

Math 140: Calculus II: Spring '22 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:

[https://web.williams.edu/Mathematics/sjmiller/
public_html/140Sp22/](https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/)

Lecture 6: 2-16-22:

<https://youtu.be/H0vHoX8UtLw> (slides [here](#))

Logarithms: watch <https://www.youtube.com/watch?v=-SsbkPaB6j8> (22 minutes)

Plan for the day: Lecture 4: February 11, 2022:

Review Exponential Function and Logarithm

Applications

Compound interest definition, series definition

Log Laws

Introduction to Logarithms

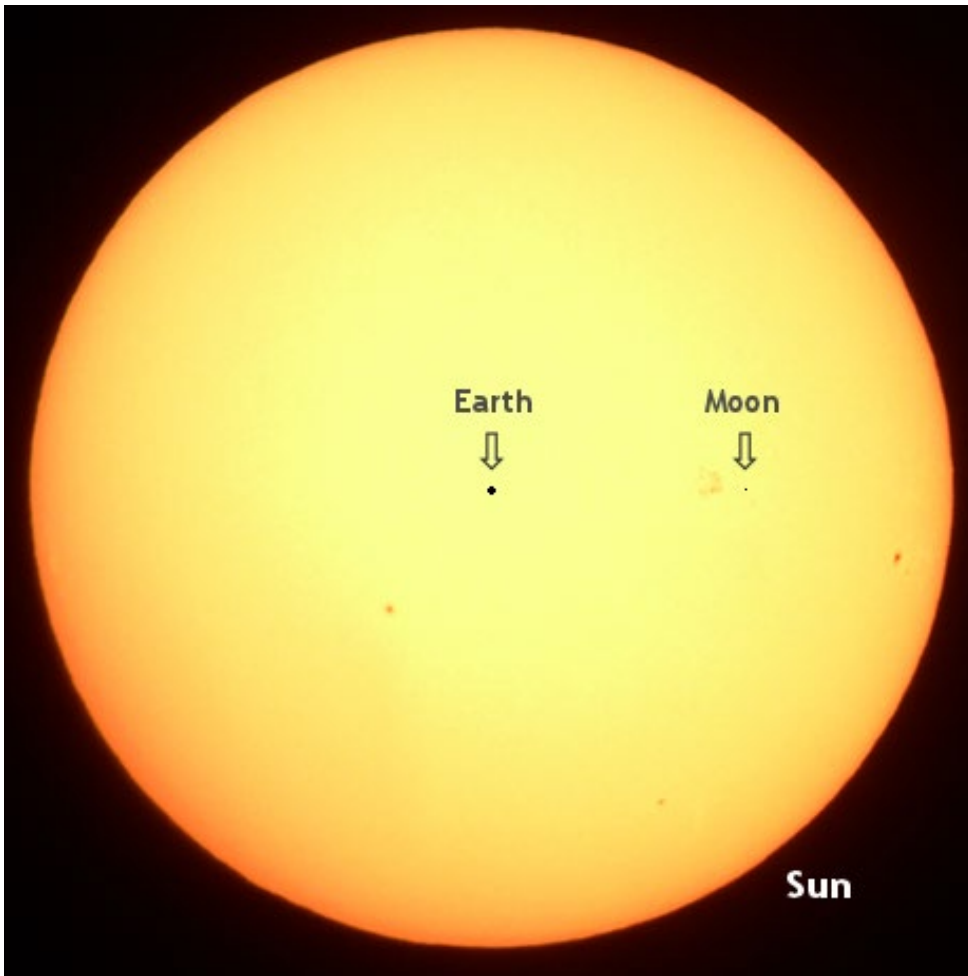
Steven Miller, Williams College

sjm1@Williams.edu

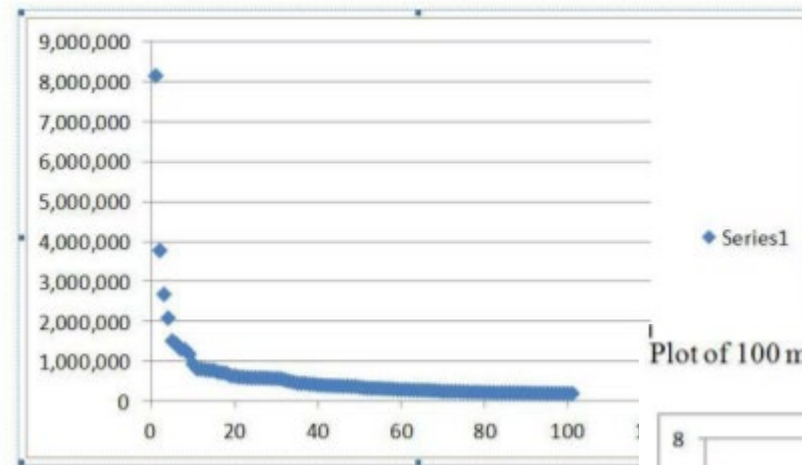
x	0	1	2	3	4	5	6	7	8	9	ADD								
10	.0000	0043	0086	0128	0170	0212	0255	0297	0340	0382	4	8	12	17	21	25	29	34	38
11	.0414	0457	0499	0541	0583	0625	0667	0709	0751	0793	4	8	12	17	21	24	28	32	36
12	.0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	4	7	11	15	19	22	26	30	33
13	.1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	7	10	13	16	20	24	27	31
14	.1461	1492	1523	1554	1584	1614	1644	1673	1702	1731	3	6	10	13	16	19	22	26	29
15	.1761	1790	1818	1846	1874	1902	1929	1956	1983	2010	3	6	9	11	14	17	20	22	25
16	.2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	.2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	.2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	10	12	14	17	19	22
19	.2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	6	9	11	13	15	18	20
20	.3010	3032	3054	3075	3096	3118	3139	3160	3181	3202	2	4	6	8	10	13	15	17	19
21	.3222	3243	3263	3284	3304	3324	3345	3365	3385	3405	2	4	6	8	10	12	14	16	18
22	.3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	11	13	15	17
23	.3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	5	7	9	11	13	14	16
24	.3802	3820	3838	3856	3874	3892	3910	3928	3945	3963	2	4	5	7	9	11	13	14	16
25	.3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	.4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	6	8	10	11	13	14
27	.4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	10	11	13	14
28	.4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	.4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13

Why do we care about Logarithms

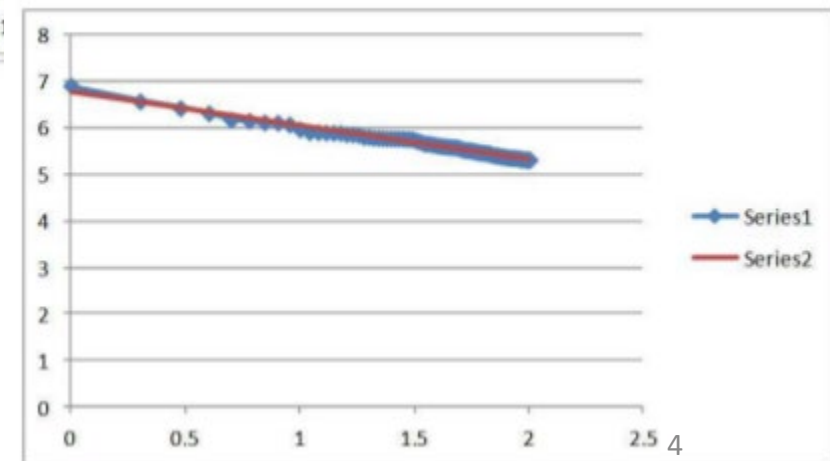
- Discuss objects across many orders of magnitude.
- Linearize many non-linear functions (calculus becomes available).



