Math 140: Calculus II: Spring ‘22 (Williams)

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Homepage:

https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/

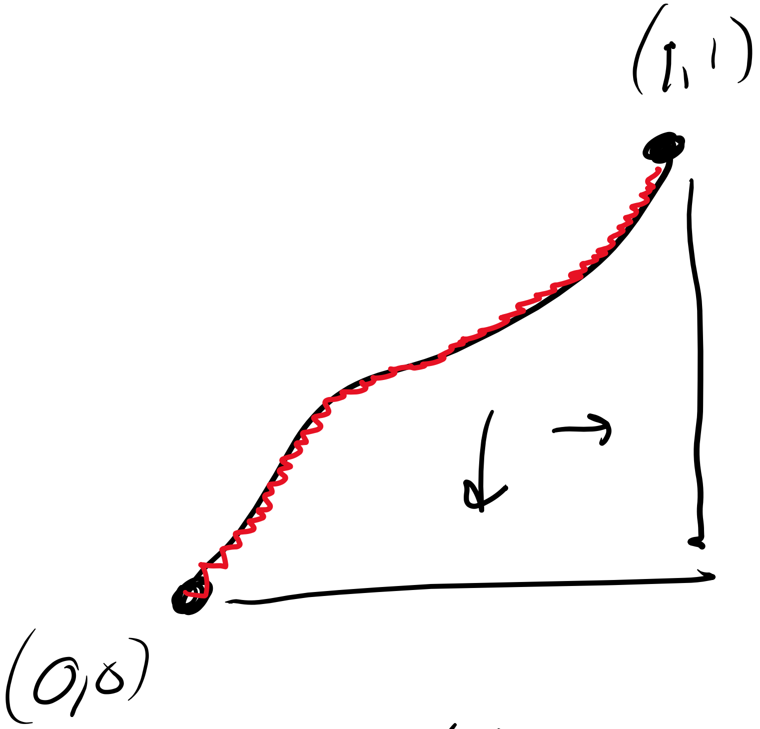
Lecture 17: 3-16-22: https://youtu.be/weksElaciXQ

https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/talks2022/140Sp22_lecture17.pdf

Plan for the day: Lecture 17: March 16, 2022:

Topics

Length of curves

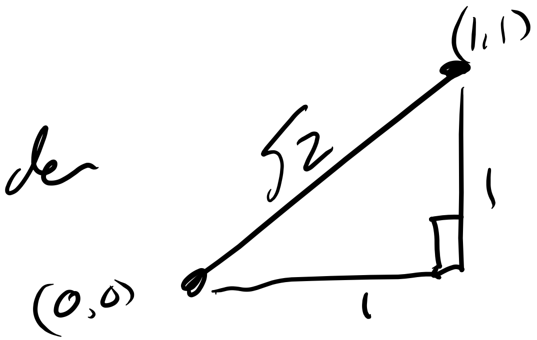


as sizes go to zero, polygonal approx converges to curve
length converges to length of curve

