Math 140: Calculus II: Spring ‘22 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/

Lecture 19: 4-4-22: https://youtu.be/SuJ7ly2RVQE

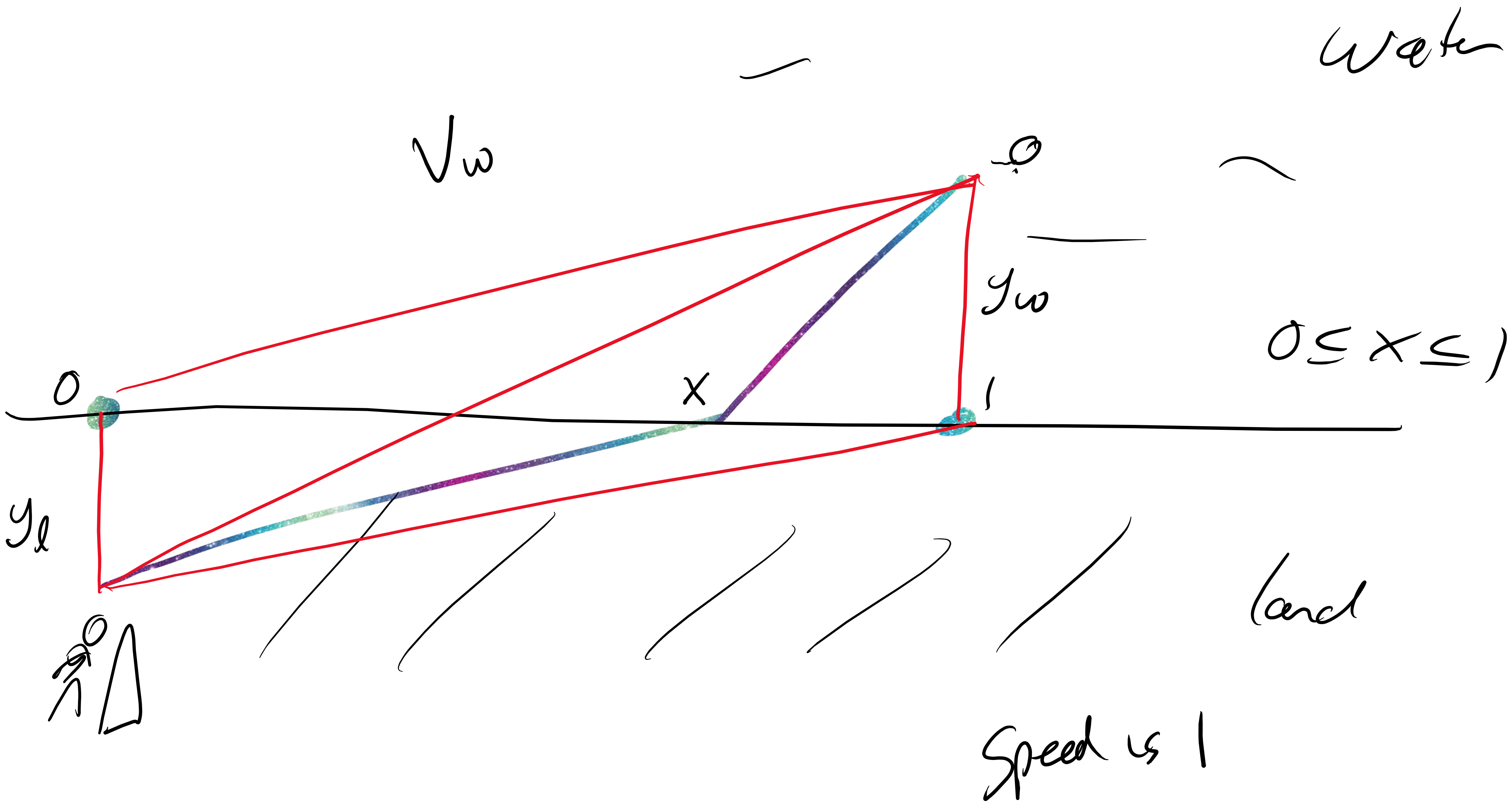
https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/talks2022/140Sp22_lecture19.pdf

Plan for the day: Lecture 19: April 4, 2022:

