Math 140: Calculus II: Spring '22 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/

Lecture 9: 2-25-22:
https://youtu.be/Wj9P1NcLECA

Areas between curves, volumes, and trig substitution

Plan for the day: Lecture 9: February 25, 2022:

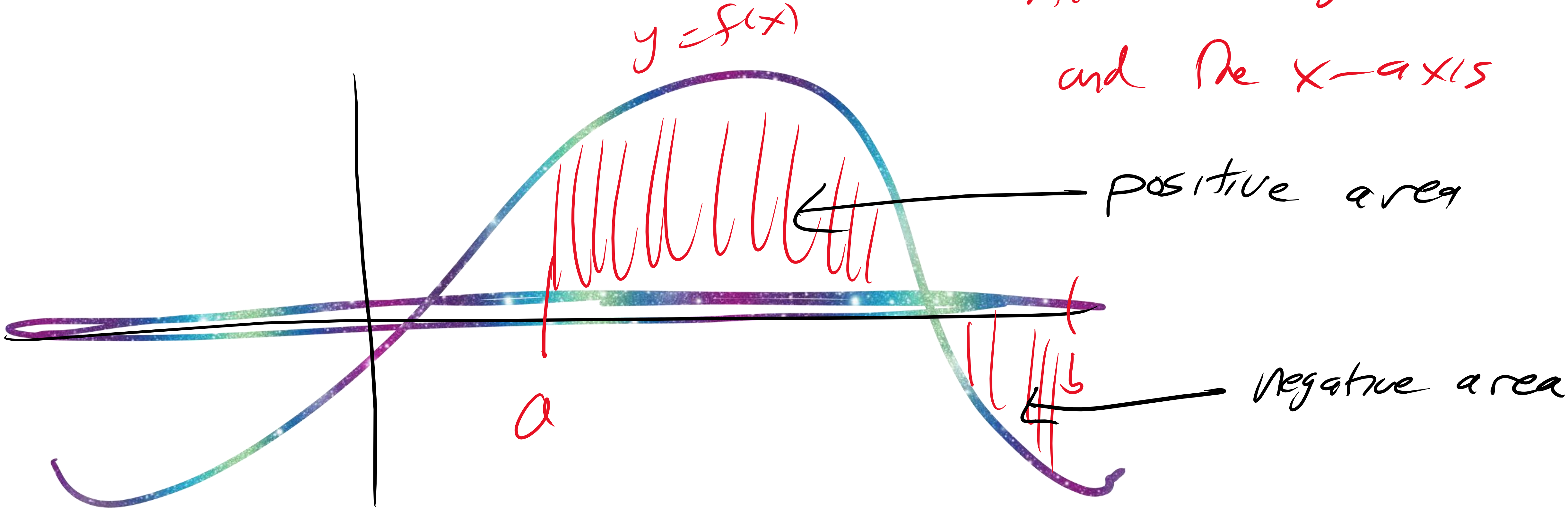
Area between curves

Volumes

Trig Substitution

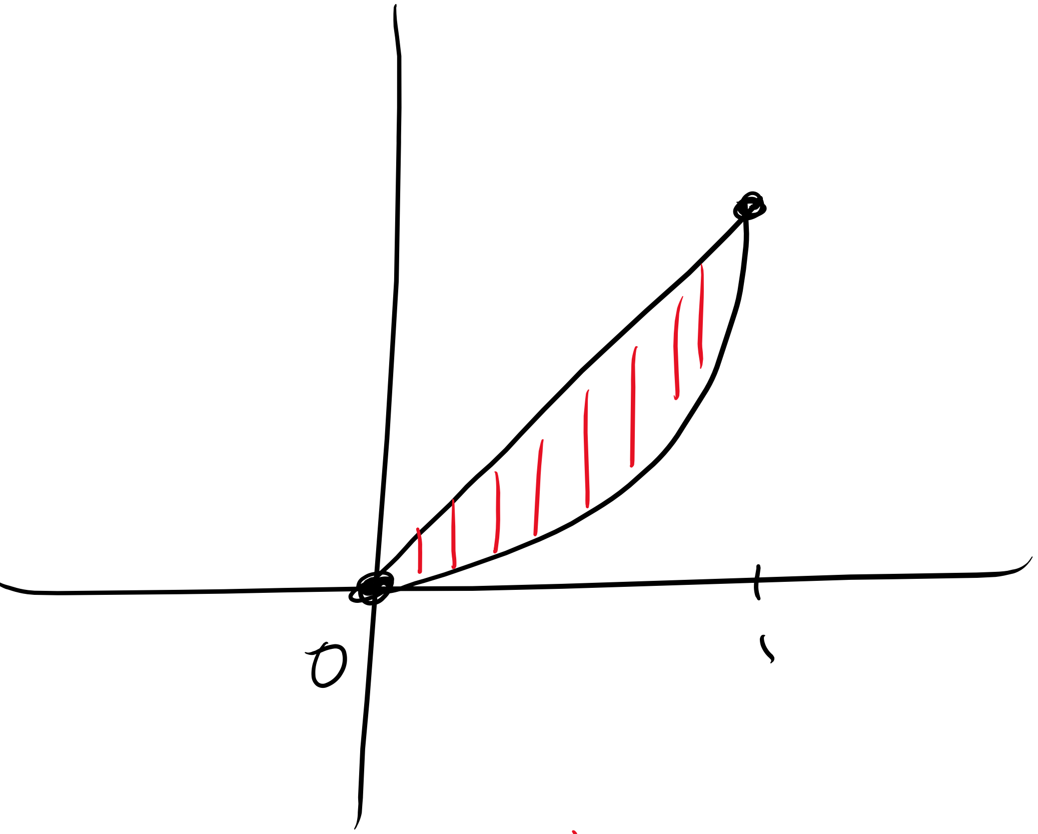
Area b/w curves

Area b/w $y=f(x)$ and the x-axis



$\int_a^b f(x)\,dx$ signed area