sum(horiz) = 1, sum(vertical) = 1
$\Rightarrow$ curve is of length 2

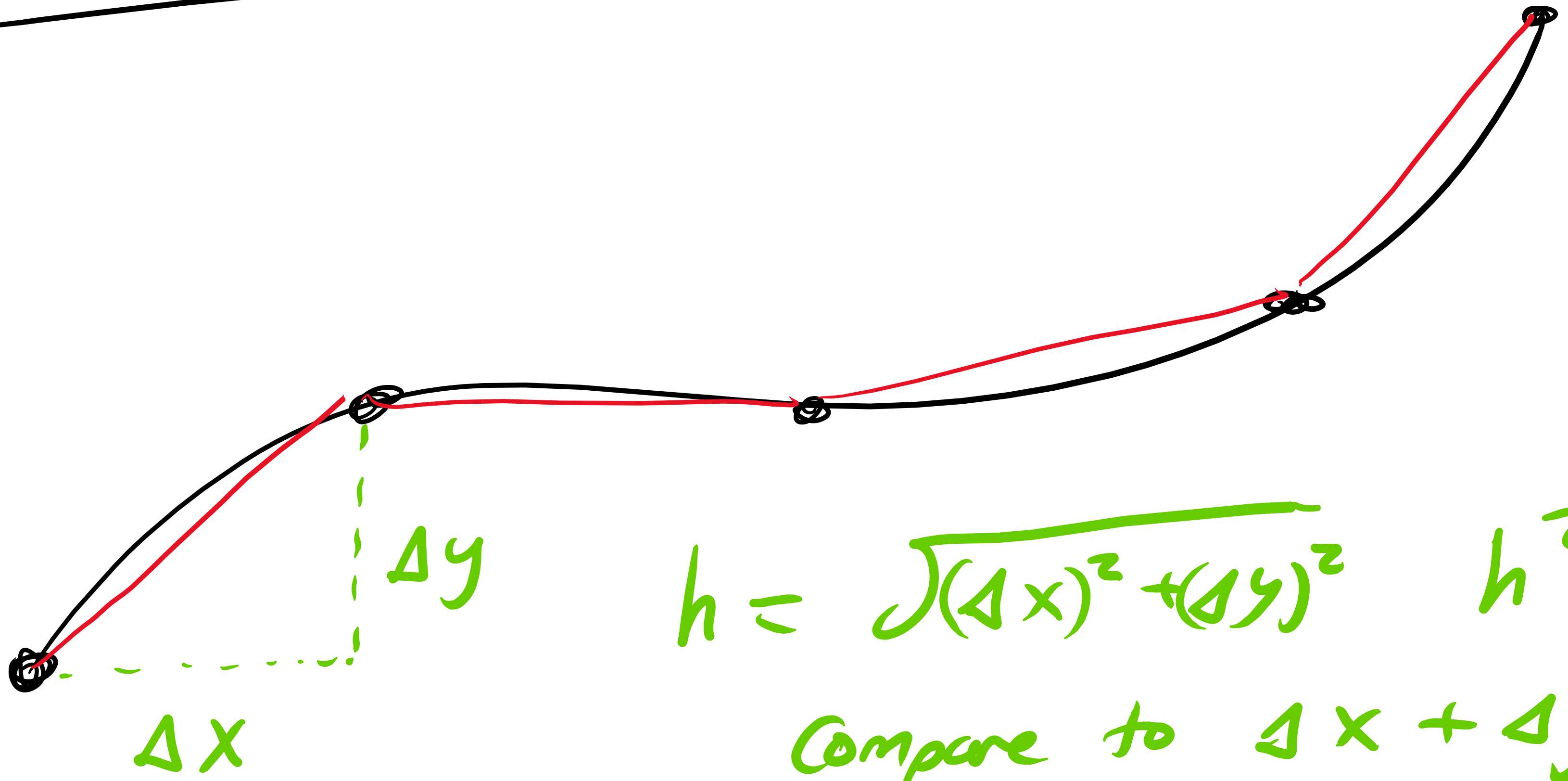
small errors accumulate

Consider



length is $\sqrt{2}$ by Pythagoras

Lengths of Curves

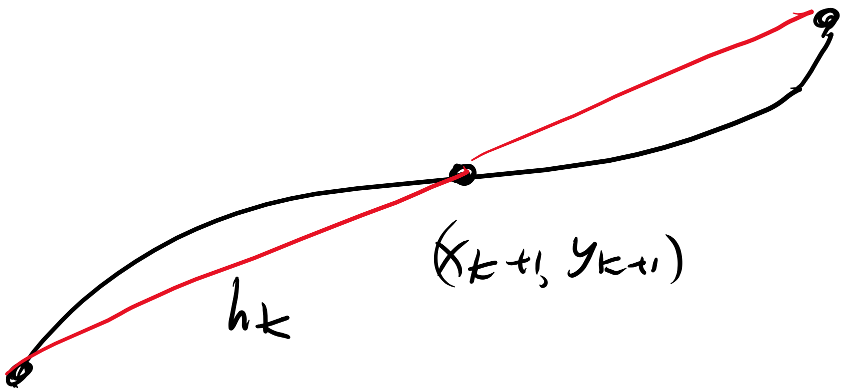


Using hypotenuses not sides

$h = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ $\quad h^2 = (\Delta x)^2 + (\Delta y)^2$