Plot of 100 most populous cities



Plot of 100 most populous cities: log-log plot



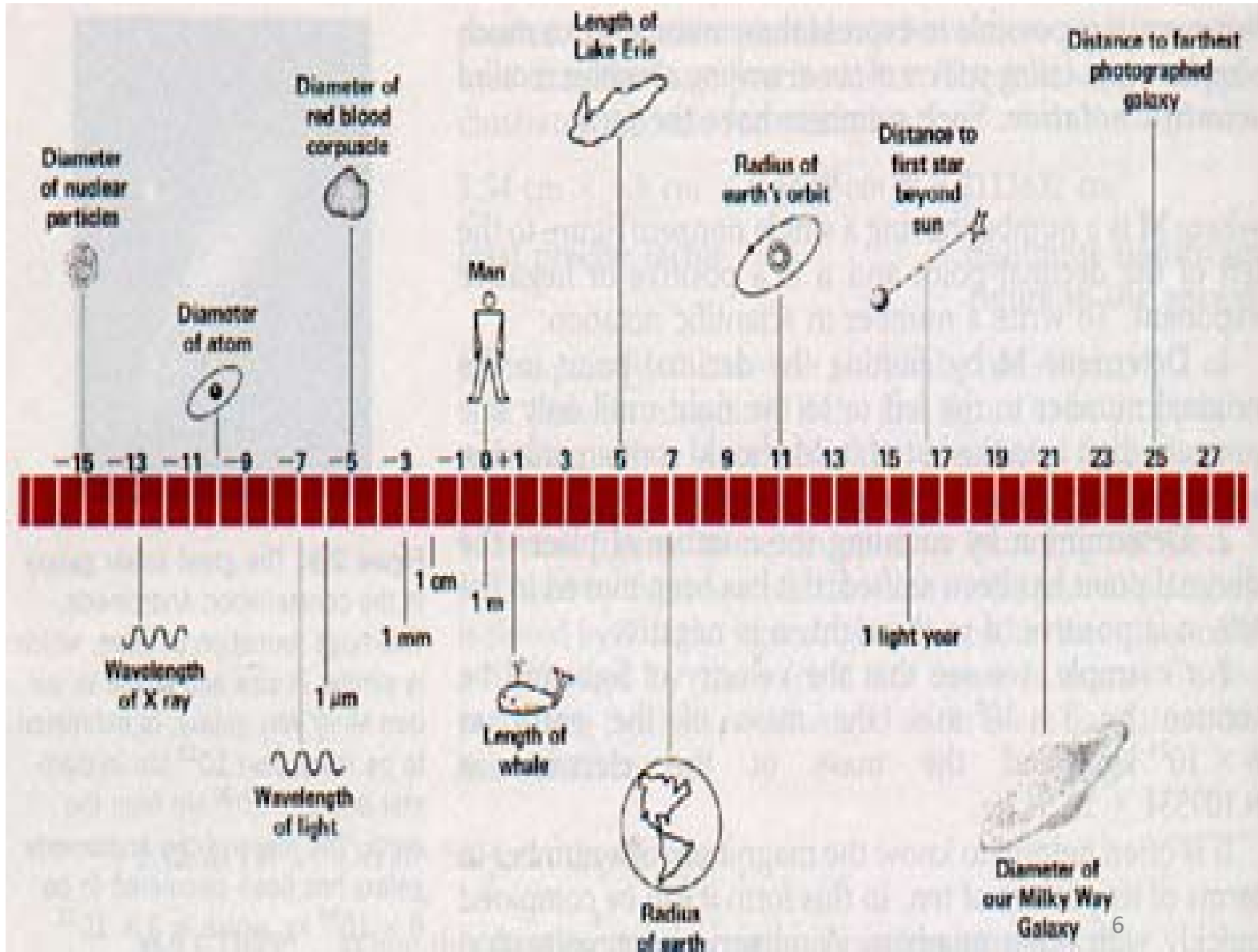
Definition of Logarithms

- **If $x = b^y$ then $\log_b x = y$.**
- Read as the logarithm of x base b is y .
- Often use base 10, and some authors suppress the subscript 10.
- Other popular bases are 2 for computers, and e for calculus; many sources write $\ln x$ for the natural logarithm of x , which is its logarithm base e (e is approximately 2.71828).
- **Examples: $\log_b x = y$ means we need y powers of b to get x .**
 - $100 = 10^2$ becomes $\log_{10} 100 = 2$. In base e it is about 4.6.
 - $1 = 10^0$ becomes $\log_{10} 1 = 0$. In base e it is still 0.
 - $.001 = 10^{-3}$ becomes $\log_{10} .001 = -3$. In base e it is about -6.9.

Examples of Logarithms

Order of Magnitude of some Lengths

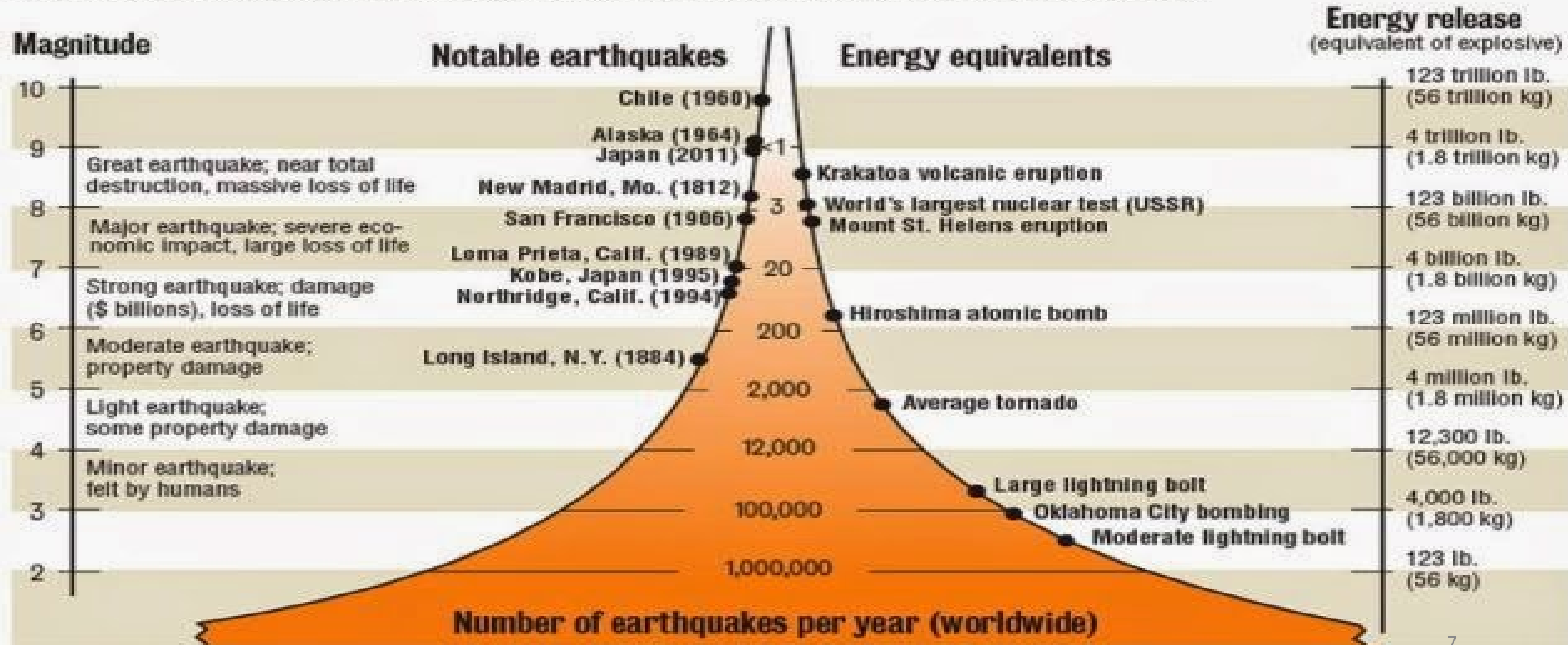
LENGTH	meters
radius of proton	10^{-15}
radius of atom	10^{-10}
radius of virus	10^{-7}
radius of amoeba	10^{-4}
height of human being	10^0
radius of earth	10^7
radius of sun	10^9
earth-sun distance	10^{11}
radius of solar system	10^{13}
distance of sun to nearest star	10^{16}
radius of milky way galaxy	10^{21}
radius of visible Universe	10^{26}



