Math 140: Calculus II: Spring ‘22 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/

Lecture 10: 2-28-22: https://youtu.be/9W-qVOSl7h0

https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/talks2022/140Sp22_lecture10.pdf

Plan for the day: Lecture 10: February 28, 2022:

Geometric Series Formula (convergence, hoops later…)

Derivative relations between inverse functions (especially tan-arctan)

Gregory – Leibniz Formula

Theory (derivative of x^r or proofs by induction)

Other possibilities: Optimization: Farmer Brown Problem, Product-Sum

$$\int_0^1 \frac{1}{1+x^2}\,dx = \arctan(1) - \arctan(0)$$

$$A(x) = f(g(x)) = x \qquad \text{inverse functions}$$

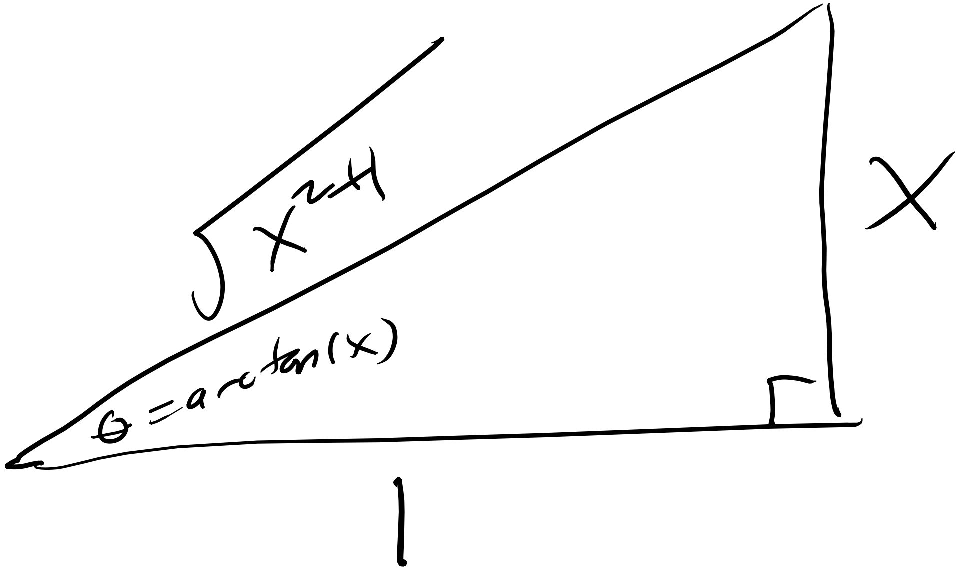
$$\tan(\arctan(x)) = x$$

$$f'(g(x))\, g'(x) = 1 \implies g'(x) = \frac{1}{f'(g(x))}$$

$$\tan'(\arctan(x)) \cdot \arctan'(x) = 1$$

$$\text{So } \arctan'(x) = \frac{1}{\tan'(\arctan x)} = \frac{1}{\sec^2(\arctan x)}$$

$$\arctan'(x) = \cos^2(\arctan x)$$



$$\cos\theta = \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$

so $\cos^2(\arctan x) = \dfrac{1}{1+x^2}$

tan is $\dfrac{\text{opp}}{\text{adj}}$

$$\sqrt{a^2+b^2} \neq \sqrt{a^2} + \sqrt{b^2}$$

$$\sqrt{a^2} = a, \quad \sqrt{b^2} = b \quad\Rightarrow\quad a + b$$

$$(a+b)^2 = a^2 + \underline{\underline{2ab}} + b^2$$

$\sqrt{3^2+4^2}$ vs $\sqrt{3^2} + \sqrt{4^2}$

$= \sqrt{25} = 5$ vs $3+4$

$$f(g(x)) = x \implies g'(x) = \frac{1}{f'(g(x))}$$

Consider $\exp(\ln x) = x$

$$\implies (\ln x)' = \frac{1}{\exp'(\ln x)}$$

$$= \frac{1}{\exp(\ln x)}$$

$$(\ln x)' = \frac{1}{x}$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Geometric Series Formula:

$$S(n) = 1 + r + r^2 + r^3 + \cdots + r^n$$

$$\frac{r\,S(n) = \qquad r + r^2 + \cdots \cdots + r^n + r^{n+1}}{(1-r)\,S(n) = 1 - r^{n+1}}$$

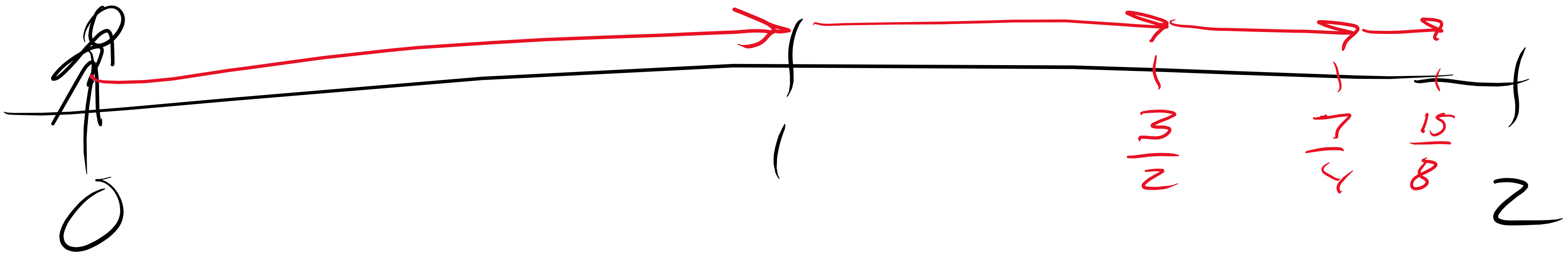
(telescoping sum)

$$S(n) = \frac{1-r^{n+1}}{1-r} = \frac{1}{1-r} - \frac{r^{n+1}}{1-r} \xrightarrow[\text{if } |r|<1]{n\to\infty} \frac{1}{1-r}$$

