Math 140: Calculus II: Spring '22 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/

Lecture 23: 4-13-22: https://youtu.be/5J9g8_9gbmQ

https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/talks2022/140Sp22_lecture23.pdf

Plan for the day: Lecture 23: April 13, 2022:

Topics

•Taylor Series Introduction, Trig Identities:
https://youtu.be/MqAV2OhPoD0 slides: pdf

•Taylor Series II: https://youtu.be/yc_ihgQw7GI slides: pdf

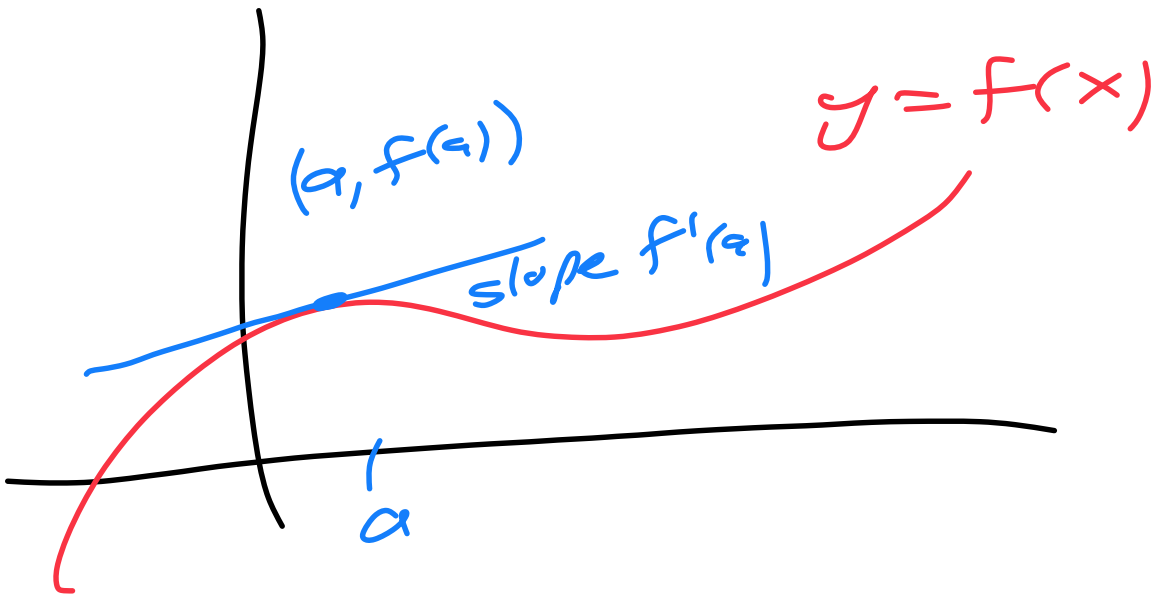
Taylor Series

Tangent Lines

Point-Slope:

$$y - f(a) = f'(a)(x-a)$$

$$y = \underbrace{f(a)}_{\text{start}} + \underbrace{f'(a)}_{\text{initial speed}}\underbrace{(x-a)}_{\text{elapsed time}}$$



Taylor Series

will do at $a=0$ and have x^n, more generally $(x-a)^n$

$$(T_N f)(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

agrees w. $f(x)$ for the first N derivs at $x=0$

$$\cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$$(T_N f)(0) = 0$$

$$(T_N f)'(x) = f'(0) + \frac{f''(0)\cdot 2}{2!}x + \cdots + \frac{f^{(n)}(0)\cdot n}{n!}x^{n-1}$$

$$(T_N f)'(0) = f'(0)$$

$$(T_N f)''(x) = \frac{f''(0)\cdot 2\cdot 1}{2!} + \cdots + \frac{f^{(n)}(0)\cdot n(n-1)}{n!}x^{n-2}$$

$$(T_N f)''(0) = f''(0)$$

Taylor Series

Goal is to see how well Taylor Series approximate functions,

how listtle later terms change approximation ....

