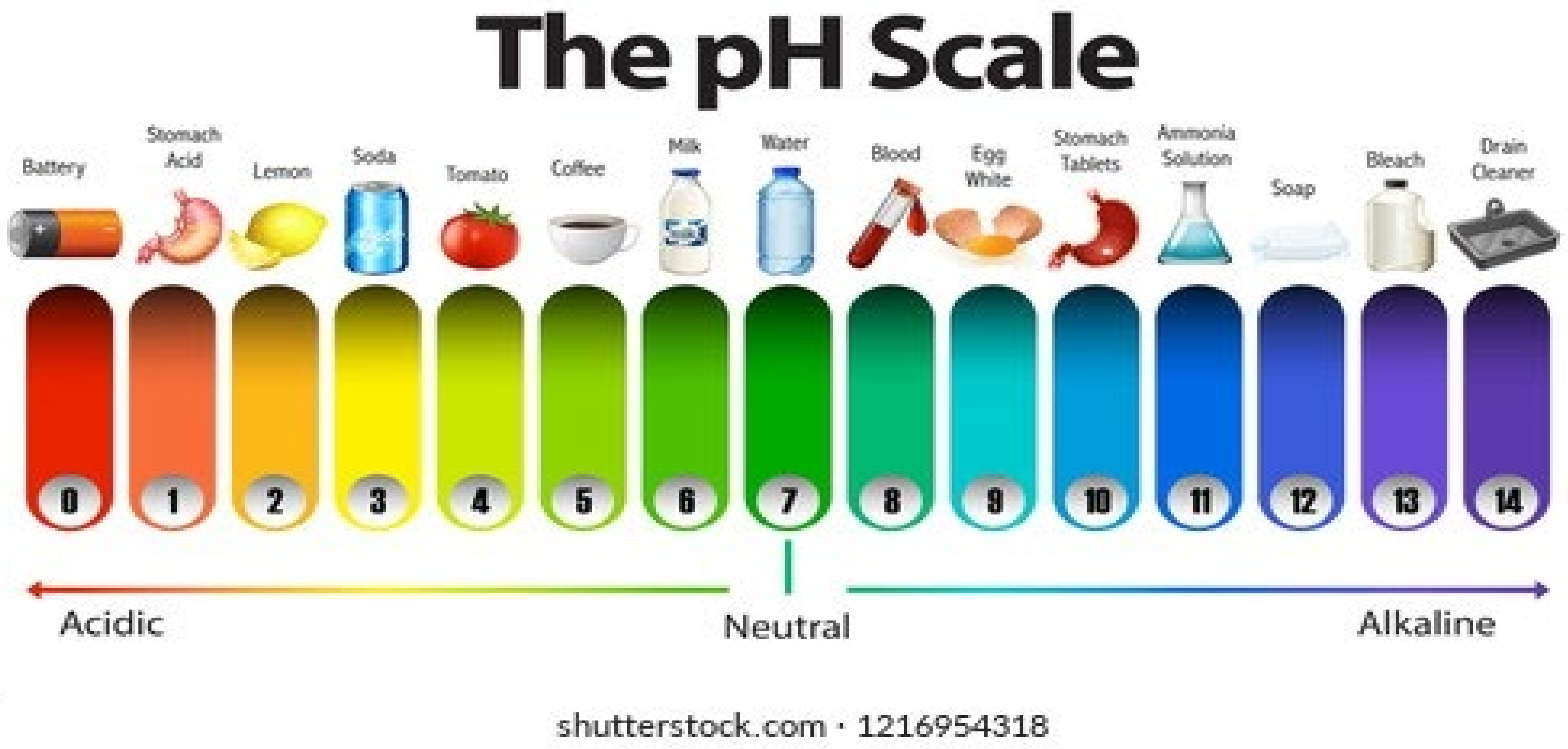
Examples of Logarithms



Examples of Logarithms



Examples of Logarithms



Recall: Definition of Logarithms

- **If $x = b^y$ then $\log_b x = y$.**
- Read as the logarithm of x base b is y.
- Often use base 10, and some authors suppress the subscript 10.
- Other popular bases are 2 for computers, and e for calculus; many sources write $\ln x$ for the natural logarithm of x, which is its logarithm base e (e is approximately 2.71828).

