Math 140: Calculus II: Spring '22 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/

Lecture 16: 3-14-22: https://youtu.be/Kw3K6EElo8Y

https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/talks2022/140Sp22_lecture16.pdf

Lecture: More partial fractions, u-substitution

Plan for the day: Lecture 16: March 24, 2022:

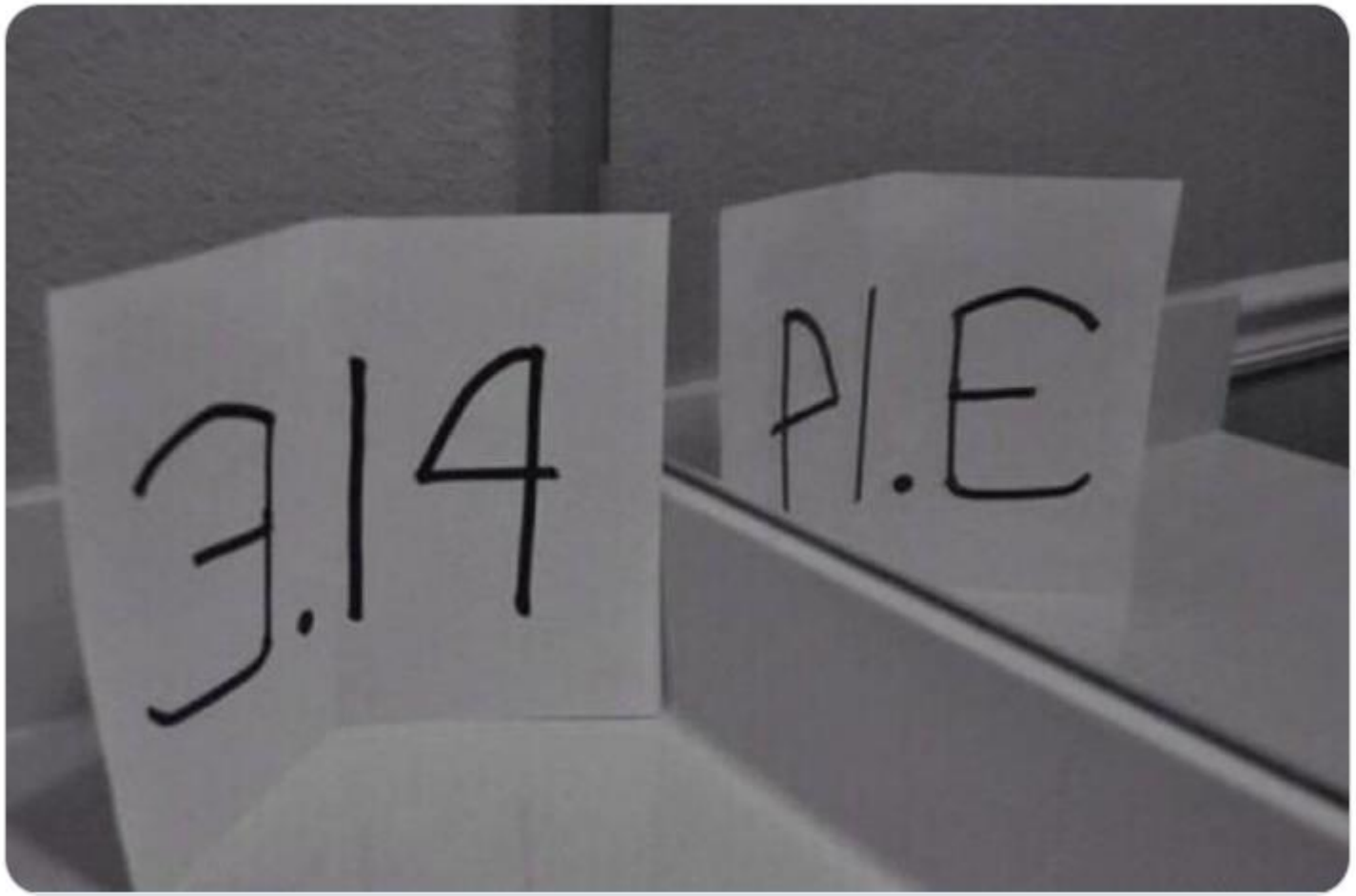
Topics

Review:

partial fractions

u-substitution

The secret of π revealed #PiDay



http://math.oxford.emory.edu/site/math111/uSubstitution/#:~:text=Let%20us%20consider%20a%20few%20examples%20of%20this,that%20d%20x%20%3D%201%203%20d%20u.

Find $\int \sqrt{3x+4}\,dx$ Find $\int t(5+3t^2)^8\,dt$ Find $\int x^2\sqrt{1+x}\,dx$

https://www.math.ucdavis.edu/~kouba/CalcTwoDIRECTORY/usubdirectory/USubstitution.html

$\int \frac{x^2+1}{x^3+3x}\,dx$. $\int \frac{\sin(\ln x)}{x}\,dx$. $\int \frac{3}{x\ln x}\,dx$. $\int (x+3)(x-1)^5\,dx$

$\int \frac{x+5}{2x+3}\,dx$. $\int \frac{(3+\ln x)^2(2-\ln x)}{4x}\,dx$ $\int_0^9 \sqrt{4-\sqrt{x}}\,dx$.

