Math 140: Calculus II: Spring '22 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/

Lecture 13: 3-7-22: https://youtu.be/Vhk_25kBy8U

https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/talks2022/140Sp22_lecture13.pdf

Lecture: The Birthday Problem

Plan for the day: Lecture 12: March 4, 2022:

Topics

Review problems (u-substitution)

Applications of Calculus: Birthday Problem

http://math.oxford.emory.edu/site/math111/uSubstitution/#:~:text=Let%20us%20consider%20a%20few%20examples%20of%20this,that%20d%20x%20%3D%201%203%20d%20u.

Find $\int \sqrt{3x+4}\,dx$

Find $\int t(5+3t^2)^8\,dt$

Find $\int x^2\sqrt{1+x}\,dx$

https://www.math.ucdavis.edu/~kouba/CalcTwoDIRECTORY/usubdirectory/USubstitution.html

$\int \frac{x^2+1}{x^3+3x}\,dx$.

$\int \frac{\sin(\ln x)}{x}\,dx$.

$\int \frac{3}{x\ln x}\,dx$.

$\int (x+3)(x-1)^5\,dx$

$\int \frac{x+5}{2x+3}\,dx$.

$\int \frac{(3+\ln x)^2(2-\ln x)}{4x}\,dx$

$\int_0^9 \sqrt{4-\sqrt{x}}\,dx$.

Birthday Problem

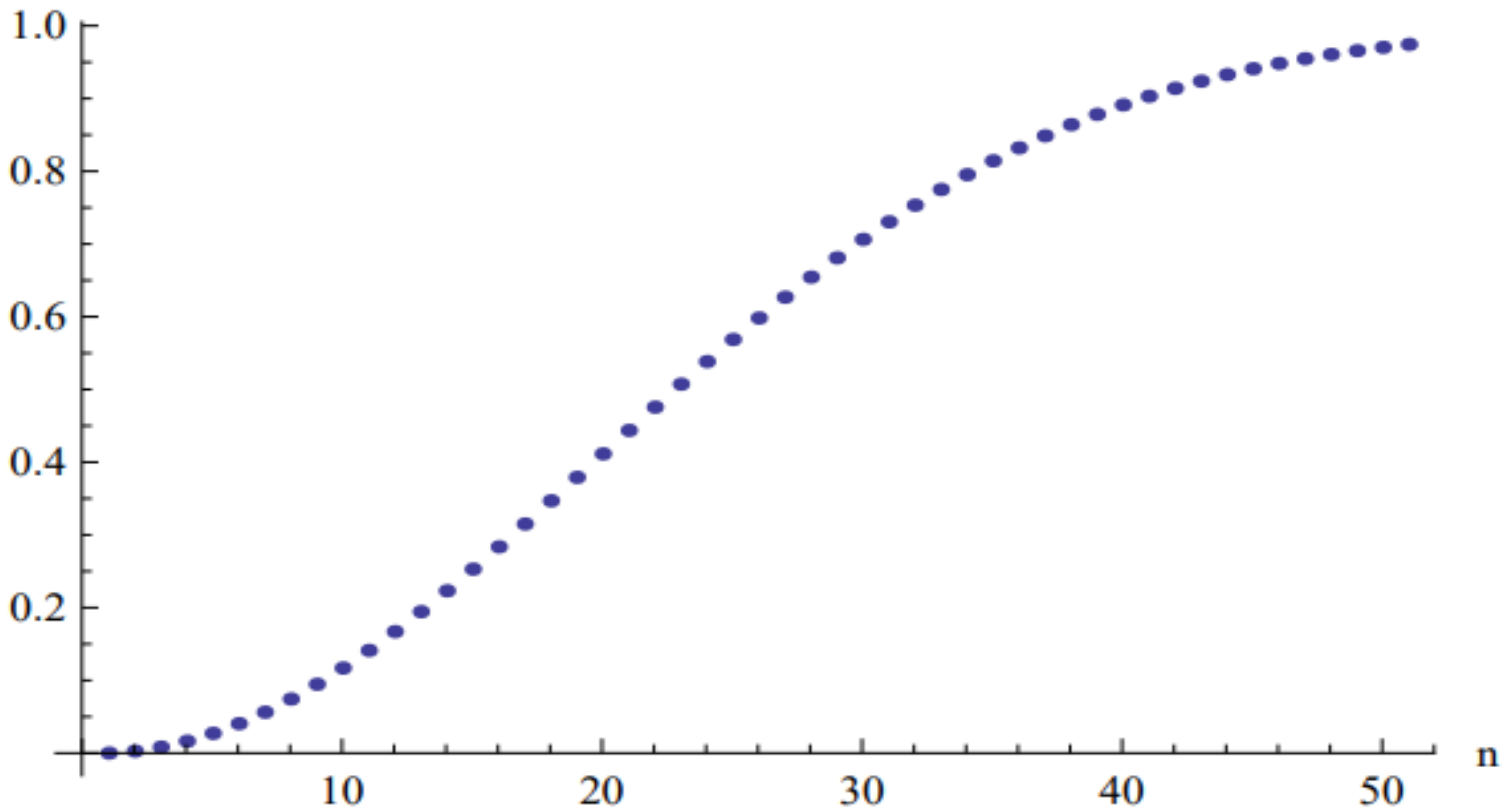
How large must N be for there to be at least a 50% probability that two of the N people share a birthday?