- **Examples: $\log_b x = y$ means we need y powers of b to get x.**
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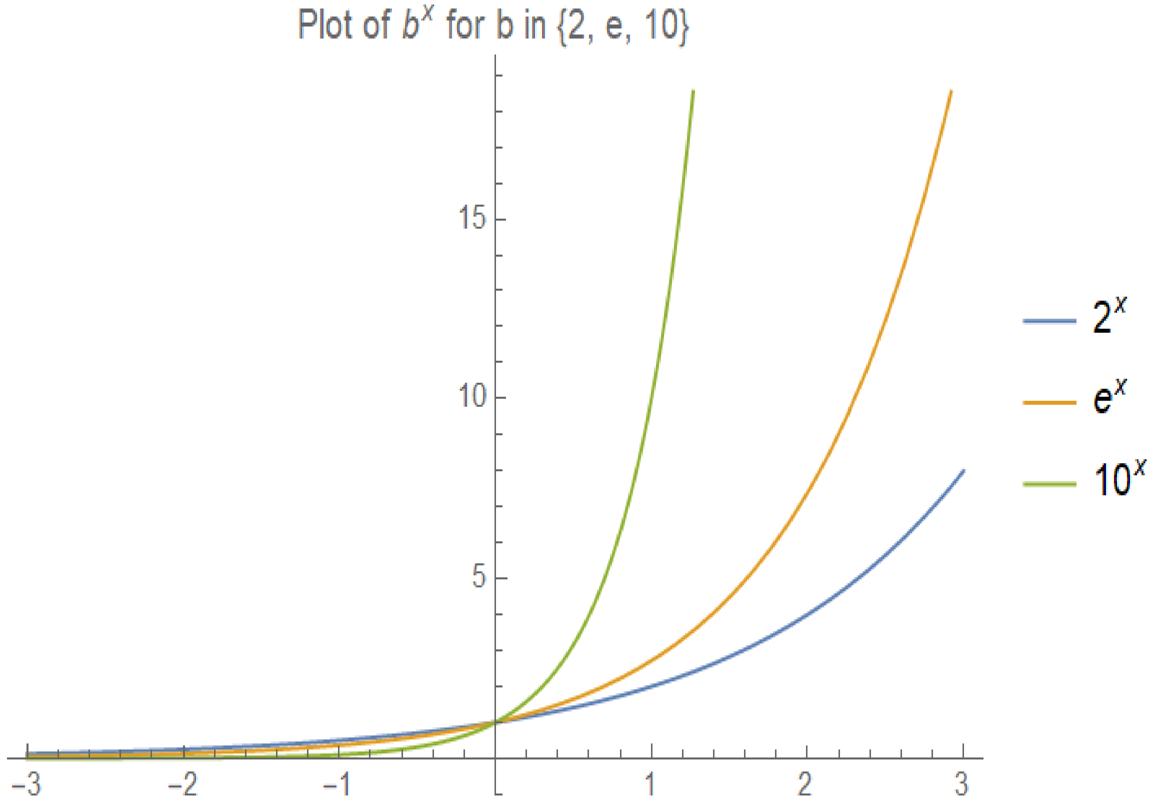
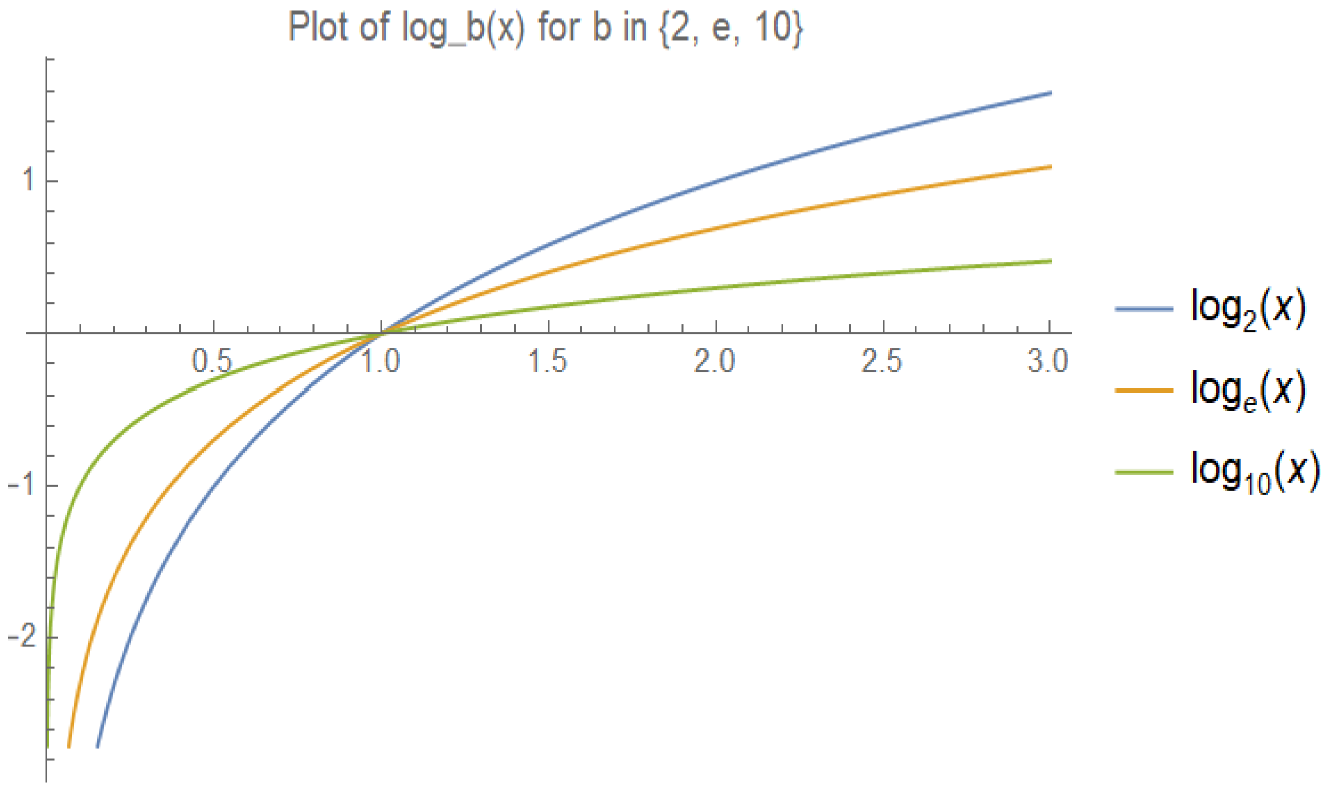
Plots of Exponentiation and Logarithms

- If $x = b^y$ then $\log_b x = y$.
- Read as the logarithm of x base b is y.

