Math 140: Calculus II: Spring ‘22 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/

Lecture 15: 3-11-22:
https://youtu.be/Mpe9w4TlH14

https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/talks2022/140Sp22_lecture10.pdf

Plan for the day: Lecture 15: March 11, 2022:

Topics

Partial Fractions (Continued)

u-Substitution

http://math.oxford.emory.edu/site/math111/uSubstitution/#:~:text=Let%20us%20consider%20a%20few%20examples%20of%20this,that%20d%20x%20%3D%201%203%20d%20u.

Find $\int \sqrt{3x+4}\,dx$

Find $\int t(5+3t^2)^8\,dt$

Find $\int x^2\sqrt{1+x}\,dx$

https://www.math.ucdavis.edu/~kouba/CalcTwoDIRECTORY/usubdirectory/USubstitution.html

$\displaystyle\int \frac{x^2+1}{x^3+3x}\,dx$.

$\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$.

$\displaystyle\int \frac{3}{x\ln x}\,dx$.

$\displaystyle\int (x+3)(x-1)^5\,dx$

$\displaystyle\int \frac{x+5}{2x+3}\,dx$.

$\displaystyle\int \frac{(3+\ln x)^2(2-\ln x)}{4x}\,dx$

$\displaystyle\int_0^9 \sqrt{4-\sqrt{x}}\,dx$.

Partial Fractions

$$\int \frac{1}{f(x)^n}\,dx$$ where $f(x)$ is not factorable

More generally: $$\int \frac{g(x)}{f(x)^n}\,dx$$

Simple: $f(x) = x$

$$\int \frac{1}{x^n}\,dx = \int x^{-n}\,dx = \begin{cases} -\dfrac{x^{-n+1}}{n-1} & n \neq 1 \\ \ln(x) & n = 1 \end{cases}$$

Harder

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\int \frac{1}{x+c}\,dx \quad (c = b/a)$$

$u = ax+b$

$\frac{du}{dx} = a \Rightarrow du = a\,dx$

or $dx = \frac{1}{a}\,du$

or int $= \int \frac{1}{u}\frac{1}{a}\,du$

$= \frac{1}{a}\int \frac{du}{u}$

$= \frac{1}{a}\ln(u)$

$= \frac{1}{a}\ln(ax+b)$

Did $f(x) = x^n$, $ax+b$, try $f(x) = ax^2 + c$

Note: $\int \frac{1}{ax^2+c}\,dx = \frac{1}{a}\int \frac{1}{x^2+\left(\frac{c}{a}\right)}\,dx$

let $x = \sqrt{\left|\frac{c}{a}\right|}\,u$ so $x^2 = \left|\frac{c}{a}\right| u^2$

assume $c/a > 0$ $\qquad$ $c/a < 0$

$\int \frac{1}{x^2+A}\,dx$ $\qquad$ $\int \frac{1}{x^2-A}\,dx$ $\quad A > 0$

$x = \sqrt{A}\,u \quad dx = \sqrt{A}\,du \quad x^2 = A u^2$

Get $\frac{\sqrt{A}}{A}\int \frac{1}{u^2+1}\,du$ or $\frac{\sqrt{A}}{A}\int \frac{1}{u^2-1}\,du$

$$\int \frac{1}{1+u^2}\,du = \arctan(u)$$

Card do: $u = \tan\theta$ $\quad du = \sec^2\theta\, d\theta$

$\cos^2\theta + \sin^2\theta = 1$ Divide by $\cos^2\theta$

$1 + \tan^2\theta = \sec^2\theta$

Substitute: $\displaystyle\int \frac{1}{\sec^2\theta}\sec^2\theta\, d\theta$

$$= \int 1\, d\theta = \theta = \arctan(u)$$

since $u = \tan\theta$

$\rightarrow \arctan(u) = \arctan(\tan\theta)$

$= \theta$

Do: $\int \frac{1}{u^2-1}\,du$ Try $u=\sec\theta$

$\sec^2\theta-1=\tan^2\theta \ldots$?