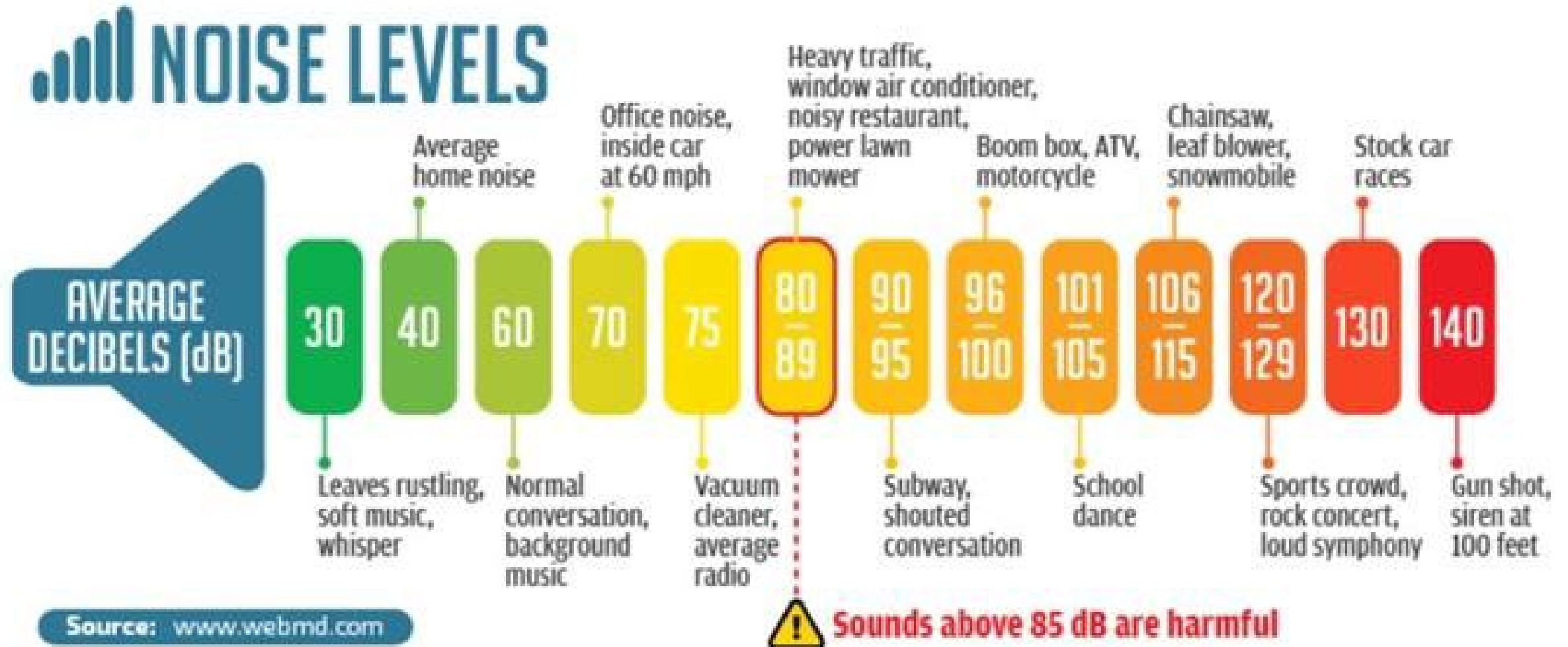
Examples of Logarithms

Earthquake frequency and destructive power

The left side of the chart shows the magnitude of the earthquake and the right side represents the amount of high explosive required to produce the energy released by the earthquake. The middle of the chart shows the relative frequencies.