actual area $\int_a^b |f(x)|\,dx$



$y_1(x) = x$

$y_2(x) = x^2$

Set $y_1(x) = y_2(x)$

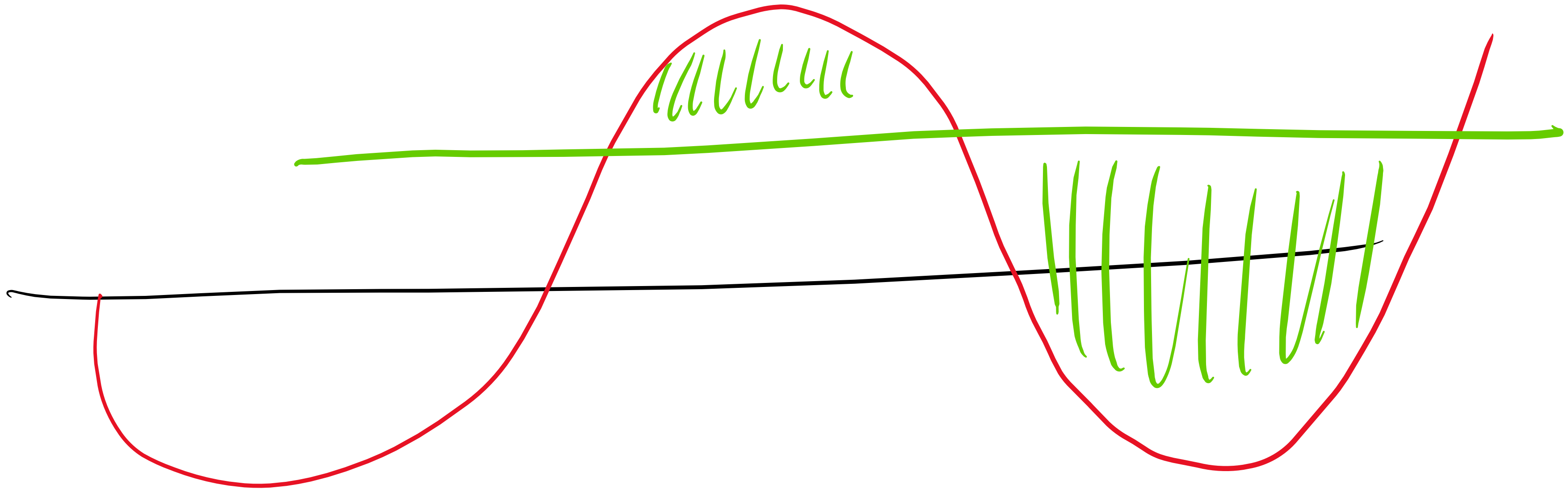
So $x = x^2$

$\Rightarrow x^2 - x = 0$

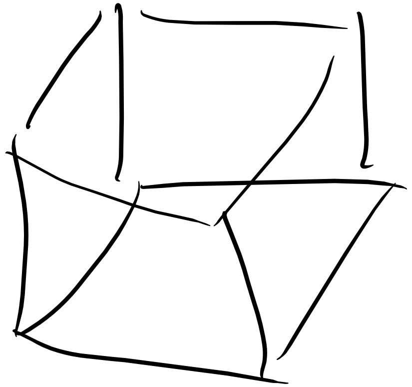
$x(x-1) = 0 \Rightarrow x \in \{0, 1\}$

$$\text{Area} = \int_0^1 [y_1(x) - y_2(x)]\,dx = \int_0^1 (x - x^2)\,dx$$

$$= \int_0^1 x\,dx - \int_0^1 x^2\,dx = \frac{x^2}{2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$