Partial Fractions

(1) remove multiples of denom from numerator until top has smaller deg than bottom

$$\frac{g(x)}{f(x)} \quad \text{with } \deg(g) < \deg(f)$$

(2) factor denominator: $f_1(x)^{r_1} \cdots f_k(x)^{r_k}$

(3) $$\frac{g(x)}{f(x)} = \sum_{\ell=1}^{k} \left(\sum_{m=1}^{r_\ell} \frac{\text{poly of deg 1 less than } f_\ell}{f_\ell(x)^m} \right)$$

base poly

$$\frac{1}{x^n} = \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \cdots + \frac{a_n}{x^n}$$

all base polynomials are x, and of deg 1, so numerators are constant

$$= \frac{a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n}{x^n}$$

numerator is an arbitrary poly of deg $n-1$, one less than denominator

Reduce to $\frac{1}{x}$ or $\frac{1}{x^m}$, $\frac{1}{x^2+1}$, $\frac{1}{1-x^2}$, $\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$

$\llcorner\ x^{-m}$

$$\int x^{-n}\,dx = \frac{x^{-n+1}}{-n+1} \quad \text{if } n \neq 1$$

$$\int x^{-1}\,dx = \ln(x)$$

$$\int \frac{1}{x^2+1}\,dx = \arctan(x)$$

$$\int \frac{3x^4+x-2}{x^2(x+1)}\,dx = \int \frac{3x^4+x-2}{x^3+x^2}\,dx$$

$$x^3+x^2 \overline{\big)\, 3x^4+0x^3+0x^2+x-2}$$

$$3x^4+x-2 = (3x^4+3x^3)-3x^3+x-2$$

$$= \underbrace{\int \frac{3x(x^3+x^2)}{x^3+x^2}\,dx}_{\frac{3}{2}x^2} + \int \frac{-3x^3+x-2}{x^3+x^2}\,dx$$

$$-3x^3+x-2 = -3x^3-3x^2+3x^2+x-2$$

$$\int \frac{3x^2+x-2}{x^2(x+1)}\,dx = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+1}$$

$$\frac{3x^2+x-2}{x^2(x+1)} = \frac{ax(x+1) + b(x+1) + cx^2}{x^2(x+1)}$$

$$3x^2+x-2 = ax^2 + ax + bx + b + cx^2$$

$$3x^2+x-2 = \underbrace{(a+c)}_{3}x^2 + \underbrace{(a+b)}_{1}x + \underset{-2}{b}$$

(must be equal for all x...)

$b=-2$ Then $a+b=1 \Rightarrow a-2=1 \Rightarrow a=3$ Then $a+c=3 \Rightarrow 3+c=3$ so $c=0$

$$\int \frac{3}{x}\,dx - \int \frac{2}{x^2}\,dx$$

$$= 3\int \frac{1}{x}\,dx - 2\int x^{-2}\,dx$$

$$= 3\ln(x) + 2x^{-1} + C$$

$$\int \underbrace{\frac{f(x)}{x^3(x^2+1)(x^3+2)^2}} \Rightarrow \frac{\alpha}{x} + \frac{\beta}{x^2} + \frac{\gamma}{x^3} + \frac{a_1 x + a_0}{x^2+1} + \frac{b_2 x^2 + b_1 x + b_0}{x^3+2} + \frac{c_2 x^2 + c_1 x + c_0}{(x^3+2)^2}$$

poly has deg 11

↳ 12 coefficients!

↳ deg(f) < 11

so f has 11 coefficients

11 coefficients

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$$\int \frac{(3+\ln x)^2(2-\ln x)}{4x}\,dx$$

Possible u's: $3+\ln x$, $2-\ln x$, $\ln x$, x, $\frac{1}{x}$

Could do $3+\ln x = -(2-\ln x)+5$

$$\frac{d(\ln x)}{dx} = \frac{1}{x} \quad \text{or} \quad d(\ln x) = \frac{dx}{x}$$

$$\int (\ln x)^{2022}\,\frac{1}{x}\,dx = \frac{(\ln x)^{2023}}{2023}$$

$$\begin{pmatrix} u = \ln x \\ du = \frac{1}{x}dx \end{pmatrix}$$

$$\int \frac{(3+\ln x)^2(2-\ln x)}{4x}\,dx$$

Try $u = \ln(x)$ so $\frac{du}{dx} = \frac{1}{x}$ or $du = \frac{1}{x}\,dx$

Integral is $\displaystyle\int \frac{(3+u)^2(2-u)}{4}\,du$

$$= \frac{1}{4}\int (9+6u+u^2)(2-u)\,du$$

$$= \frac{1}{4}\int \left[18 + \begin{matrix}12u + 2u^2 \\ -9u - 6u^2\end{matrix} - u^3\right] du$$

$$= \frac{1}{4}\int \left[18 - 3u - 4u^2 - u^3\right] du = \frac{1}{4}\left[18u - \frac{3}{2}u^2 - \frac{4}{3}u^3 - \frac{u^4}{4}\right]$$

now write $\ln(x)$ for u

GRE Practice #9: The following is Problem #14 from https://www.ets.org/s/gre/pdf/practice_book_math.pdf: Suppose g is a continuous real-valued function such that

$$3x^5 + 96 = \int_c^x g(t)dt = G(x) - G(c)$$

for each $x \in \mathbb{R}$, where c is a constant. What is the value of c?

(a) -96 (b) -2 (c) 4 (d) 15 (e) 32.

$$3c^5 + 96 = 0 \implies c^5 = -\frac{96}{3} = -32$$

$$c = -2$$