Examples of Logarithms



Examples of Logarithms

The pH Scale



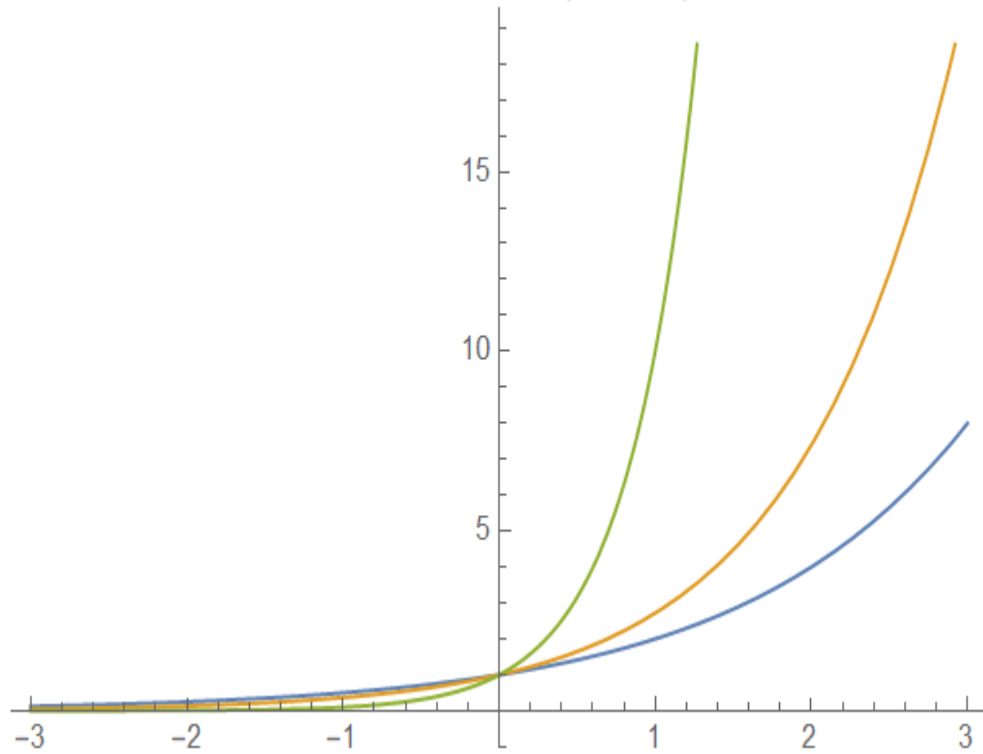
Recall: Definition of Logarithms

- **If $x = b^y$ then $\log_b x = y$.**
- Read as the logarithm of x base b is y .
- Often use base 10, and some authors suppress the subscript 10.
- Other popular bases are 2 for computers, and e for calculus; many sources write $\ln x$ for the natural logarithm of x , which is its logarithm base e (e is approximately 2.71828).
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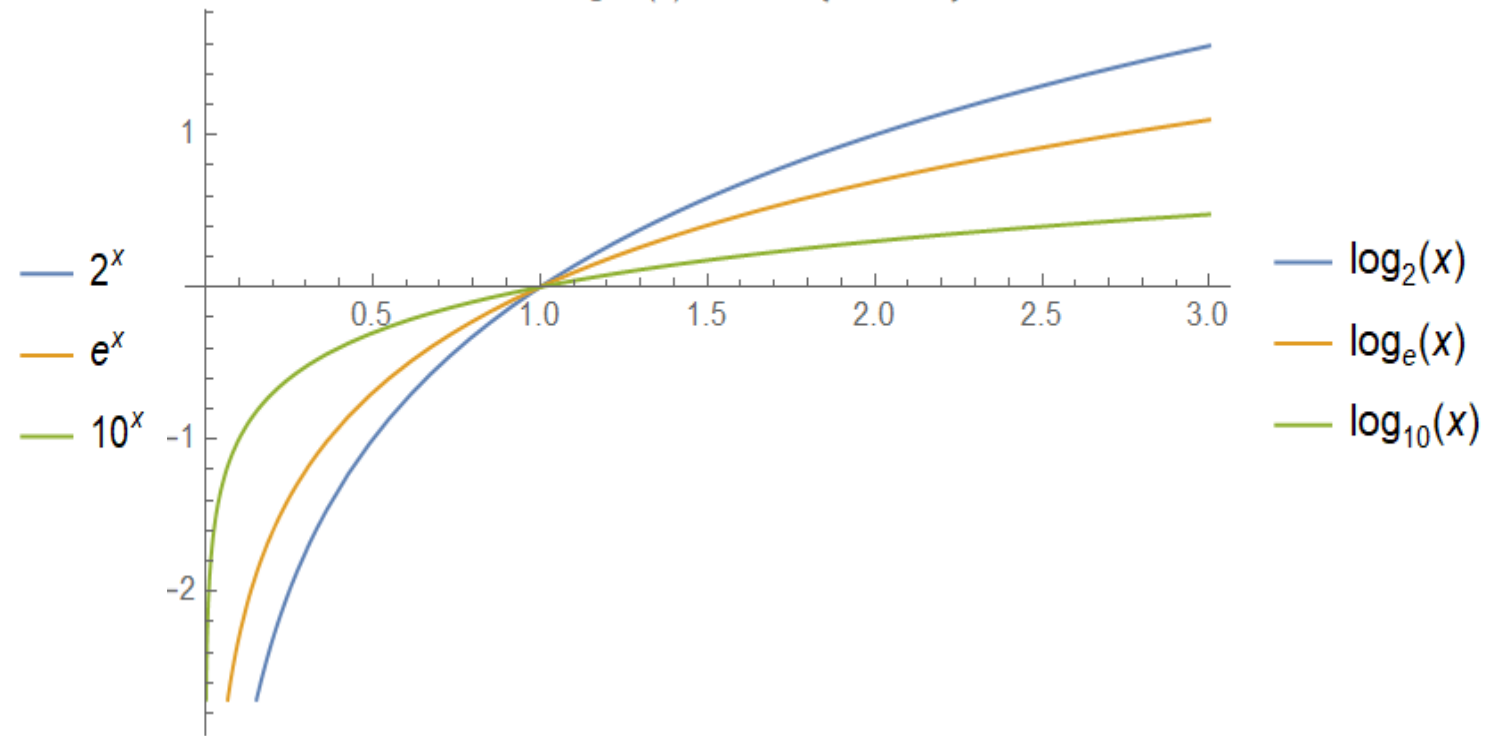
Plots of Exponentiation and Logarithms

- If $x = b^y$ then $\log_b x = y$.
- Read as the logarithm of x base b is y .

Plot of b^x for b in $\{2, e, 10\}$



Plot of $\log_b(x)$ for b in $\{2, e, 10\}$

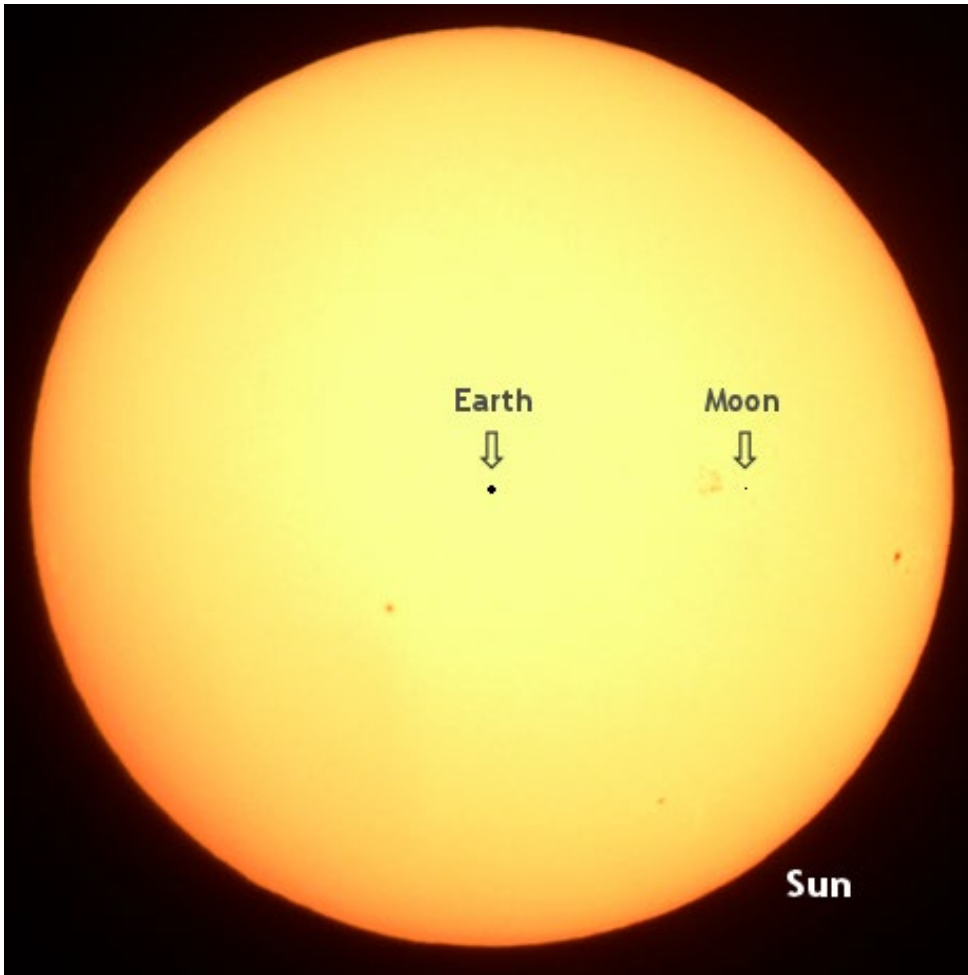


$$10^n \leq x \leq 10^{n+1}$$

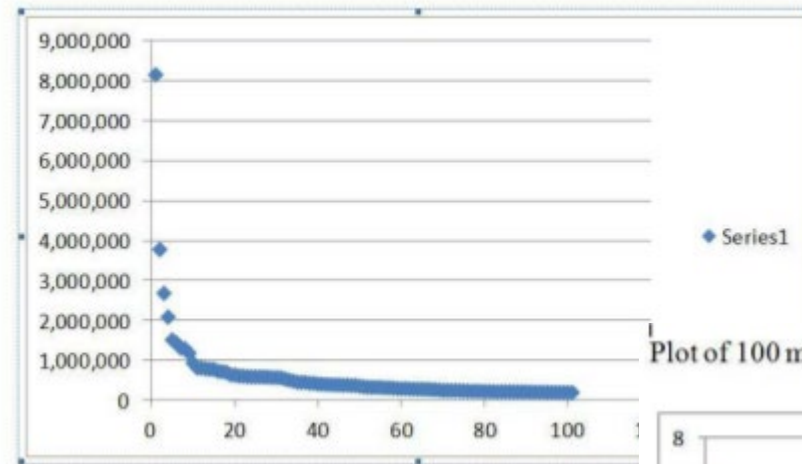
implies $n \leq \log_{10} x \leq n + 1.$