- (A) 11 people
- (B) 22 people
- (C) 33 people
- (D) 44 people
- (E) 90 people
- (F) 180 people
- (G) 365 people
- (H) 500 people.

On page 23, Gladwell substitutes the birthdays for the players names: "It no longer sounds like the championship of Canadian junior hockey. It now sounds like a strange sporting ritual for teenage boys born under the astrological signs Capricorn, Aquarius, and Pisces. *March 11 starts around one side of the Tigers' net, leaving the puck for his teammate January 4, who passes it to January 22, who flips it back to March 12, who shoots point-blank at the Tigers' goalie, April 27. April 27 blocks the shot, but it's rebounded by Vancouver's March 6. He shoots! Medicine Hat defensemen February 9 and February 14 dive to block the puck while January 10 looks on helplessly. March 6 scores!.*"

Birthday Problem

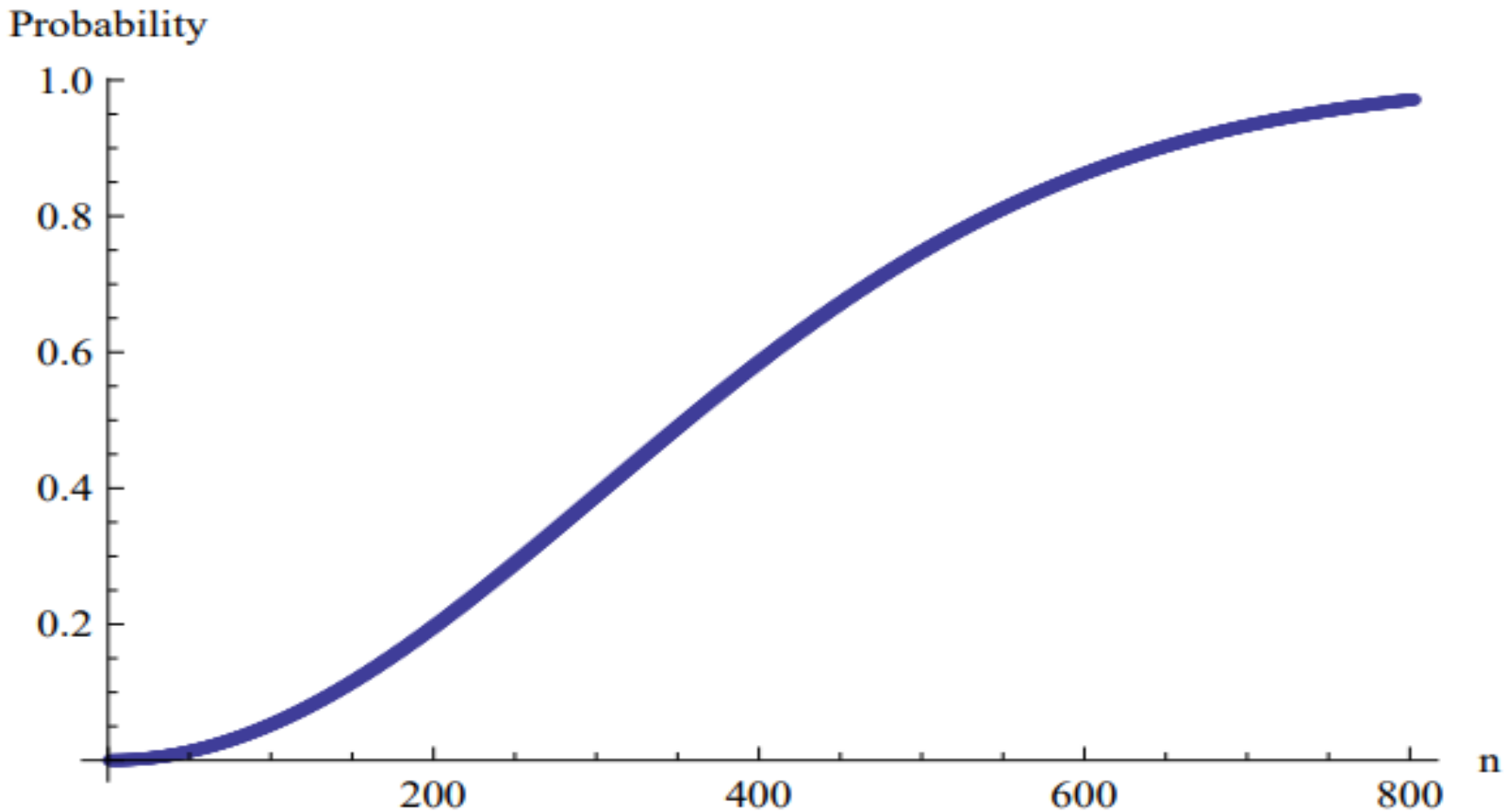
How large must N be for there to be at least a 50% probability that two of the N people share a birthday?



How large must N be for there to be at least a 50% probability that two of N Plutonians share a birthday? 'Recall' one Plutonian year is about 248 Earth years (or 90,520 days).

- (A) 110 people