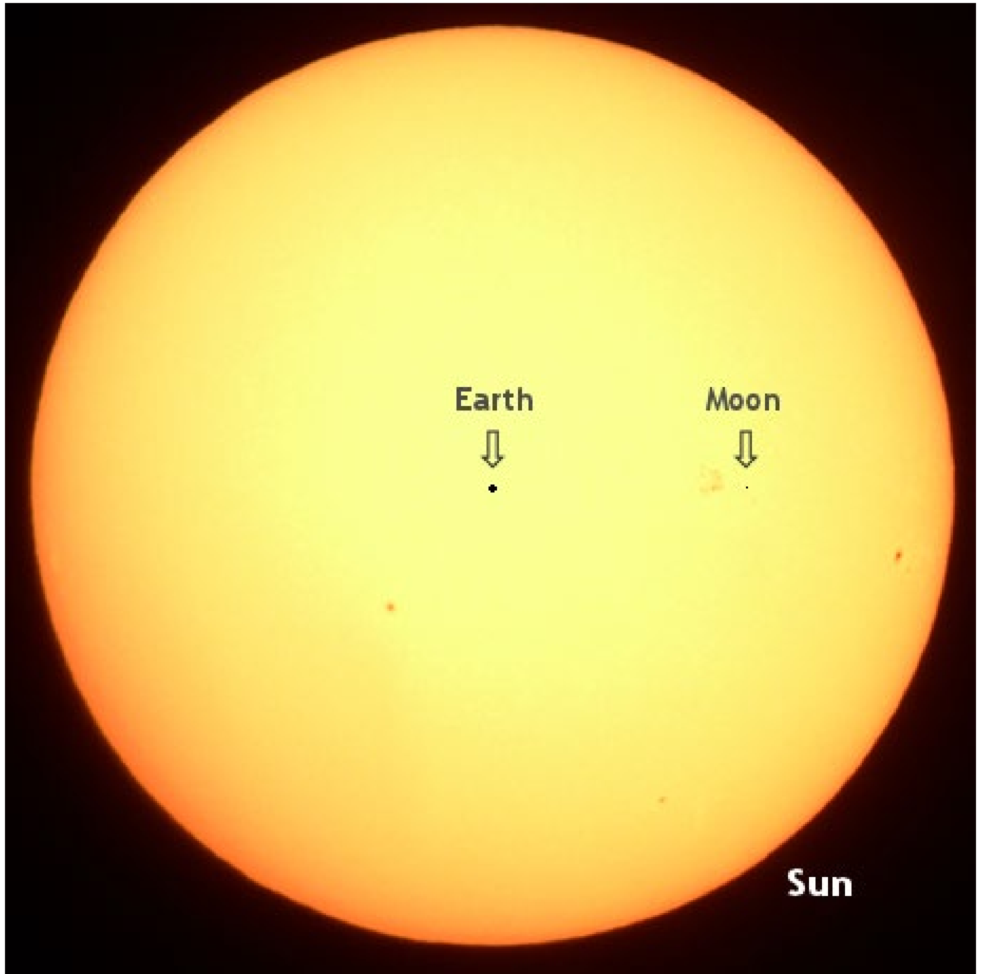




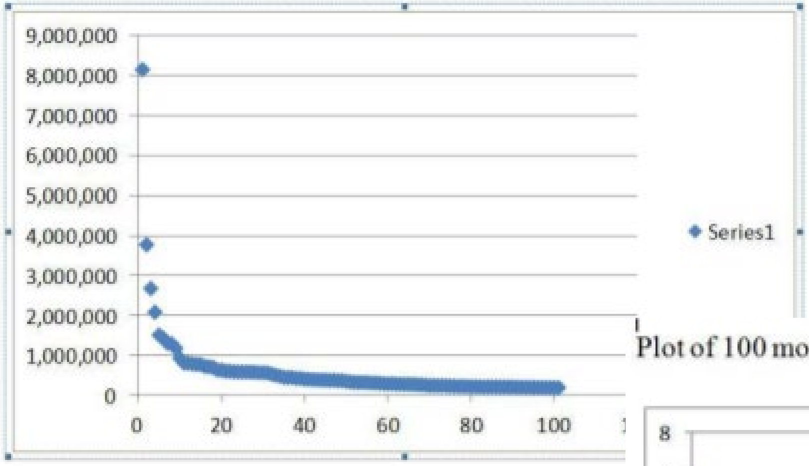
$$10^n \leq x \leq 10^{n+1} \quad \text{implies} \quad n \leq \log_{10} x \leq n + 1.$$

Why do we care about Logarithms

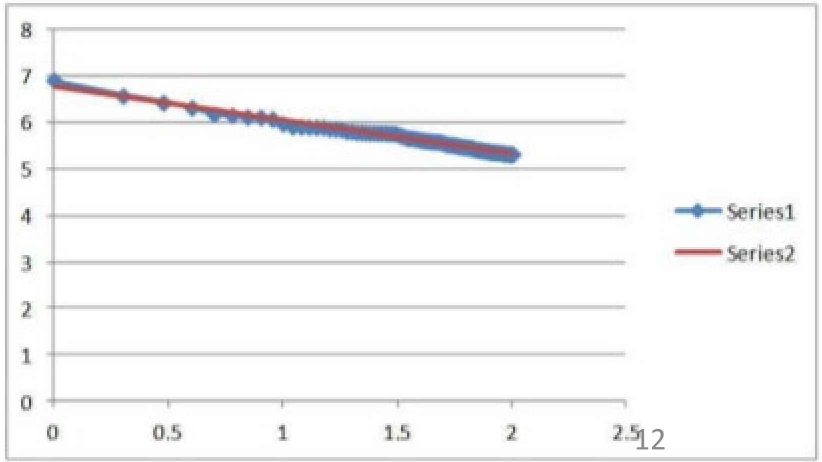
- Discuss objects across many orders of magnitude.
- Linearize many non-linear functions (calculus becomes available).