Why do we care about Logarithms

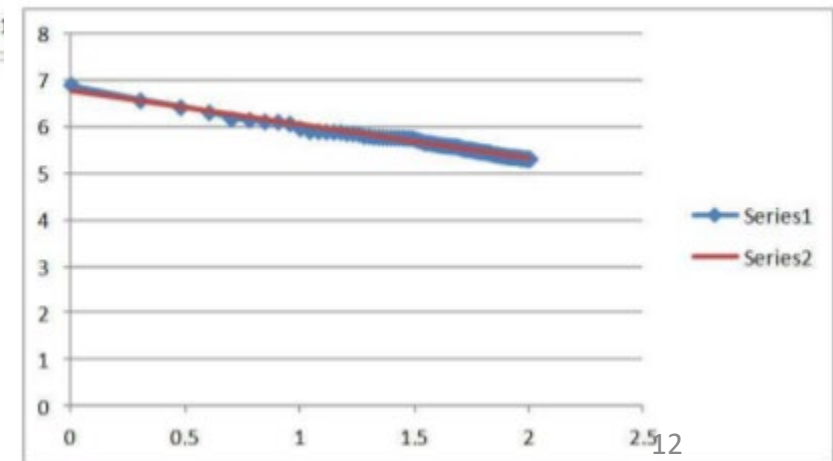
- Discuss objects across many orders of magnitude.
- Linearize many non-linear functions (calculus becomes available).



Plot of 100 most populous cities



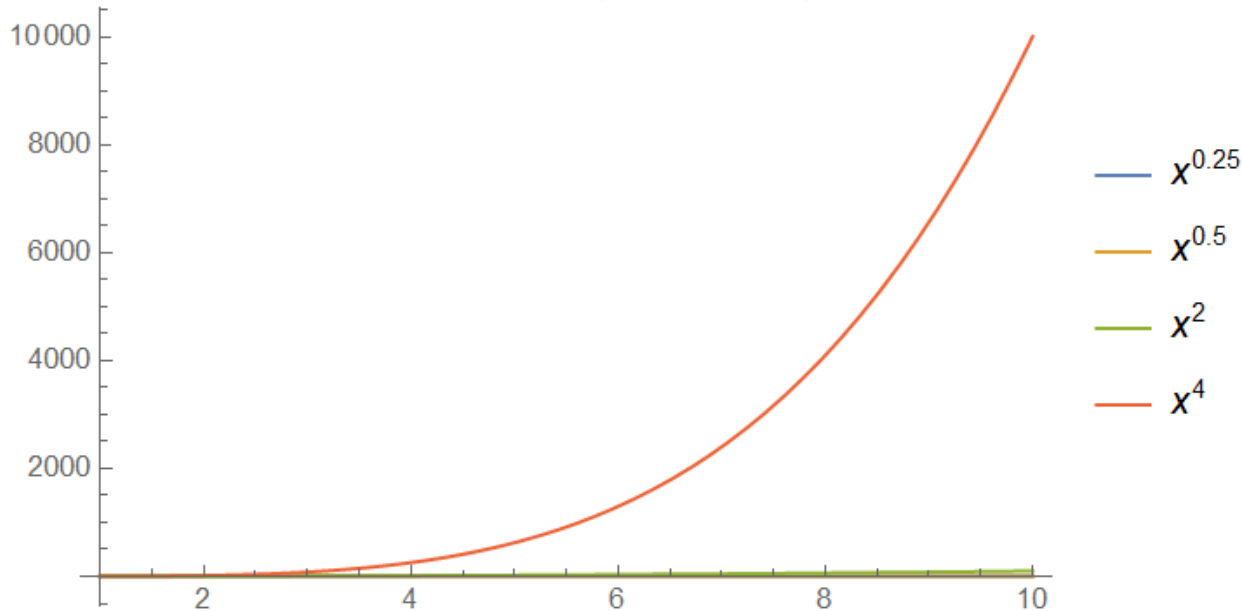
Plot of 100 most populous cities: log-log plot



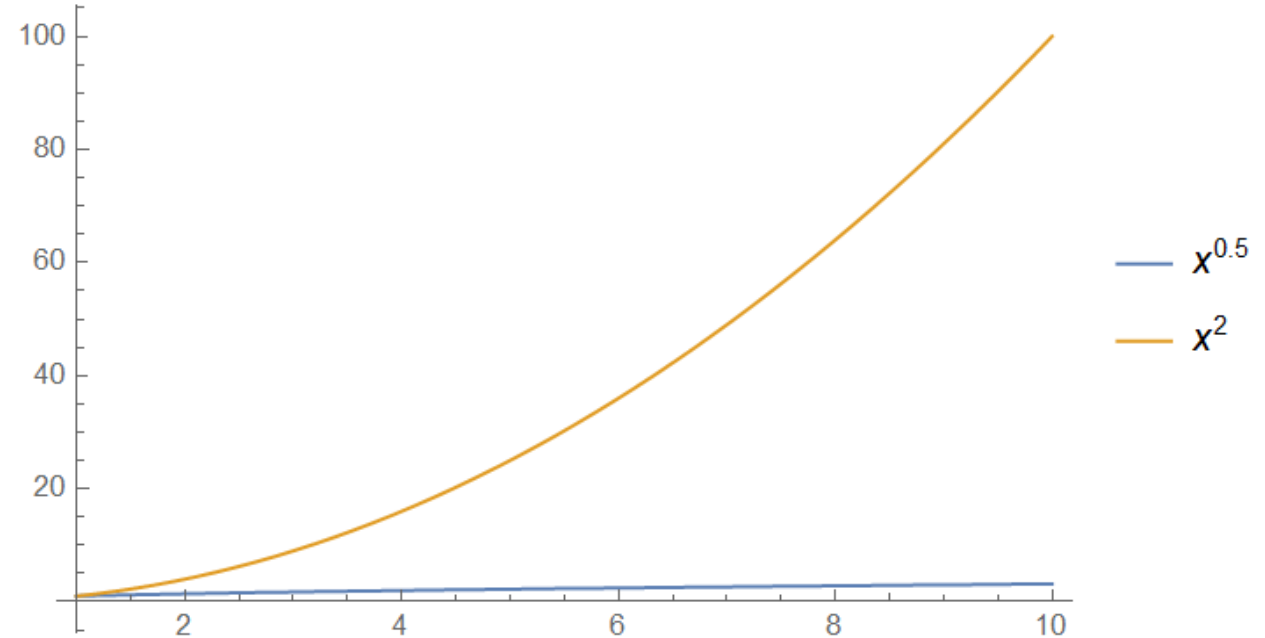
Why do we care about Logarithms

- Linearize many non-linear functions (calculus becomes available).