- (B) 220 people
- (C) 330 people
- (D) 440 people
- (E) 1,000 people
- (F) 5,000 people
- (G) 10,000 people
- (H) 20,000 people
- (I) more than 30,000 people.

How large must N be for there to be at least a 50% probability that two of N Plutonians share a birthday? 'Recall' one Plutonian year is about 248 Earth years (or 90,520 days).



Want at least 2 bdays same, N people, D days per year

Calculate the prob no 2 people of N share a bday

Get that is 50%, Then prob at least 2 share is 50%

$$\left(1\right)\left(\frac{D-1}{D}\right)\cdots\left(\frac{D-(N-1)}{D}\right)$$

$$\left(1-\frac{0}{D}\right)\left(1-\frac{1}{D}\right)\left(1-\frac{N-1}{D}\right) \approx .5$$

Find N st equals $1/2$

$$1/2 \approx \prod_{n=0}^{N-1} \left(1 - \frac{n}{D}\right) = \left(1 - \frac{0}{D}\right)\left(1 - \frac{1}{D}\right) \cdots \left(1 - \frac{N-1}{D}\right)$$

products multiplying: Pavlovian Reaction: LOG

$$\log(1/2) \approx \log \prod_{n=0}^{N-1} \left(1 - \frac{n}{D}\right)$$

$$= \sum_{n=0}^{N-1} \log\left(1 - \frac{n}{D}\right)$$

log(prod) = sum(logs)

If $n \ll D$, $\frac{n}{D} \approx 0$ so $1 - \frac{n}{D} \approx 1$

So $\log\left(1 - \frac{n}{D}\right) \approx 0$

Q: how close?