Plot of 100 most populous cities

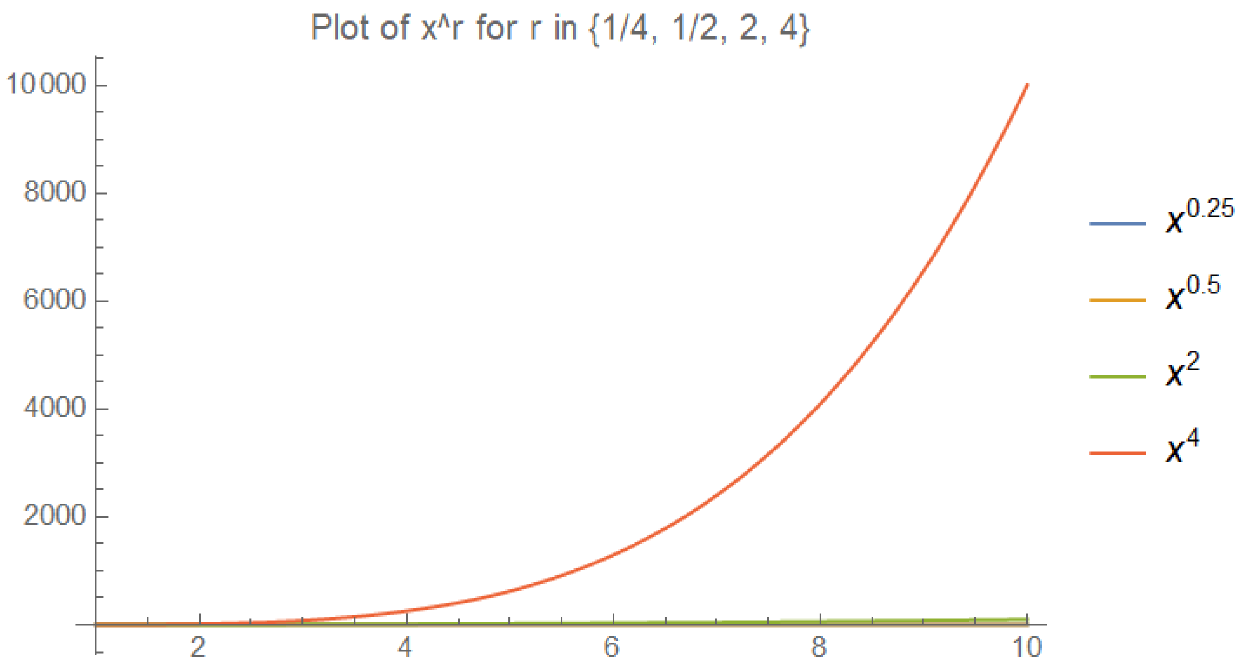


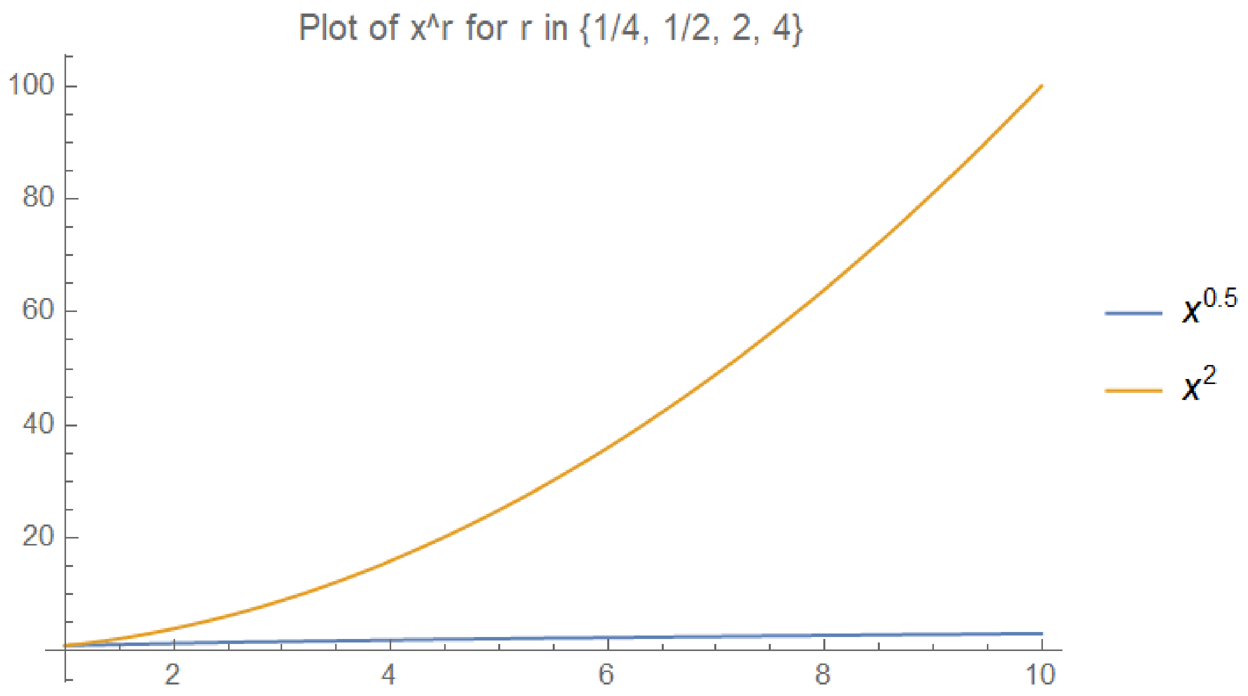
Plot of 100 most populous cities: log-log plot



Why do we care about Logarithms

- Linearize many non-linear functions (calculus becomes available).