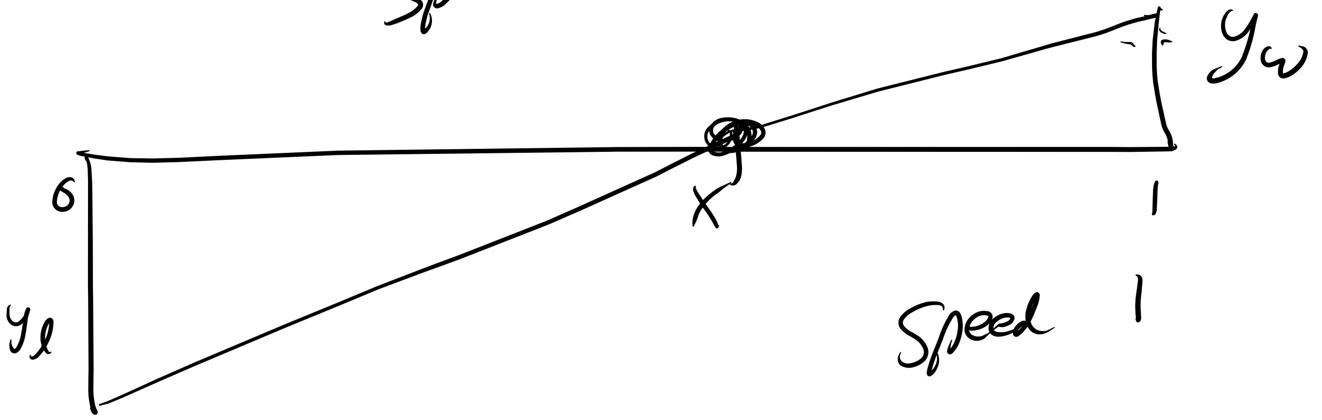
Topics

Review Day

Sequences and Series: Integral Test, Ratio Test


water
V_w
y_w
$0 \le x \le 1$
0
x
1
y_l
land
Speed is 1

Speed $v_w < 1$



land dist: $\sqrt{x^2+y_l^2}$

land time: $\dfrac{\sqrt{x^2+y_l^2}}{1}$

waterdist: $\sqrt{(1-x)^2+y_w^2}$

water time $\dfrac{\sqrt{(1-x)^2+y_w^2}}{v_w}$

minimize $T(x) = \sqrt{x^2+y_l^2} + \left(\sqrt{(1-x)^2+y_w^2}\right)\frac{1}{v_w}$

subject to $0 \le x \le 1$

$x=0$ bad, $x=1$ in the "running"

minimize $T(x) = \sqrt{x^2 + y_l^2} + \left(\sqrt{(1-x)^2 + y_w^2}\right)\frac{1}{v_w}$

subject to $0 \le x \le 1$

$$T'(x) = \frac{1}{2} \frac{2x}{(x^2+y_l^2)^{1/2}} + \frac{1}{v_w} \frac{1}{2} \frac{2(1-x)(-1)}{((1-x)^2 + y_w^2)^{1/2}}$$

$T'(x) = 0$ means $\dfrac{x}{(x^2+y_l^2)^{1/2}} = \dfrac{1-x}{v_w((1-x)^2+y_w^2)^{1/2}}$