Cube

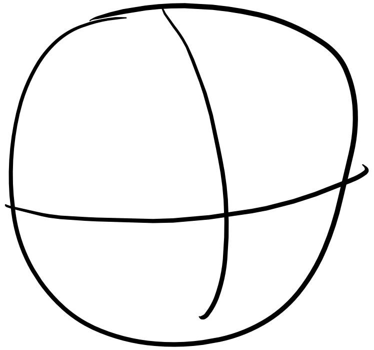


box of size 1 x 1
thickness dx

Volume $\int_{x=0}^{1} 1^2 \, dx$

Generally $\int_{x=0}^{h} L * W \, dx$

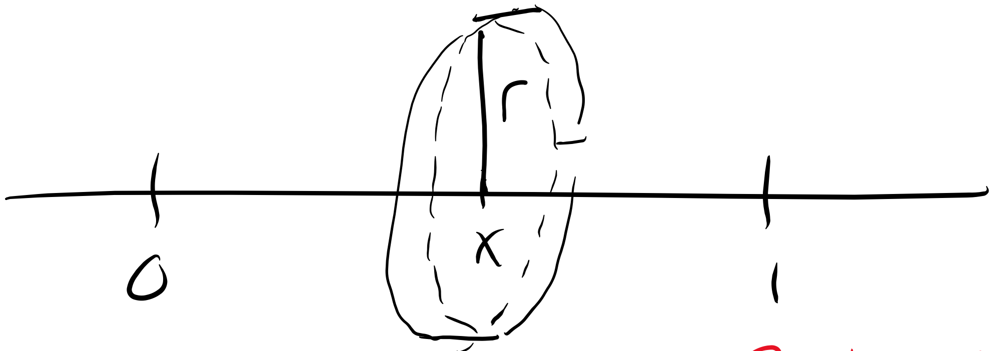
Sphere



do circular slices

Sphere is $x^2+y^2+z^2 = \cancel{r^2}\,1$ $\quad r=1$ special case

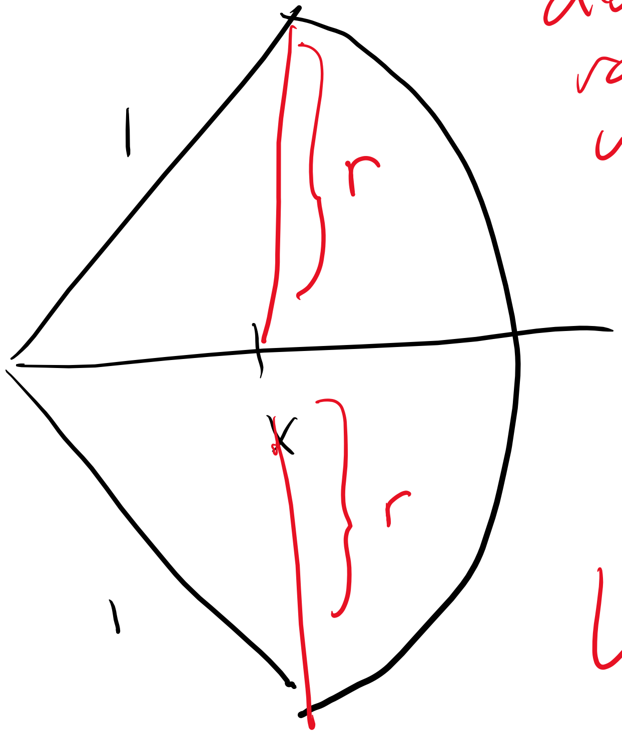
bunch of circular rings



disk of thickness dx
radius r, has
volume $\pi r^2 dx$

when $x=1$, $r=0$

when $x=0$, $r=1$



General $x^2+r^2=1$

so $r^2 = 1-x^2$

$r = \sqrt{1-x^2} = r(x)$

$$\text{Vol} = 2\int_{x=0}^{1} \pi r(x)^2\,dx = 2\int_{x=0}^{1} \pi(1-x^2)\,dx$$

Compute $2\int_0^1 \pi(1-x^2)\,dx$ (if radius R, $2\int_0^R \pi(R^2-x^2)\,dx$)

$$= 2\pi \int_0^1 (1-x^2)\,dx$$

$$= 2\pi \int_0^1 1\,dx - 2\pi \int_0^1 x^2\,dx \qquad \Big\| \qquad 2\pi \int_0^R R^2\,dx - 2\pi \int_0^R x^2\,dx$$

$$= 2\pi x\Big|_0^1 - 2\pi \frac{x^3}{3}\Big|_0^1 \qquad \Big| \qquad 2\pi R^2 x\Big|_0^R - 2\pi \frac{x^3}{3}\Big|_0^R$$

$$= 2\pi - \frac{2\pi}{3} \qquad \text{OR} \qquad 2\pi R^3 - 2\pi R^3/3$$

$$= \frac{4}{3}\pi \qquad \text{or} \qquad \frac{4}{3}\pi R^3$$

volume of a sphere!