Compare to $\Delta x + \Delta y$ from before

$(\text{before})^2 = (\Delta x)^2 + (\Delta y)^2 + \underline{\underline{2\,\Delta x\,\Delta y}}$



$$(x(t_k), y(t_k))$$

$$h_k = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}$$

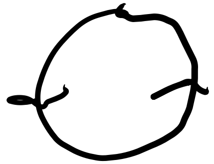
$$\text{Length}(n) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}$$

$$c : [0,1] \to \mathbb{R}^2$$

$$c(t) = (x(t), y(t))$$

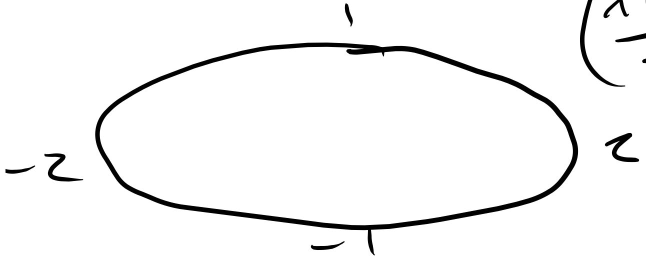
parametrization of a curve

$$c(t) = (\cos(t), \sin(t))$$



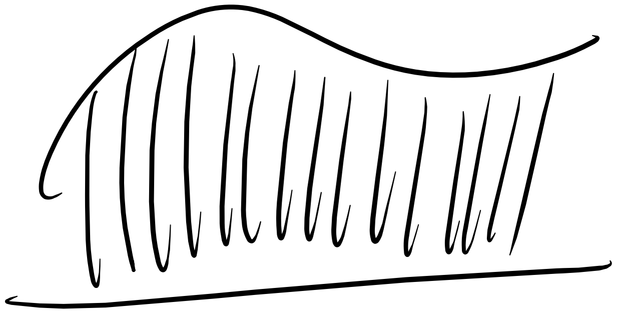
$$c(t) = (2\cos t, \sin t)$$

$$\left(\frac{x(t)}{2}\right)^2 + y(t)^2 = 1$$



$$\text{Length}(n) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1}-x_k)^2 + (y_{k+1}-y_k)^2} \cdot 1$$

need $\frac{1}{n}$. $\quad \dfrac{x_{k+1}-x_k}{(x_{k+1}-x_k)} = 1 = \dfrac{x_{k+1}-x_k}{\sqrt{(x_{k+1}-x_k)^2}}$



$$U(n) = \sum_{k=0}^{n-1} f(u_k)\,\frac{1}{n}$$

$$\text{Length}(n) = \sum_{k=0}^{n-1} \frac{\sqrt{(x_{k+1}-x_k)^2 + (y_{k+1}-y_k)^2}}{\sqrt{(x_{k+1}-x_k)^2}} \cdot (x_{k+1}-x_k) \qquad \frac{1}{n} = x_{k+1}-x_k$$

If $a, b > 0$ then $\sqrt{a}/\sqrt{b}$ is $\sqrt{a/b}$

$$\text{Length}(n) = \sum_{k=0}^{n-1} \sqrt{1 + \frac{y_{k+1}-y_k}{x_{k+1}-x_k}} \cdot \Delta x_k \xrightarrow[n\to\infty]{} \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$y'(x_k) = dy/dx$

$$\text{Length}(n) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}$$

$$\text{Length}(n) = \sum_{k=0}^{n-1} \sqrt{\left(x(t_{k+1}) - x(t_k)\right)^2 + \left(y(t_{k+1}) - y(t_k)\right)^2} \quad \frac{t_{k+1} - t_k}{\sqrt{(t_{k+1} - t_k)^2}}$$