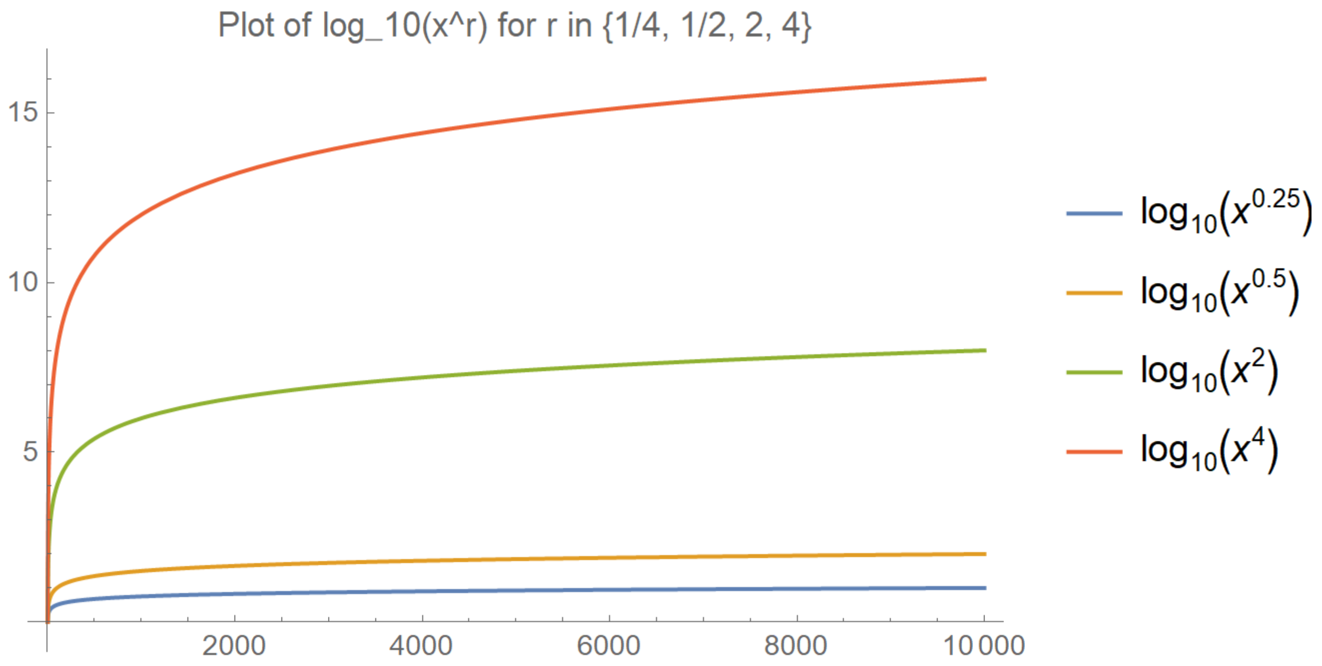


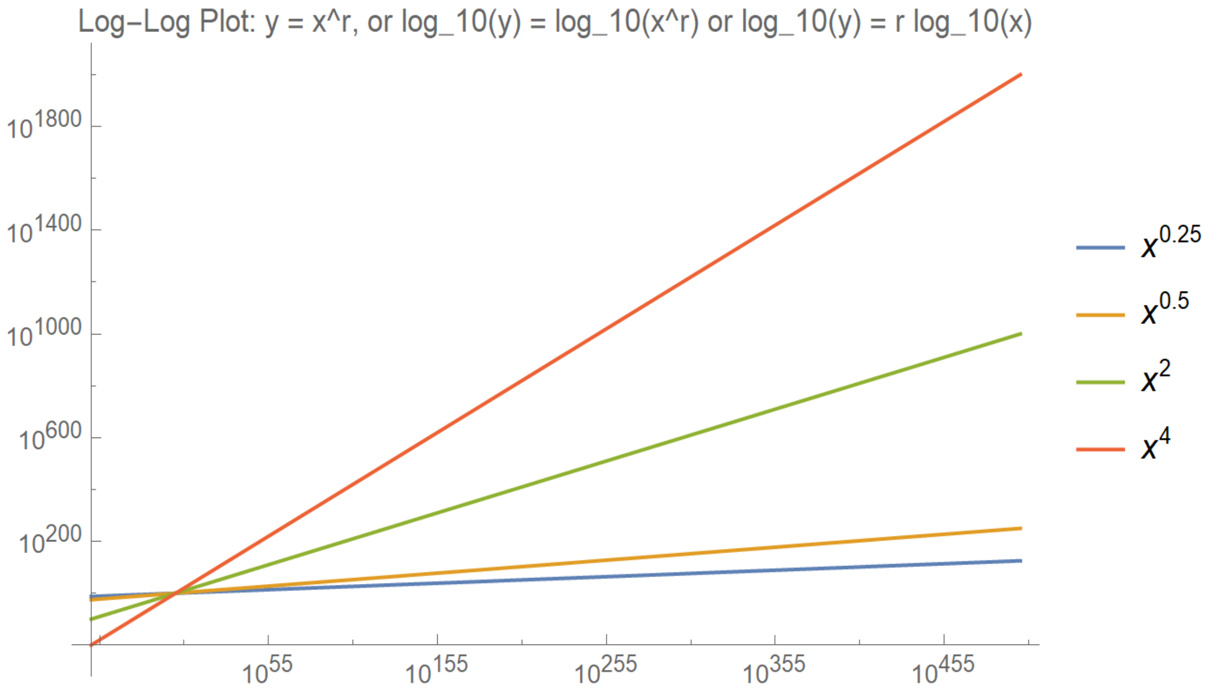


Notice that even on a small range, from 1 to 10, the polynomial of highest degree drowns out the others and can barely see.

Why do we care about Logarithms

- Linearize many non-linear functions (calculus becomes available).





Left: Semi-log plot: $y = \log x^r$. Right: log-log plot: $\log y = \log x^r$.
Note that we can now see the four functions on one plot, and the log-log plot now has linear relations.