Circle

Area πr^2

$\xrightarrow{\text{deriv}}$

perim $2\pi r$

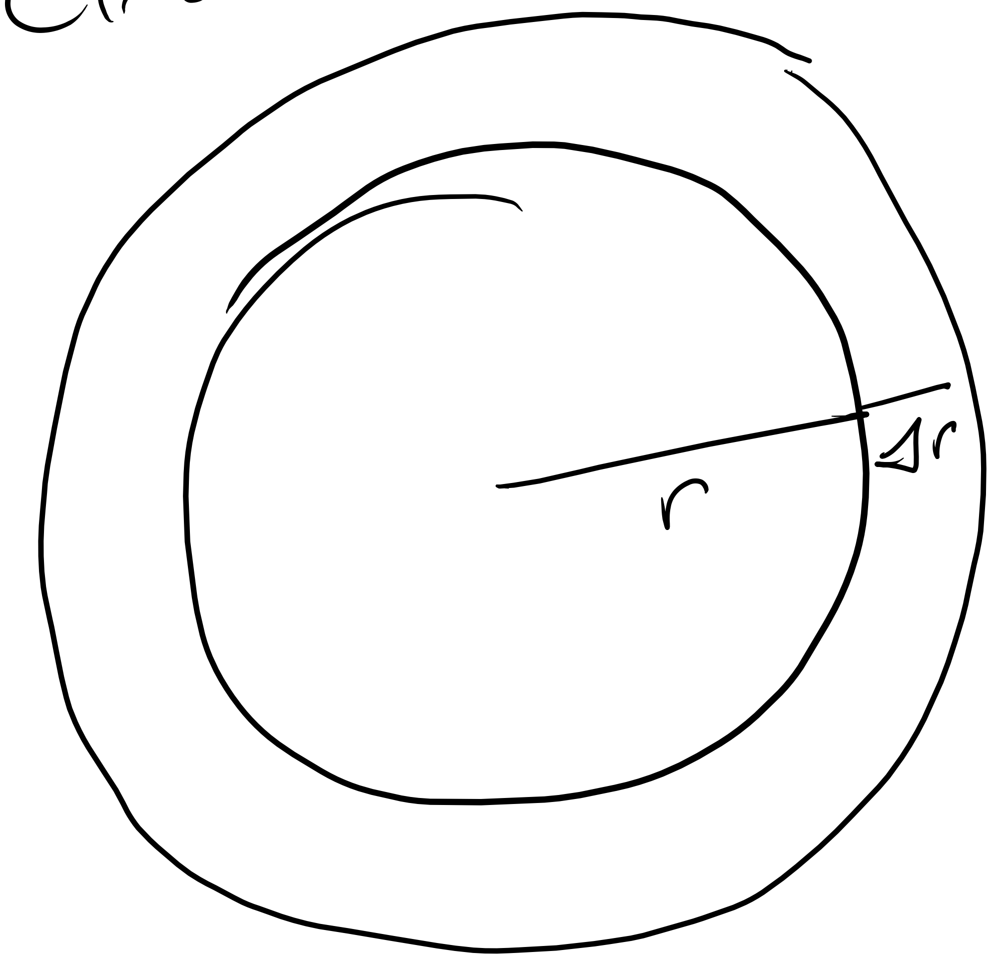
Sphere

Volume $\frac{4}{3}\pi r^3$

$\xrightarrow{\text{deriv}}$

Surface Area $4\pi r^2$

Circle



Big circle has area $\pi(r+\Delta r)^2$

Smaller circle area πr^2

diff is the ring

area $\pi(r^2+2r\Delta r+(\Delta r)^2) - \pi r^2$

$$= 2\pi r\Delta r + \pi(\Delta r)^2$$

perimeter * thickness $\approx$ area of the ring

$$p(r)\,\Delta r \approx 2\pi r\Delta r + \pi(\Delta r)^2$$

$$\text{Perim}(r) \approx 2\pi r + \underbrace{\pi\Delta r}_{\text{Goes to 0 as } \Delta r \to 0}$$

Trig Substitution

u-substitution

$$\left[f(g(x))\right]' = f'(g(x))\,g'(x)$$

$$\underbrace{\int \left[f(g(x))\right]' dx}_{f(g(x))} = \int f'(g(x))\,g'(x)\,dx$$

$$f(g(x)) = \int f'(g(x))\,g'(x)\,dx$$

$u = g(x)$ $\quad \frac{du}{dx} = g'(x)$ or $du = g'(x)dx$

$$f(g(x)) = \int f'(u)\,du = f(u) = f(g(x))$$

as $u = g(x)$

$$\int x^2 e^{x^3}\,dx$$

$$\left(e^{x^3}\right)' = e^{x^3}\,(x^3)' = e^{x^3}\cdot 3x^2$$

$u = x^3$ $\qquad \frac{du}{dx} = 3x^2$ $\qquad$ or $\quad du = 3x^2\,dx$

or $\quad x^2\,dx = \frac{1}{3}\,du$

$$\int x^2 e^{x^3}\,dx = \int e^u \cdot \frac{1}{3}\,du$$

$$= \frac{1}{3}\int e^u\,du$$

$$= \frac{1}{3}e^u = \frac{1}{3}e^{x^3}$$ as $u = x^3$

Trig Substitution

$$\int_0^1 \frac{1}{1+x^2}\,dx \quad \text{or} \quad \int_0^\infty \frac{1}{1+x^2}\,dx$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\boxed{\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \tan^2\theta + 1 &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}}$$