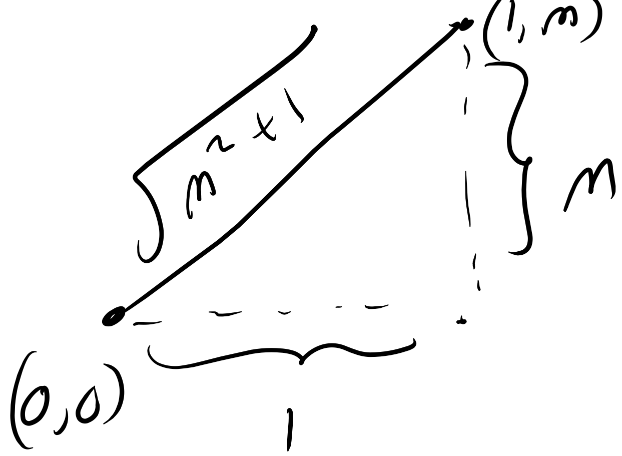
$$\text{Length}(n) = \sum_{k=0}^{n-1} \sqrt{\left(\frac{x(t_{k+1}) - x(t_k)}{t_{k+1} - t_k}\right)^2 + \left(\frac{y(t_{k+1}) - y(t_k)}{t_{k+1} - t_k}\right)^2} \cdot (t_{k+1} - t_k)$$

$$\text{Length} = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2} \, dt$$

if $(x(t), y(t)) = c(t)$

otherwise use $\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$

$y = mx + \cancel{b}$ $\quad x: 0 \to 1$ $\quad$ Length is



$$\frac{dy}{dx} = m$$

$$\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\int_0^1 \sqrt{1+m^2}\, dx = \sqrt{1+m^2} \int_0^1 dx$$

$$= \sqrt{1+m^2} \cdot 1$$

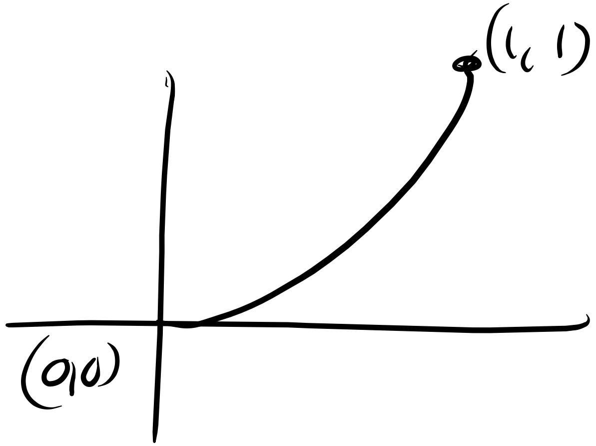
$$y(x) = x^2$$

$$\frac{dy}{dx} = 2x$$

$$\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + 4x^2}\, dx$$

$$= \frac{1}{2} \int_{u=0}^{2} \sqrt{1 + u^2}\, du$$



$u = 2x$

$du = 2\,dx$ or $dx = \frac{1}{2} du$

As $x: 0 \to 1$, $u: 0 \to 2$

Need $\int_0^2 \sqrt{1+u^2}\,du$

$u = \tan\theta$ $\qquad \frac{du}{d\theta} = \sec^2\theta \rightarrow du = \sec^2\theta\, d\theta$

$u : 0 \rightarrow 2, \quad \theta : 0, \arctan(2)$

Integral is $\displaystyle\int_{\theta=0}^{\arctan(2)} \sqrt{1+\tan^2\theta} \cdot \sec^2\theta\, d\theta$

$$= \int_{\theta=0}^{\arctan(2)} \sec^3\theta\, d\theta = \int_{\theta=0}^{\arctan(2)} \frac{1}{\cos^3\theta}\, d\theta$$

HELP!