Factor! $u^2-1=(u-1)(u+1)$

$$\frac{1}{u^2-1}=\frac{A}{u-1}+\frac{B}{u+1}=\frac{A(u+1)+B(u-1)}{u^2-1}$$

$$1=\underbrace{(A+B)}_{0}u+\underbrace{(A-B)}_{1}$$ true for all u!

$0 \Rightarrow A+B=0$ or $A=-B$

$1 \Rightarrow A-B=1$ or $A=B+1$

So $-B=B+1 \Rightarrow -1=2B$

So $B=-1/2$ $A=1/2$

$$\int \frac{1}{u^2-1}\,du = \int \frac{1/2}{u-1}\,du + \int \frac{-1/2}{u+1}\,du$$

$$= \frac{1}{2}\ln(u-1) - \frac{1}{2}\ln(u+1)$$

Careful! must make sure $u-1$ and $u+1$ are positive as taking logs...

Consider $f(x) = ax^2+bx+c = a(x-h)^2+k$

Vertex form: vertex is (h,k)

$$\int \frac{dx}{ax^2+bx+c} = \int \frac{dx}{a(x-h)^2+k}$$

$u = x-h \qquad du = dx$

int is $\displaystyle\int \frac{du}{au^2+k}$ just did!

Nice: $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$

$u = f(x) \qquad du = f'(x)\,dx$

integral $= \displaystyle\int \frac{1}{u}\,du = \ln(u)$ worry about sign

$= \ln(f(x))$

Ex: $$\int \frac{2x+3}{x^2+x+1}\,dx = \underbrace{\int \frac{2x+1}{x^2+x+1}\,dx}_{\ln(x^2+x+1)} + \underbrace{\int \frac{2\,dx}{x^2+x+1}}_{\text{help!}}$$

Consider $\int \frac{g(x)}{f(x)}\,dx$

If $\deg(g) \geq \deg(f)$ write $g(x) = q(x)f(x) + r(x)$

q for quotient

r for remainder : $\deg(r) < \deg(f)$

$$\int \frac{q(x)f(x)+r(x)}{f(x)}\,dx = \int q(x)\,dx + \int \frac{r(x)}{f(x)}\,dx$$

Ex: $\frac{x^3+4}{x^2+x}$: $x^3+4 = (x\ldots)(x^2+x)$?

$$\frac{x^3+4}{x^2+x}$$

$$
\begin{array}{r}
x \quad -1 \qquad\qquad\qquad \\
x^2+x \,\big)\, \overline{x^3+0x^2+0x+4} \\
-(x^3+x^2) \qquad\qquad\quad \\
\hline
-x^2+0x \qquad \\
-(-x^2-x) \qquad \\
\hline
x+4
\end{array}
$$

$$(x^3+4) = (x-1)(x^2+x) + (x+4)$$

Partial Fractions: deg(num) < deg(denom)

$$\int \frac{r(x)}{f_1(x)^{d_1} \cdots f_k(x)^{d_k}} = ?$$

numerators are one deg less than base poly below

Ex: $$\int \frac{r(x)}{x^2(x-1)^3(x^2+2)} = ?$$

$$\frac{r(x)}{x^2(x-1)^3(x^2+2)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1} + \frac{d}{(x-1)^2} + \frac{f}{(x-1)^3} + \frac{gx+h}{x^2+2}$$

$(x)^2$ $(x-1)^2$ $(x-1)^3$

$$\frac{a}{x} + \frac{bx+c}{x^2} \quad \text{vs} \quad \frac{A}{x} + \frac{B}{x^2} = \frac{3x+17}{x^2}$$

$$\frac{ax+bx+c}{x^2}$$

$$= \frac{(a+b)x+c}{x^2}$$

$\hookrightarrow$ $a+b=3$

$c=17$

b not needed!

$$\frac{Ax+B}{x^2}$$

$$\frac{Ax+B}{x^2}$$

$A=3$

$B=17$

Goal is

$$\frac{r(x)}{x^2}$$

$r(x)$ is deg 1