Plot of x^r for r in $\{1/4, 1/2, 2, 4\}$



Plot of x^r for r in $\{1/4, 1/2, 2, 4\}$

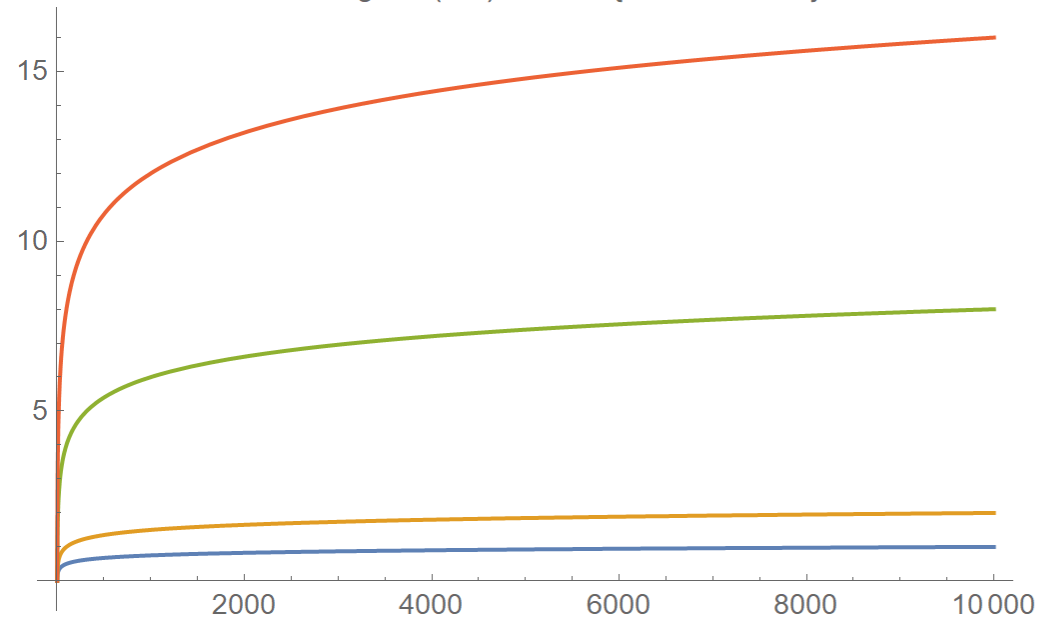


Notice that even on a small range, from 1 to 10, the polynomial of highest degree drowns out the others and can barely see.

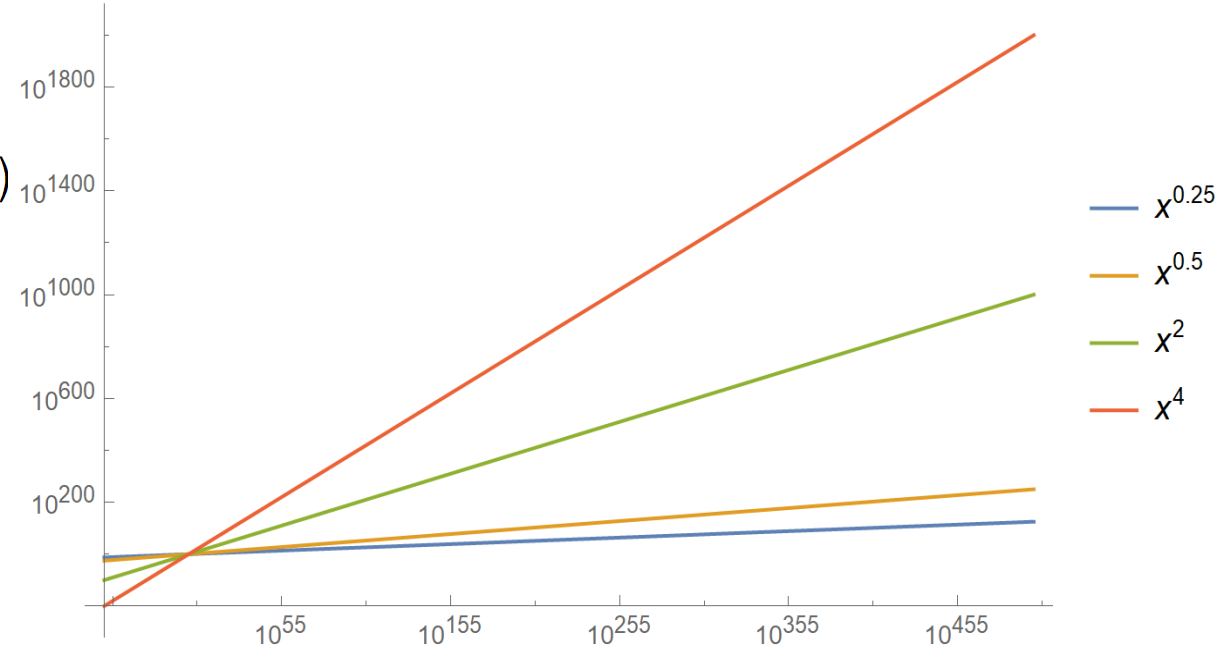
Why do we care about Logarithms

- Linearize many non-linear functions (calculus becomes available).

Plot of $\log_{10}(x^r)$ for r in $\{1/4, 1/2, 2, 4\}$



Log-Log Plot: $y = x^r$, or $\log_{10}(y) = \log_{10}(x^r)$ or $\log_{10}(y) = r \log_{10}(x)$



Left: Semi-log plot: $y = \log x^r$. Right: log-log plot: $\log y = \log x^r$.

Note that we can now see the four functions on one plot, and the log-log plot now has linear relations.

Review: Exponent Laws

Laws

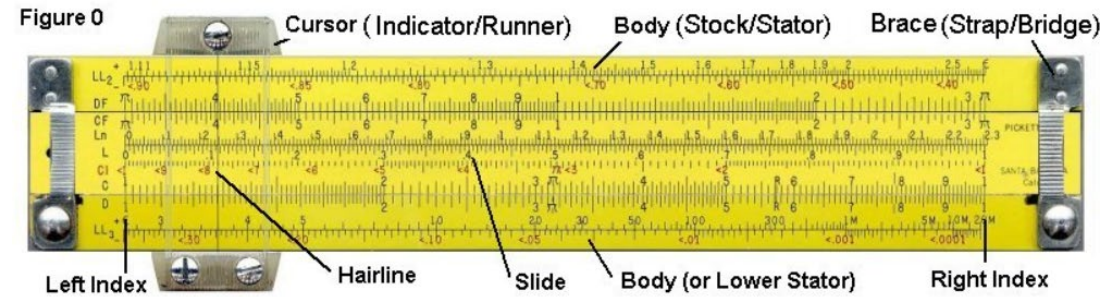
- $b^m b^n = b^{m+n}$
- $b^m / b^n = b^{m-n}$
- $(b^m)^n = b^{mn}$

$$b^2 b^3 = (b \cdot b) \cdot (b \cdot b \cdot b) = b^5$$
$$b^3 / b^2 = \frac{b \cdot b \cdot b}{b \cdot b} = b$$
$$(b^2)^3 = (b \cdot b) \cdot (b \cdot b) \cdot (b \cdot b) = b^6$$

Examples

- $10^3 10^2 = (10 * 10 * 10) * (10 * 10) = 10^5$
- $10^3 / 10^2 = (10 * 10 * 10) / (10 * 10) = 10^1$
- $(10^3)^2 = 10^3 * 10^3 = (10 * 10 * 10) * (10 * 10 * 10) = 10^6$