Circle! $x^2+y^2=1$ or $y=\sqrt{1-x^2} = y(x)$

$$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$$ ARG

$$c(t) = (x(t), y(t)) = (\cos t, \sin t)$$

$$c'(t) = (x'(t), y'(t)) = (-\sin t, \cos t)$$

so $x'(t)^2 + y'(t)^2 = \sin^2 t + \cos^2 t = 1$

$$\text{length} = \int_{t=0}^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_{t=0}^{2\pi} \sqrt{1}\, dt = 2\pi$$

Circle: $x^2+y^2=1$ or $y=\sqrt{1-x^2} = y(x)$

$$\frac{dy}{dx} = \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x) \qquad \text{ARG}$$

$$4\int_{x=0}^{1} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 4\int_{x=0}^{1} \sqrt{1+\frac{x^2}{(1-x^2)}}\,dx$$

$$= 4\int_{x=0}^{1} \sqrt{\frac{1}{1-x^2}}\,dx$$

$$x = \sin(t)$$

$$= 4\int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$$

Simpson's rule

From Wikipedia, the free encyclopedia

For Simpson's voting rule, see Minimax Condorcet.

In numerical integration, **Simpson's rules** are several approximations for definite integrals, named after Thomas Simpson (1710–1761).

The most basic of these rules, called **Simpson's 1/3 rule**, or just **Simpson's rule**, reads

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

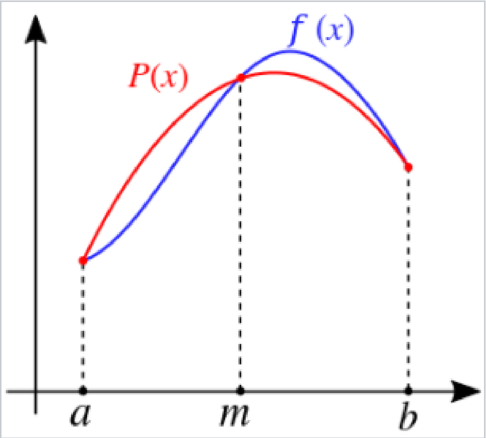
In German and some other languages, it is named after Johannes Kepler, who derived it in 1615 after seeing it used for wine barrels (barrel rule, *Keplersche Fassregel*). The approximate equality in the rule becomes exact if *f* is a polynomial up to 3rd degree.

If the 1/3 rule is applied to *n* equal subdivisions of the integration range [*a*, *b*], one obtains the **composite Simpson's rule**. Points inside the integration range are given alternating weights 4/3 and 2/3.

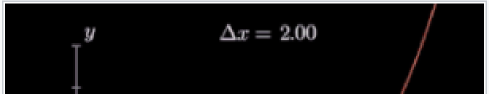
Simpson's 3/8 rule, also called **Simpson's second rule**, requires one more function evaluation inside the integration range and gives lower error bounds, but does not improve on order of the error.

Simpson's 1/3 and 3/8 rules are two special cases of closed Newton–Cotes formulas.

In naval architecture and ship stability estimation, there also exists **Simpson's third rule**, which has no special importance in general numerical analysis, see Simpson's rules (ship stability).



Simpson's rule can be derived by approximating the integrand *f* (*x*) (in blue) by the quadratic interpolant *P*(*x*) (in red).



https://en.wikipedia.org/wiki/Simpson%27s_rule

Trapezoidal Rule Formula

https://byjus.com/maths/trapezoidal-rule/

Let f(x) be a continuous function on the interval [a, b]. Now divide the intervals [a, b] into n equal subintervals with each of width,

Δx = (b-a)/n, Such that $a = x_0 < x_1 < x_2 < x_3 < \ldots.. < x_n = b$

Then the Trapezoidal Rule formula for area approximating the definite integral $\int_a^b f(x)dx$ is given by:

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots .2f(x_{n-1}) + f(x_n)]$$

Where, $x_i = a + i\Delta x$

If $n \to \infty$, R.H.S of the expression approaches the definite integral $\int_a^b f(x)dx$

Trapezoidal Rule Definition