Finite Geom Series formula

$r = 1/2$

$1 + 1/2 + 1/4 + 1/8 + \cdots = \frac{1}{1-\frac{1}{2}} = 2$



$\int_0^1 \frac{1}{1+x^2}\,dx$ really should do $\int_0^{1-\varepsilon} \frac{1}{1+x^2}\,dx$ send $\varepsilon \to 0$

Consider $1 + r + r^2 + r^3 + \cdots = \frac{1}{1+x^2} = \frac{1}{1-r}$

What is r? $r = -x^2$ need $|r| < 1$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^1 \left(1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots\right) dx$$

$$= \int_0^1 1\,dx - \int_0^1 x^2\,dx + \int_0^1 x^4\,dx - \int_0^1 x^6\,dx + \cdots$$

$$= x\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 + \frac{x^5}{5}\Big|_0^1 - \frac{x^7}{7}\Big|_0^1 + \cdots$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}$$

Gregory-Leibniz formula!

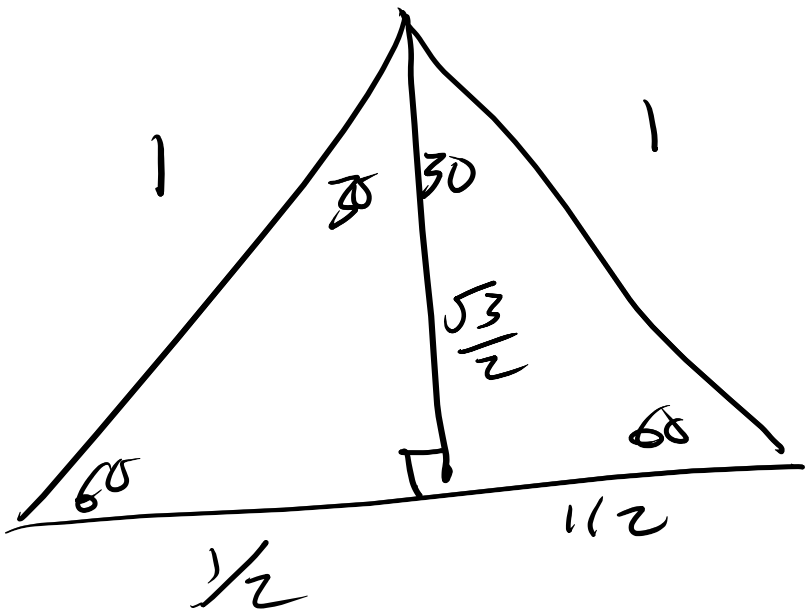
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \approx .785$$

over estimate

under estimate

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} < \frac{\pi}{4} < 1 - \frac{1}{3} + \frac{1}{5}$$

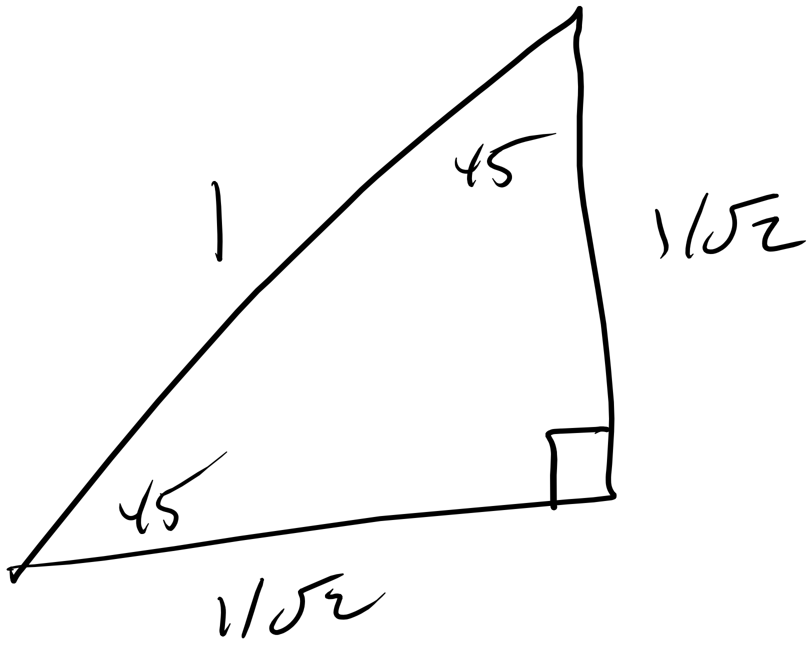
Approximations: replace something complicated with a small number of nice terms sin(x), cos(x), tan(x)



$$\tan(30^\circ) = \frac{1/2}{\sqrt{3}/2} = 1/\sqrt{3} = \frac{\sqrt{3}}{3}$$

$$\cos(30^\circ) = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$$

$$\sin(30^\circ) = \frac{1/2}{1} = \frac{1}{2}$$



$$\cos(45^\circ) = \frac{1/\sqrt{2}}{1} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$= \sin(45^\circ)$$

$$\tan(45^\circ) = 1$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

Take $y = x$

$$\cos(2x) = \cos^2 x - \sin^2 x = \cos^2 x - (1-\cos^2 x) = 2\cos^2 x - 1$$

$$\Longrightarrow \frac{\cos(2x) + 1}{2} = \cos^2 x$$

or $\cos x = \sqrt{\dfrac{\cos(2x) + 1}{2}}$ $\qquad x = \theta/2$

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{\cos\theta + 1}{2}}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$