Logarithm Laws



Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

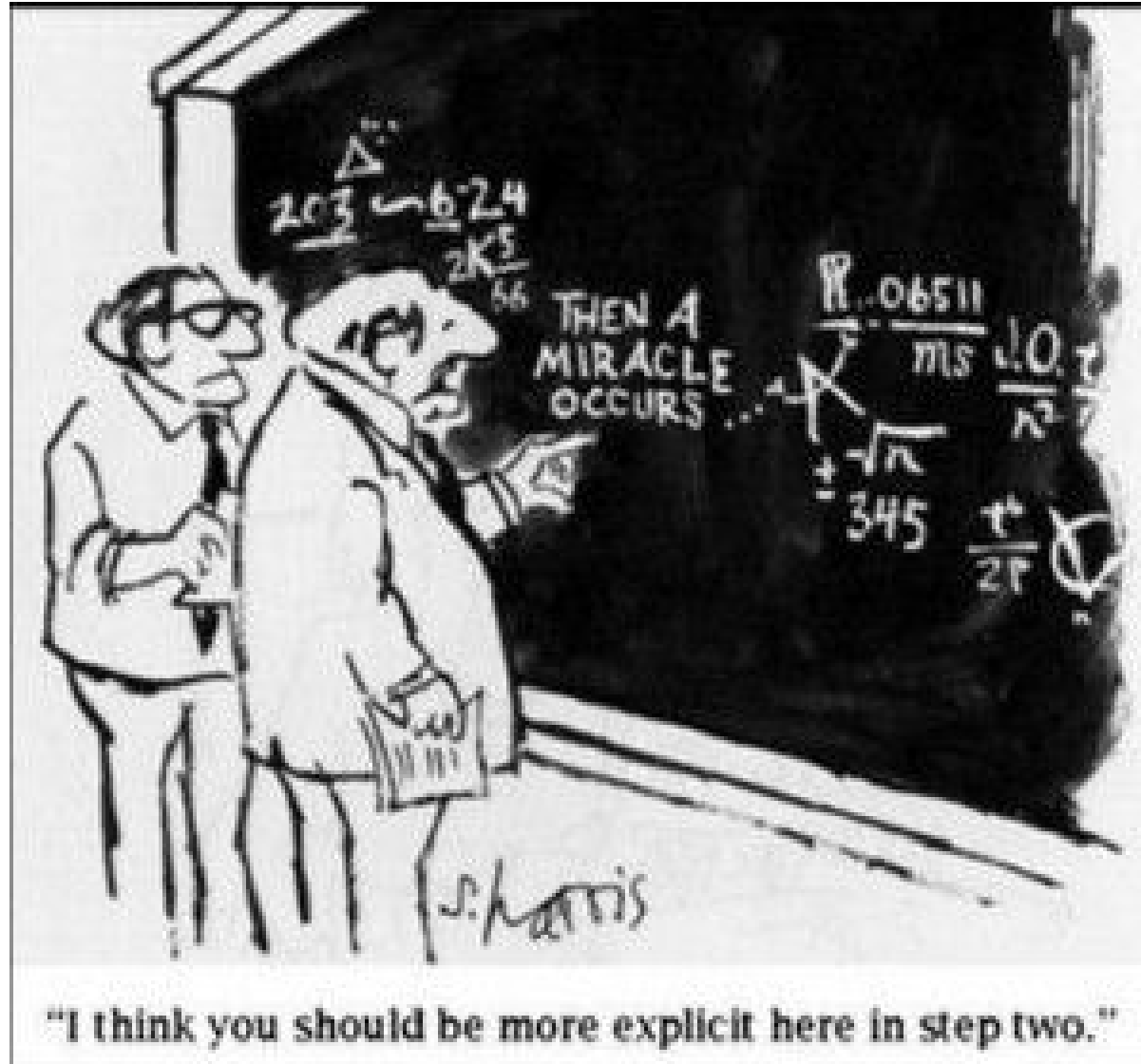
These allow us to simplify computations with logarithms.

THEOREM

$$\log_b (x^{-1}) = -\log_b (x)$$

- $\log_b (x^n) = n \log_b x$. Log of a power is that power times the log.
- $\log_b (x_1 x_2) = \log_b (x_1) + \log_b (x_2)$. Log of a product is the sum of the logs.
- $\log_b (x_1 / x_2) = \log_b (x_1) - \log_b (x_2)$. Log of a quotient is the difference of the logs.
- $\log_b x = \log_c x / \log_c b$. If know logs in one base, know in all.

OPTIONAL – PROOFS OF THE LOG LAWS



Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

- **$\log_b(x^n) = n \log_b x$.** Log of a power is that power times the log.

Proof:

- $\log_b x = y$ means $x = b^y$.
- Thus $x^n = (b^y)^n = b^{ny}$.
- Taking logarithms: $\log_b(x^n) = ny = n \log_b x$. ■

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$.

- **$\log_b(x_1 x_2) = \log_b(x_1) + \log_b(x_2)$.** Log of a product is the sum of the logs.

Proof:

- As $\log_b x_1 = y_1$ and $\log_b x_2 = y_2$, we have $x_1 = b^{y_1}$ and $x_2 = b^{y_2}$.
- Thus $x_1 x_2 = b^{y_1} b^{y_2} = b^{y_1 + y_2}$.
- Therefore $\log_b(x_1 x_2) = y_1 + y_2 = \log_b x_1 + \log_b x_2$. ■

Logarithm Laws: Proofs

Remember if $x = b^y$ then $\log_b x = y$.

Below assume $\log_c x = u$ (so $x = c^u$) and $\log_c b = v$ (so $b = c^v$).

- **$\log_b x = \log_c x / \log_c b$** . Know logs in one base, know in all.

Proof:

- As $\log_b x = y$ have $x = b^y$. Similarly $x = c^u$ and $b = c^v$.
- Thus $x = b^y = (c^v)^y = c^{vy}$.
- As also have $x = c^u$ we have $u = vy$ or $y = u/v$.
- Substituting gives $\log_b x = \log_c x / \log_c b$. ■

Example: Factorial Function:

Number ways to order *n objects when order matters:*

$$n! = n * (n - 1) * \dots * 3 * 2 * 1.$$

```
list = {}; semiloglist = {}; logloglist = {};
```

```
For[n = 1, n <= 200, n++,
```

```
{
```

```
list = AppendTo[list, {n, n!}];
```

```
semiloglist = AppendTo[semiloglist, {n, Log[n!]}];
```

```
logloglist = AppendTo[logloglist, {Log[n], Log[n!]}];
```

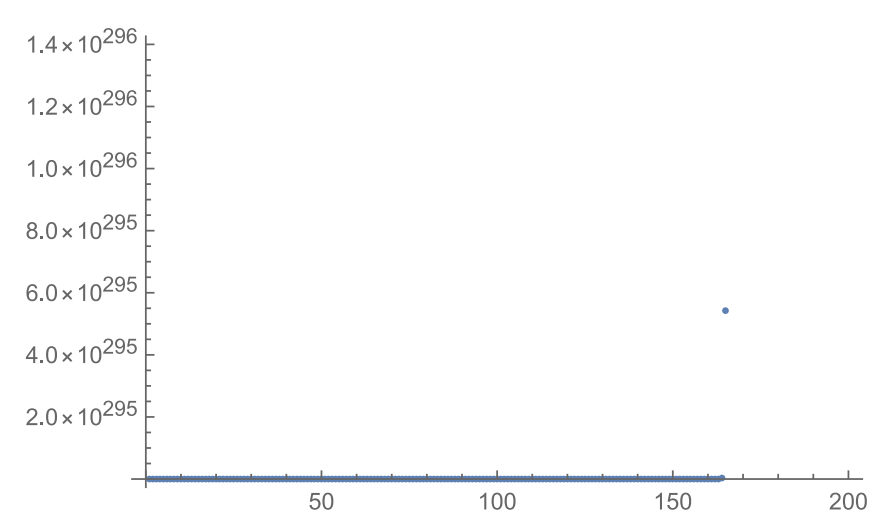
```
});
```

```
Print[ListPlot[list]]; Print[ListPlot[semiloglist]]; Print[ListPlot[logloglist]];
```