Review: Exponent Laws

Laws

- $b^m \, b^n = b^{m+n}$
- $b^m \,/\, b^n = b^{m-n}$
- $(b^m)^n = b^{mn}$

$b^2 b^3 = (b \cdot b) \cdot (b \cdot b \cdot b) = b^5$

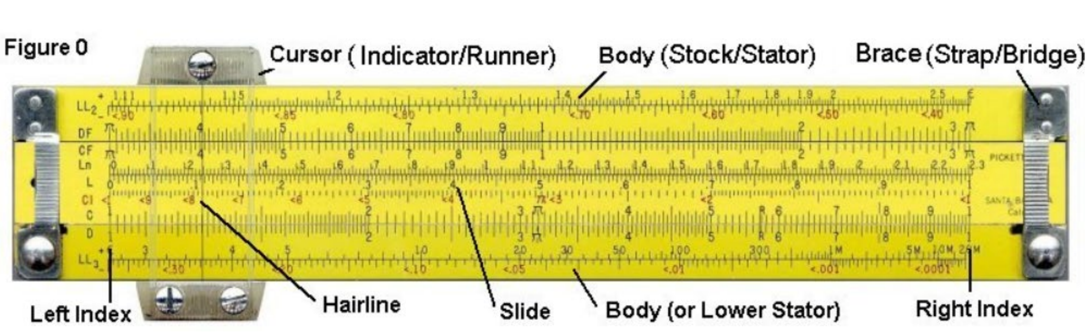
$b^3 / b^2 = \frac{b \cdot b \cdot b}{b \cdot b} = b$

$(b^2)^3 = (b \cdot b) \cdot (b \cdot b) \cdot (b \cdot b) = b^6$

Examples

- $10^3 10^2 = (10 * 10 * 10) * (10 * 10) = 10^5$
- $10^3/10^2 = (10 * 10 * 10)/(10 * 10) = 10^1$
- $(10^3)^2 = 10^3 * 10^3 = (10 * 10 * 10) * (10 * 10 * 10) = 10^6$

Logarithm Laws



Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

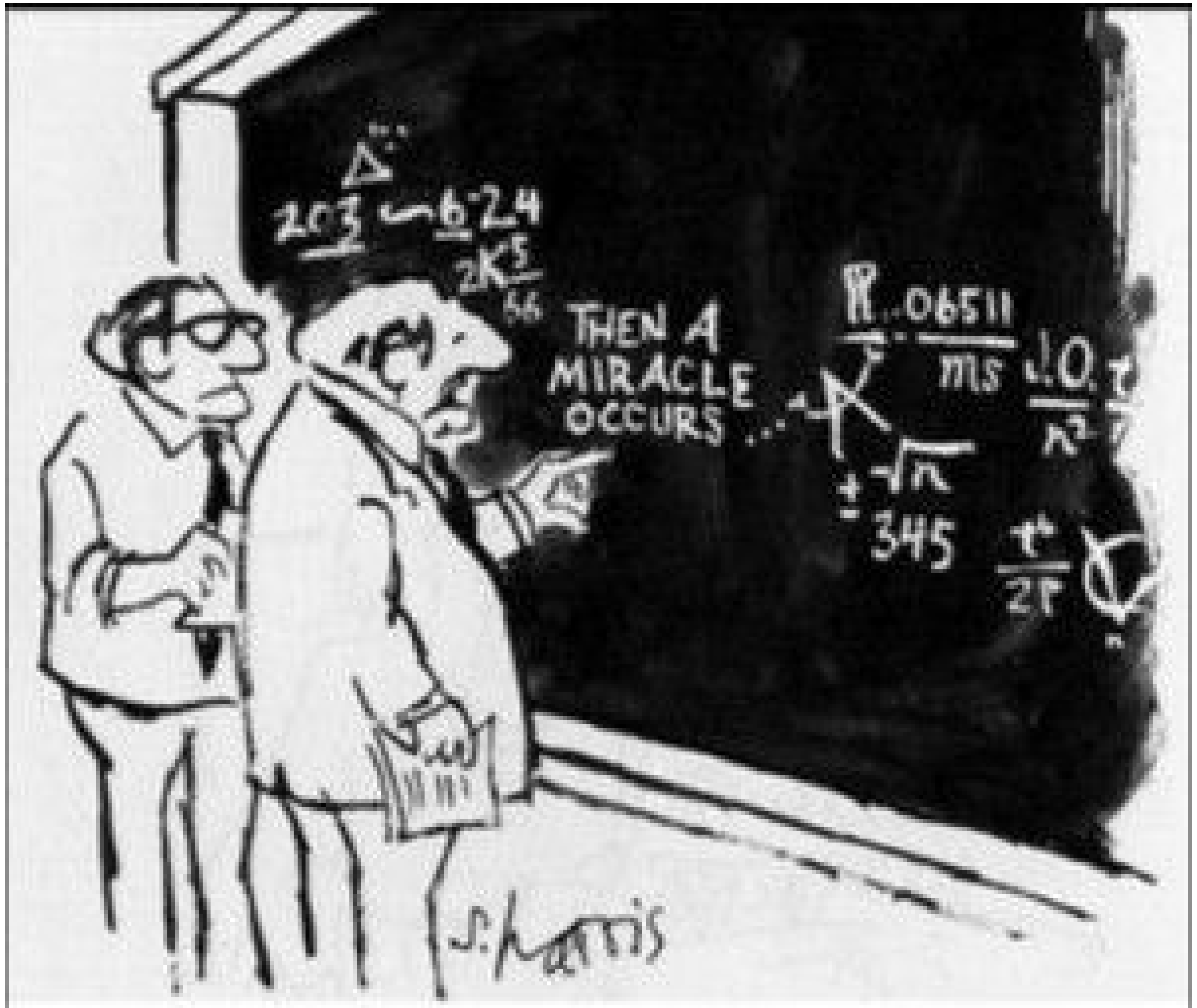
These allow us to simplify computations with logarithms.