See $1+x^2$: Think $x = \tan\theta$

$1-x^2$: Think $x = \sin\theta$ or $x = \cos\theta$

$1-\cos^2\theta = \sin^2\theta$

or $1-\sin^2\theta = \cos^2\theta$

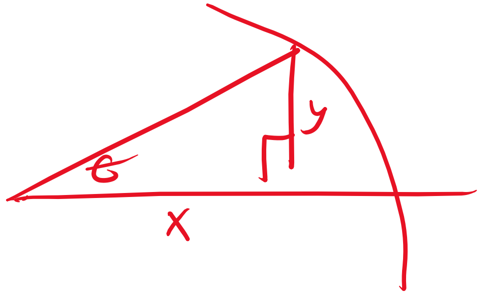
$\sin^2\theta = 1-\cos^2\theta$

or

$\cos^2\theta = 1-\sin^2\theta$

$$\int_{x=0}^{1} \frac{1}{1+x^2}\,dx \qquad x = \tan\theta \qquad x: 0 \to 1$$

$$\theta: 0 \to \frac{\pi}{4}$$



$\tan\theta = y/x$

$$\frac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta\, d\theta$$

$$\hookrightarrow \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right)$$

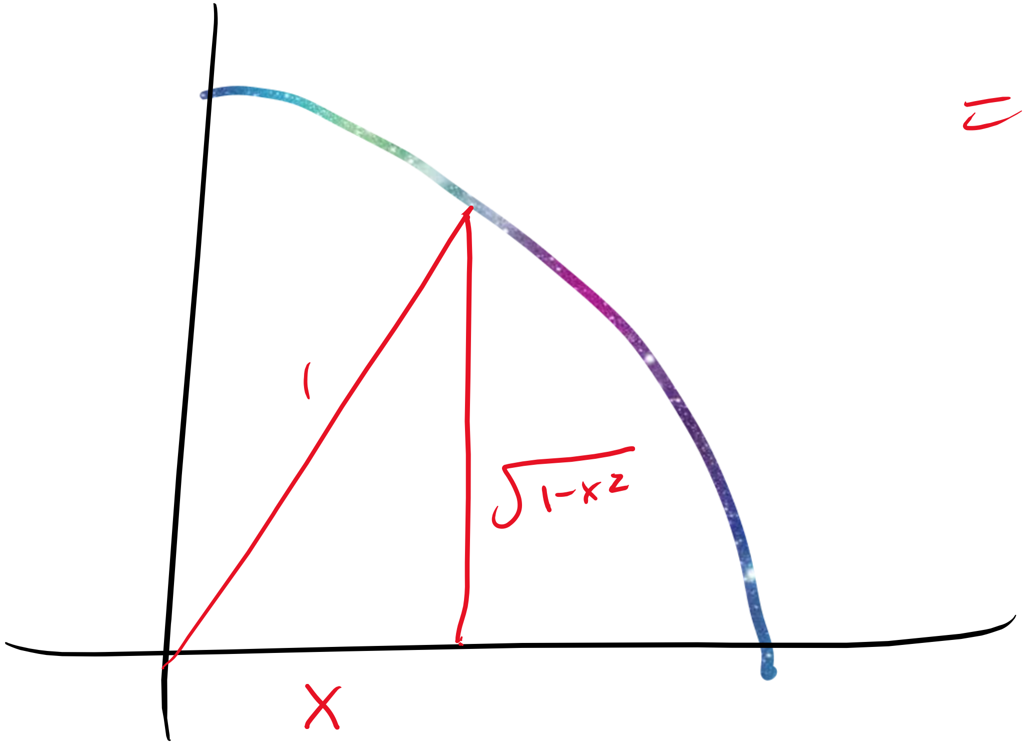
$$\int_{\theta=0}^{\pi/4} \frac{1}{1+\tan^2\theta} \cdot \sec^2\theta\, d\theta \qquad \text{but} \quad 1+\tan^2\theta = \sec^2\theta$$

$$= \int_{\theta=0}^{\pi/4} \frac{1}{\sec^2\theta} \sec^2\theta\, d\theta = \int_{\theta=0}^{\pi/4} 1\, d\theta = \theta\Big|_0^{\pi/4} = \frac{\pi}{4}$$

Later

$$\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

Think About



Area of circle

$$= 4\int_{x=0}^{1} \sqrt{1-x^2}\, dx$$

$x = \sin\theta$

$x: 0 \to 1 \quad \theta: 0 \to \pi/2$

$\frac{dx}{d\theta} = \cos\theta \quad dx = \cos\theta\, d\theta$

$$\text{Area} = 4\int_{\theta=0}^{\pi/2} \sqrt{1-\sin^2\theta}\cdot\cos\theta\, d\theta$$

$$= 4\int_{\theta=0}^{\pi/2} \cos^2\theta\, d\theta$$