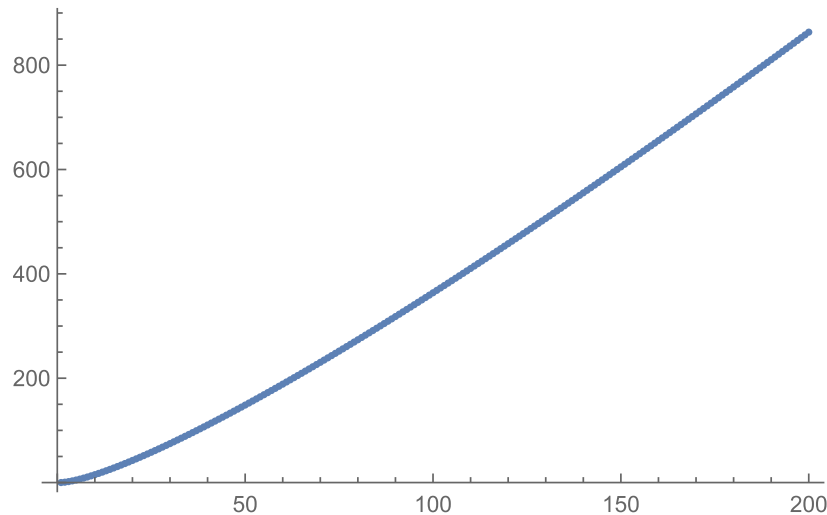
Example: Factorial Function:

Number ways to order n *objects when order matters*:

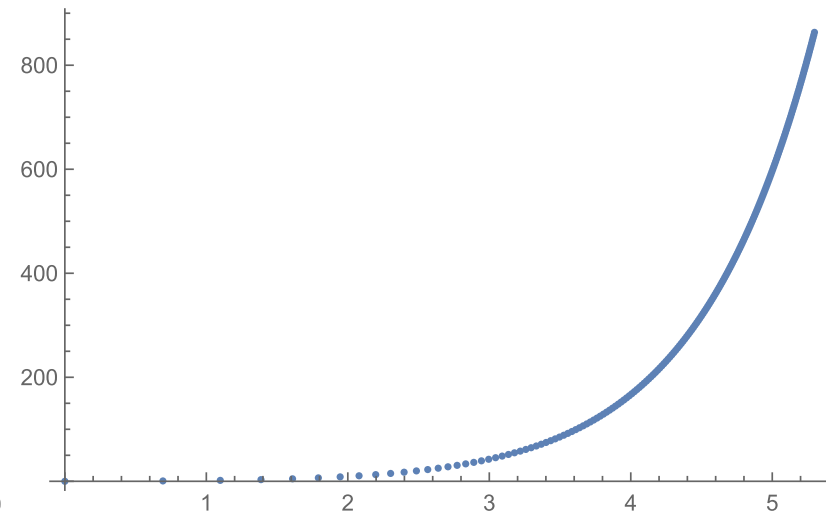
$$n! = n * (n - 1) * \dots * 3 * 2 * 1.$$



Normal Plot



Semi-log Plot



Log-Log Plot

For large n , have $n! \approx n^n e^{-n} \sqrt{2\pi n}$, so $\log n! \approx n \log n$ (plus a much smaller term).

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$n! = n(n-1) \dots 3 \cdot 2 \cdot 1$$

$$\text{with } 0! = 1$$

aside: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

ways to choose k people from n , when order does not matter

$$\binom{5}{2} = \frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3!}{2!3!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$$

$$\binom{2}{5} = 0 \stackrel{?}{=} \frac{2!}{5!(2-5)!} = \frac{2!}{5!(-3)!} \quad \text{So } (-3)! \leq \infty$$

$$\left(-\frac{1}{2}\right)! = \sqrt{\pi}$$

$$e^X = \lim_{n \rightarrow \infty} \left(1 + \frac{X}{n}\right)^n \quad \text{Compound interest}$$

Start with P_0 money, rate is $r\%$ per year

(so 100% per year is $r=1$), compound n times a year.

$$P_0 \xrightarrow{1 \text{ period}} P_0 \left(1 + \frac{r}{n}\right) \xrightarrow{2 \text{ period}} P_0 \left(1 + \frac{r}{n}\right)^2 \rightarrow \dots \rightarrow P_0 \left(1 + \frac{r}{n}\right)^n$$

Grow by a factor of $\left(1 + \frac{r}{n}\right)^n$ in one year

Once a year, $P_0 = 1$

$$1 \longrightarrow 1 + r$$

twice a year, $P_0 = 1$

$$1 \longrightarrow \left(1 + \frac{r}{2}\right) \longrightarrow \left(1 + \frac{r}{2}\right)^2 = 1 + 2 \cdot \frac{r}{2} + \frac{r^2}{4} = 1 + r + \underline{\underline{\frac{r^2}{4}}}$$

three a year, $P_0 = 1$

$$1 \longrightarrow \left(1 + \frac{r}{3}\right)^3 = 1^3 + 3 \cdot 1^2 \frac{r}{3} + 3 \cdot 1 \cdot \left(\frac{r}{3}\right)^2 + 1 \cdot \left(\frac{r}{3}\right)^3 \\ = 1 + r + \underline{\underline{\frac{r^2}{3} + \frac{r^3}{27}}}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = ?$$

$$\left(1 + \frac{r}{n}\right)^n = 1 + \binom{n}{1} \left(\frac{r}{n}\right) + \binom{n}{2} \left(\frac{r}{n}\right)^2 + \binom{n}{3} \left(\frac{r}{n}\right)^3 + \dots + \binom{n}{n} \left(\frac{r}{n}\right)^n$$

$$= 1 + n \cdot \frac{r}{n} + \frac{n(n-1)}{2!} \frac{r^2}{n^2} + \frac{n(n-1)(n-2)}{3!} \frac{r^3}{n^3} + \dots$$

$$\approx 1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \dots$$

looks like $1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{r^n}{n!} = e^r$

$$e^x e^y = e^{x+y}$$

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{y^m}{m!} \right) = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}$$

High level sketch

$$\sum_{k=0}^{\infty} \sum_{l=0}^k \frac{k! x^l y^{k-l}}{l! (k-l)!} \frac{1}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} x^l y^{k-l} = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}$$

$$\left(1 + x + \frac{x^2}{2!} + \dots \right) \left(1 + y + \frac{y^2}{2!} + \dots \right)$$

$$1 + (x+y) + \left(\frac{x^2}{2!} + xy + \frac{y^2}{2!} \right)$$

$$+ \left(\frac{x^3}{3!} + \frac{x^2 y}{2! 1!} + \frac{x y^2}{1! 2!} + \frac{y^3}{3!} \right)$$

$$+ \dots$$

Recall from Calculus

$$(e^x)' = e^x$$

Sketch of proof:

$$(e^x)' = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right)'$$
$$= \downarrow + \downarrow + \frac{2 \cdot x}{2 \cdot 1!} + \frac{3 \cdot x^2}{3 \cdot 2!} + \frac{4 \cdot x^3}{4 \cdot 3!} + \dots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x$$

$$\log x = \ln x : (\log x)' = 1/x$$

$$\text{Then: if } f(g(x)) = x \text{ then } g'(x) = \frac{1}{f'(g(x))}$$

$$\text{Proof: Chain Rule: } f'(g(x)) g'(x) = 1 \text{ so solve for } g'(x)!$$

Use it!

$$\exp(\log x) = x$$

$$\text{here } f(x) = e^x \quad g(x) = \log x$$

$$f'(x) = e^x$$

$$f'(g(x)) = e^{\log x} = x$$

$$\text{So } (\log x)' = \frac{1}{f'(g(x))} = \frac{1}{x}$$