THEOREM

$\log_b(x^{-1}) = -\log_b(x)$

- $\log_b(x^n) = n \log_b x$. Log of a power is that power times the log.
- $\log_b(x_1 \, x_2) = \log_b(x_1) + \log_b(x_2)$. Log of a product is the sum of the logs.
- $\log_b(x_1 / x_2) = \log_b(x_1) - \log_b(x_2)$. Log of a quotient is the difference of the logs.
- $\log_b x = \log_c x / \log_c b$. If know logs in one base, know in all.

OPTIONAL – PROOFS OF THE LOG LAWS



"I think you should be more explicit here in step two."

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

- $\boldsymbol{\log_b(x^n) = n \log_b x}$. **Log of a power is that power times the log.**

Proof:

- $\log_b x = y$ means $x = b^y$.
- Thus $x^n = (b^y)^n = b^{ny}$.
- Taking logarithms: $\log_b(x^n) = ny = n \log_b x$. ∎

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

- $\mathbf{\log_b(x_1\ x_2) = \log_b(x_1) + \log_b(x_2)}$. **Log of a product is the sum of the logs.**

Proof:

- As $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$, we have $x_1 = b^{y_1}$ and $x_2 = b^{y_2}$.
- Thus $x_1\ x_2 = b^{y_1} b^{y_2} = b^{y_1 + y_2}$.
- Therefore $\log_b(x_1\ x_2) = y_1 + y_2 = \log_b x_1 + \log_b x_2$. ■

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_c x = u$ (so $x = c^u$) and $\log_c b = v$ (so $b = c^v$).

- $\boldsymbol{\log_b x = \log_c x / \log_c b}$. **Know logs in one base, know in all.**

Proof:

- As $\log_b x = y$ have $x = b^y$. Similarly $x = c^u$ and $b = c^v$.
- Thus $x = b^y = (c^v)^y = cv^y$.
- As also have $x = c^u$ we have $u = vy$ or $y = u/v$.
- Substituting gives $\log_b x = \log_c x / \log_c b$. ■

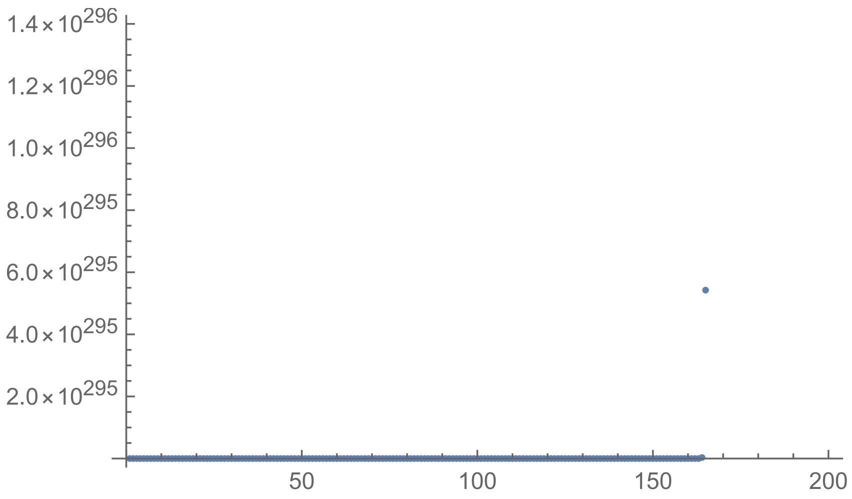
Example: Factorial Function:
Number ways to order *n objects when order matters:*

$$n! = n * (n-1) * \bullet\bullet\bullet * 3 * 2 * 1.$$

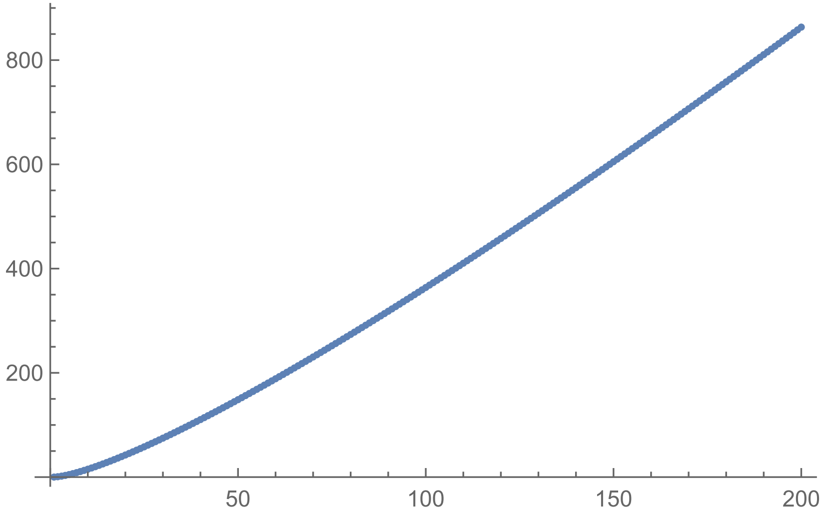
```
list = {}; semiloglist = {}; logloglist = {};
For[n = 1, n <= 200, n++,
 {
  list = AppendTo[list, {n, n!}];
  semiloglist = AppendTo[semiloglist, {n, Log[n!]}];
  logloglist = AppendTo[logloglist, {Log[n], Log[n!]}];
  }];
Print[ListPlot[list]]; Print[ListPlot[semiloglist]]; Print[ListPlot[logloglist]];
```

Example: Factorial Function:
Number ways to order n *objects when order matters:*