Taylor Series

$$f(x) = f(0) + f'(0)x + f''(0)x^2/2! + f'''(0)x^3/3! + \cdots$$

$$f(x) \approx f(0) + f'(0)x \leftarrow \text{tangent line approx}$$

Later: MVT to estimate error

$f(x) = \log(1-x)$ $\qquad f'(x) = \frac{1}{1-x}(1-x)' = \frac{-1}{1-x}$

$f(0) = \log(1-0) = 0$ $\qquad f'(0) = -1$

Get $f(x) \approx 0 - 1 \cdot x \approx -x$

So $\log\left(1 - \frac{n}{D}\right) \approx -\frac{n}{D}$ as $x = \frac{n}{D}$

$$\log(1/2) \approx -\sum_{n=0}^{N-1} \frac{n}{D}$$

$$-\log(2) \approx -\frac{1}{D}\sum_{n=0}^{N-1} n$$

$$\sum_{n=0}^{N-1} n \approx D\log 2$$

$$\underbrace{n=0}_{n=N-1}$$

$$\frac{(N-1)N}{2} \approx D\log 2$$

$$(N-1)N \approx 2D\log 2 = D\log 4$$

$S(m) = 0+1+\cdots+m$

$S(n) = n+(n-1)+\cdots+0$

$2S(n) = \underbrace{n+n+\cdots+n}_{n+1 \text{ terms}}$

$2S(n) = n(n+1)$

$S(n) = \frac{n(n+1)}{2}$

$(N-1)N \approx D\log 4$; Find N ...?

Solve $(N-1)N = D\log 4$ N variable

Quadratic: $N^2 - N - D\log 4 = 0$

$aN^2 + bN + c = 0$

$(N-1)N \approx (N-\frac{1}{2})(N-\frac{1}{2}) = N^2 - N + \frac{1}{4}$

vs $N^2 - N$

$(N-\frac{1}{2})^2 \approx D\log 4$

$\rightarrow N - \frac{1}{2} \approx \sqrt{D\log 4} \rightarrow N = D^{1/2}\sqrt{\log 4} + \frac{1}{2}$

$\sum_{n=0}^{N-1} n \longrightarrow$ This is $\frac{(N-1)N}{2} \approx \frac{(N-1/2)^2}{2}$

$\hookrightarrow$ approximated by $\int_0^{N-1} x\,dx = \frac{x^2}{2}\Big|_0^{N-1} = \frac{(N-1)^2}{2}$

$$\sum_{n=0}^{N-1} \log\left(1-\frac{n}{D}\right)$$

$$\approx \int_{x=0}^{N-1} \log\left(1-\frac{x}{D}\right) dx$$

$u = x/D \quad du = \frac{1}{D}dx \quad$ so $dx = D\,du$

$x: 0 \to N-1 \quad u: 0 \to \frac{N-1}{D}$

$$= \frac{1}{D}\int_{u=0}^{\frac{N-1}{D}} \log(1-u)\,du$$

using $\log(1-x) \approx -x$ makes life easier!

Maybe set $t = 1-x$ use next page

Before: $\int \log(x)\,dx = x\log x - x$

$$\int \log x \cdot 1\,dx$$

$u = \log x$ $\qquad$ $dv = 1$

$du = \frac{1}{x}\,dx$ $\qquad$ $v = x$

$$\text{Int} = uv - \int v\,du = x\log x - \int dx$$

$$= x\log x - x$$