$$x^2 v_w^2 \left[(1-x)^2 + y_w^2\right] = (1-x)^2 (x^2 + y_l^2)$$

Quartic

Aside

$x = 1$

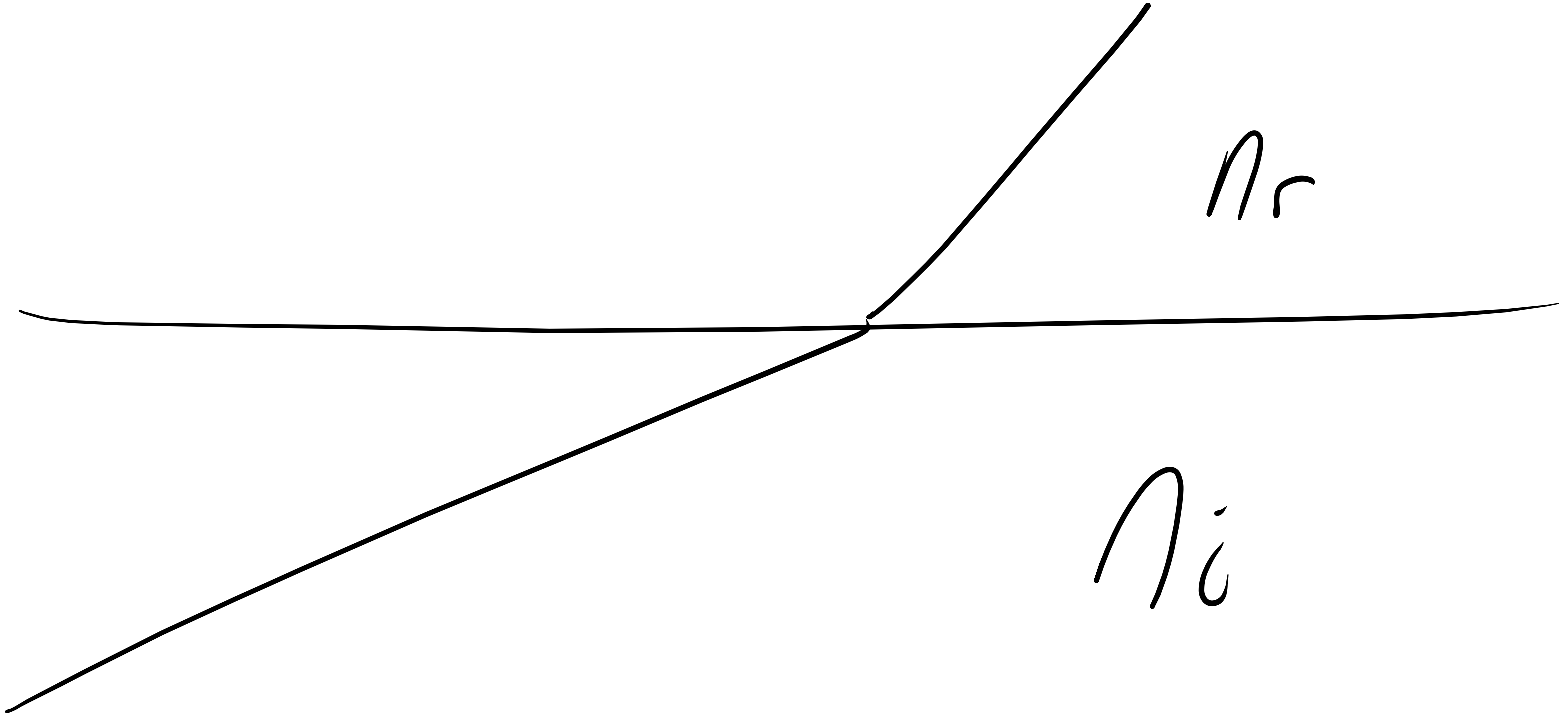
$x^2 = 1$

$x^2 - 1 = 0$

$(x-1)(x+1) = 0$

$x = 1$ or -1

Can have phantom roots from squaring

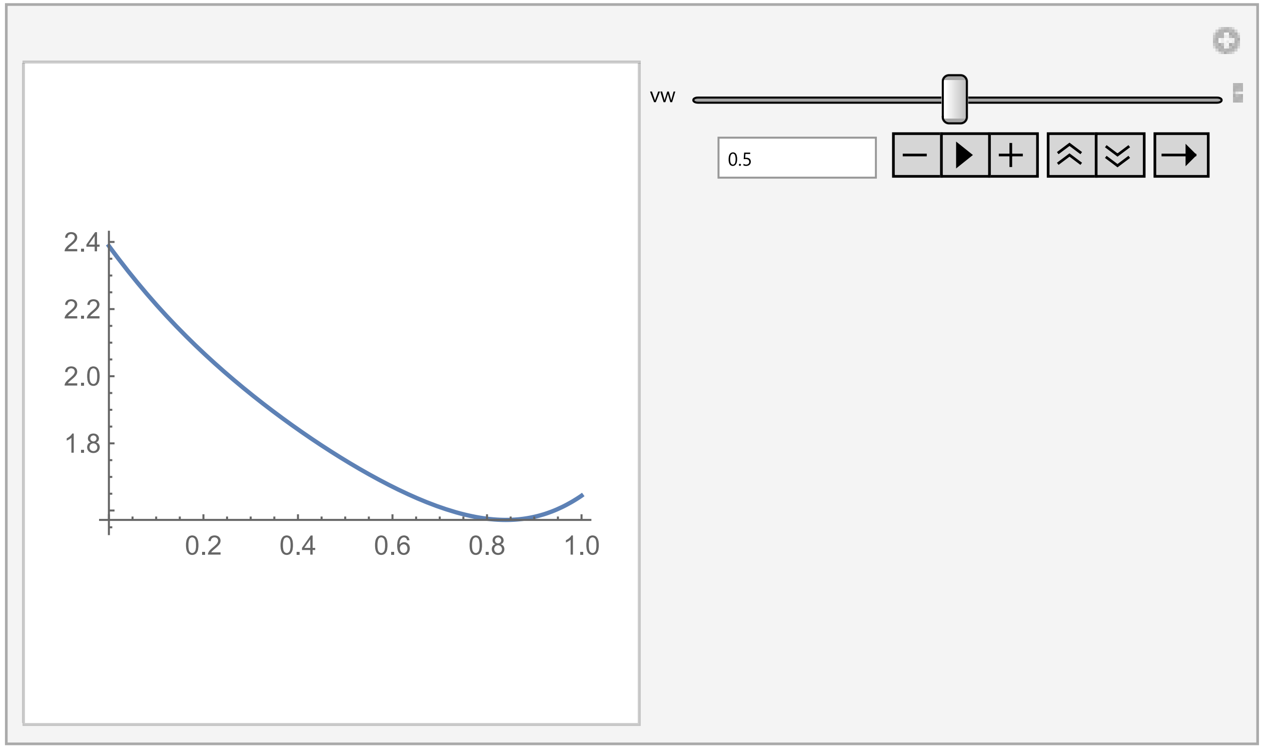


Snell's Law

T[x_, yell_, yw_, vw_]:
= Sqrt[x^2 + yell^2] + Sqrt[(1 − x)^2 + yw^2]/vw

In[•]:=
```
Manipulate[Plot[T[x, .3, .3, vw], {x, 0, 1}], {vw, .01, 1}]
```

Out[•]=



```
Tprime[x_, yell_,yw_,vw_] := x^2 vw^2 ( (1-x)^2 + yw^2) - (1-
x)^2 (x^2 + yell^2);
NSolve[Tprime[x,.3,.3,.5] == 0, x]
{{x->-0.00295643-0.350035 I},{x->-0.00295643+0.350035 I},{x-
>0.83988},{x->1.16603}}
```

$$e^x = 1 + x + x^2/2! + x^3/3! + \cdots$$

$$= \sum_{n=0}^{\infty} x^n/n!$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \frac{e^{ix} - e^{-ix}}{2i}$$

$$e^{ix} = \cos x + i \sin x$$

Geometric: $a_n = a_0 r^n$

$$a_0 + a_0 r + a_0 r^2 + a_0 r^3 + \cdots = a_0 \frac{1}{1-r} \quad \text{if } |r|<1$$

$$a_0 r^5 + a_0 r^6 + a_0 r^7 + \cdots$$

$$= a_0 r^5 \underbrace{\left(1 + r + r^2 + \cdots\right)}_{\text{Geometric Series}}$$

Necessary for $\sum_{n=0}^{\infty} a_n$ to converge:

$a_n \to 0$ (not sufficient!)

Ex: $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \cdots$

Ratio Test

Let $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

If ρ exists and ρ is $\begin{cases} < 1 & \text{converges} \\ > 1 & \text{diverges} \\ = 1 & \text{no information} \end{cases}$

Ex: $a_n = 1/n$ Harmonic Series

$$\underbrace{1}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{2}{4} = \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{4}{8} = \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{\geq \frac{8}{16} = \frac{1}{2}} + \cdots$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

$\rho = 1$ but diverges

Try: $\sum_{n=1}^{\infty} 1/n^2$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1$$

$\rho = 1$ but can show this converges

$$\sum 1/n^2 = \sum_{k=1}^{\infty} \underbrace{\left(\frac{1}{(2^k)^2} + \frac{1}{(2^{k+1})^2} + \cdots \frac{1}{(2^{k+1})^2} \right)}_{\text{largest term is first}}$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{2^{2k}} 2^k$$

$$= \sum_{k=1}^{\infty} \left(\frac{2^k}{2^{2k}} \right) = \sum_{k=1}^{\infty} \left(\frac{1}{2^k} \right)$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{2} \right)^k$$

Geometric, ratio ½! Converges