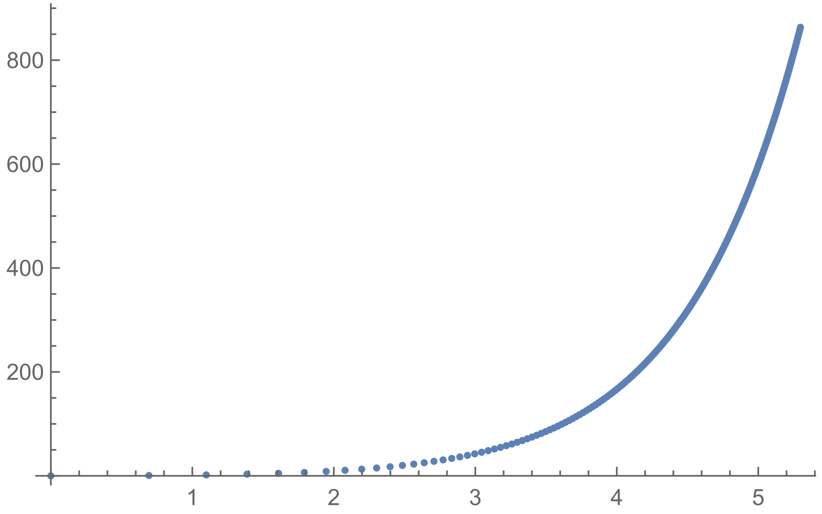
$$n! = n * (n-1) * \bullet\bullet\bullet * 3 * 2 * 1.$$



Normal Plot



Semi-log Plot



Log-Log Plot

For large n, have $n! \approx n^n e^{-n} \sqrt{2\pi n}$, so $\log n \approx \frac{n}{e} \log n$ (plus a much smaller term).

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$n! = n(n-1)\cdots 3\cdot 2\cdot 1$$

with $0! = 1$

aside: $\binom{n}{k} =$ # ways to choose k people from n when order does not matter $\frac{n!}{k!(n-k)!}$

$$\binom{5}{2} = \frac{5!}{2!\,3!} = \frac{5\cdot 4\cdot 3!}{2!\,3!} = \frac{5\cdot 4}{2\cdot 1} = 10$$

$$\binom{2}{5} = 0 \stackrel{?}{=} \frac{2!}{5!(2-5)!} = \frac{2!}{5!(-3)!}$$

so $(-3)!$ is ∞

$$\left(-\tfrac{1}{2}\right)! = \sqrt{\pi}$$

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$ Compound interest

Start with P_0 money, rate is $r\%$ per year

(so 100% per year is $r=1$), compound n times a year.

$$P_0 \xrightarrow[\text{1 period}]{} P_0\left(1 + \frac{r}{n}\right) \xrightarrow[\text{2 period}]{} P_0\left(1+\frac{r}{n}\right)^2 \longrightarrow \cdots \longrightarrow P_0\left(1+\frac{r}{n}\right)^n$$

Grow by a factor of $\left(1+\frac{r}{n}\right)^n$ in one year once a year, $P_0 = 1$

$$1 \longrightarrow 1 + r$$

twice a year, $P_0 = 1$

$$1 \longrightarrow \left(1+\frac{r}{2}\right) \longrightarrow \left(1+\frac{r}{2}\right)^2 = 1 + 2\cdot\frac{r}{2} + \frac{r^2}{4} = 1 + r + \frac{r^2}{4}$$

thrice a year, $P_0 = 1$

$$1 \longrightarrow \left(1 + \frac{r}{3}\right)^3 = 1^3 + 3\cdot 1^2 \frac{r}{3} + 3\cdot 1\cdot\left(\frac{r}{3}\right)^2 + 1\cdot\left(\frac{r}{3}\right)^3$$

$$= 1 + r + \frac{r^2}{3} + \frac{r^3}{27}$$

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = ?$$

$$\left(1+\frac{r}{n}\right)^n = 1 + \binom{n}{1}\left(\frac{r}{n}\right) + \binom{n}{2}\left(\frac{r}{n}\right)^2 + \binom{n}{3}\left(\frac{r}{n}\right)^3 + \cdots + \binom{n}{n}\left(\frac{r}{n}\right)^n$$

$$= 1 + n\cdot\frac{r}{n} + \frac{n(n-1)}{2!}\frac{r^2}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{r^3}{n^3} + \cdots$$

$$\approx 1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \cdots$$

looks like $1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{r^n}{n!} = e^r$

$$e^x e^y = e^{x+y}$$

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{y^m}{m!}\right) = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}$$

High level sketch

$$\sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{k!\, x^l y^{k-l}}{l!\,(k-l)!} \frac{1}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} x^l y^{k-l} = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}$$

$$\left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + y + \frac{y^2}{2!} + \cdots\right)$$

$$1 + (x+y) + \left(\frac{x^2}{2!} + xy + \frac{y^2}{2!}\right)$$

$$+ \left(\frac{x^3}{3!} + \frac{x^2 y}{2!\,1!} + \frac{x y^2}{1!\,2!} + \frac{y^3}{3!}\right)$$

$$+ \left(\qquad \right) + \cdots$$

Recall from Calculus

$$(e^x)' = e^x$$

Sketch of proof:

$$(e^x)' = \left(1 + x + x^2/2! + x^3/3! + \cdots\right)'$$

$$= 0 + 1 + \frac{2 \cdot x}{2 \cdot 1!} + \frac{3 \cdot x^2}{3 \cdot 2!} + \frac{4 \cdot x^3}{4 \cdot 3!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + x^3/3! + \cdots = e^x$$

$\log x = \ln x$: $(\log x)' = 1/x$

Thm: if $f(g(x)) = x$ Then $g'(x) = \dfrac{1}{f'(g(x))}$

Proof: Chain Rule: $f'(g(x))\, g'(x) = 1$ so solve for $g'(x)$!

<u>Use it!</u> $\exp(\log x) = x$

here $f(x) = e^x$ $g(x) = \log x$

$f'(x) = e^x$

$f'(g(x)) = e^{\log x} = x$

So $(\log x)' = \dfrac{1}{f'(g(x))} = \dfrac{1}{x}$