For definiteness, will do Cos[x]

```
In[1]:= coeff[x0_, n_] := If[Mod[n, 4] == 0, Cos[x0],
   If[Mod[n, 4] == 1, - Sin[x0],
    If[Mod[n, 4] == 2, - Cos[x0], Sin[x0]]
   ]];
approx[x_, x0_, n_] := Sum[coeff[x0, nn] (x - x0) ^ nn / nn!, {nn, 0, n}]

Manipulate[Plot[{Cos[x], approx[x, x0, n]}, {x, x0 - 2 Pi c, x0 + 2 Pi c},

     Epilog → {PointSize[.025], Point[{x0, Cos[x0]}]}], {x0, 0, 20 Pi / 2}, {n, 1, 40},

  {c, 4, .01}]
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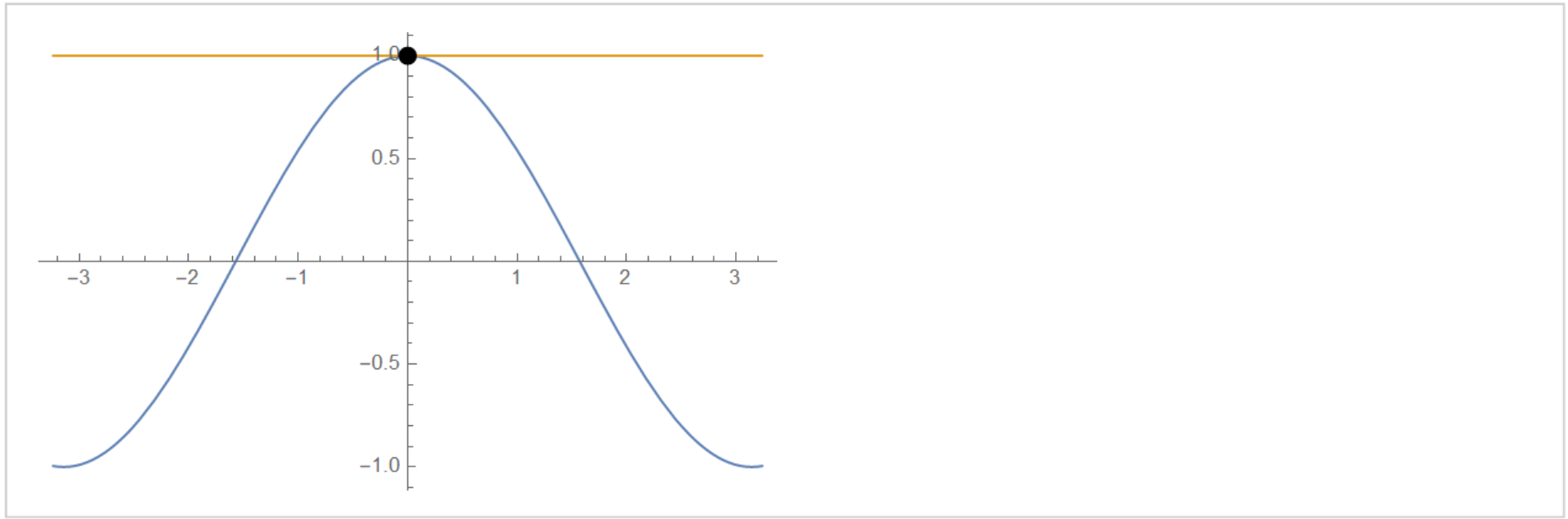


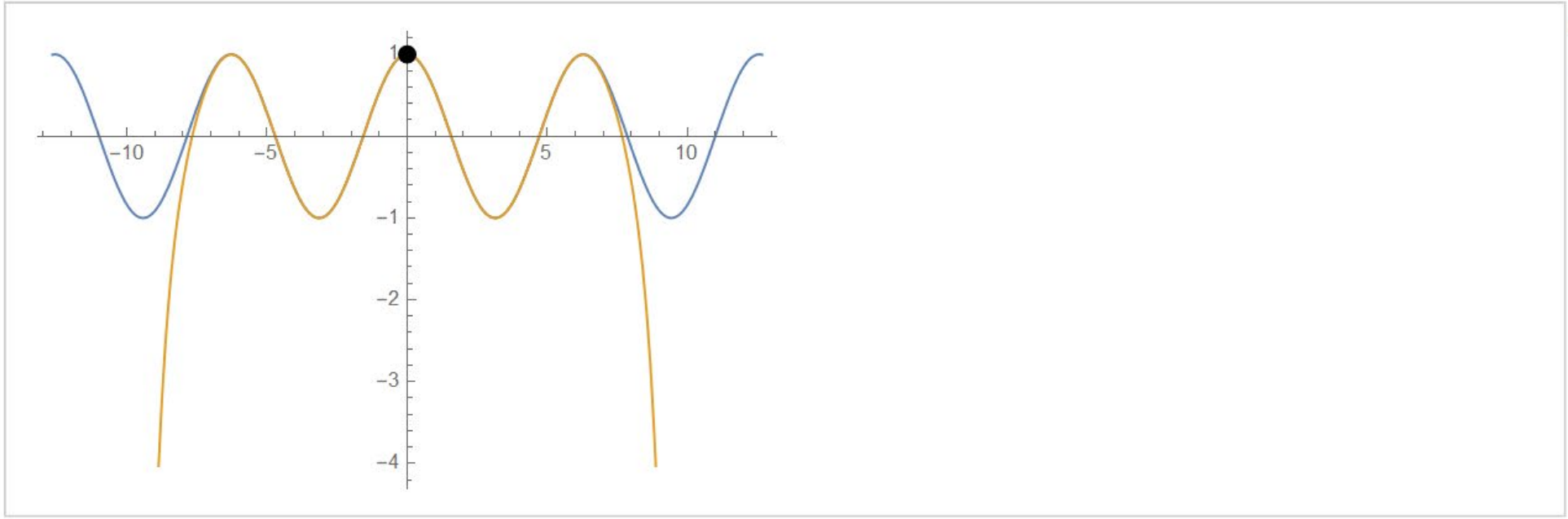
x0

n

1.

c

0.515



x0

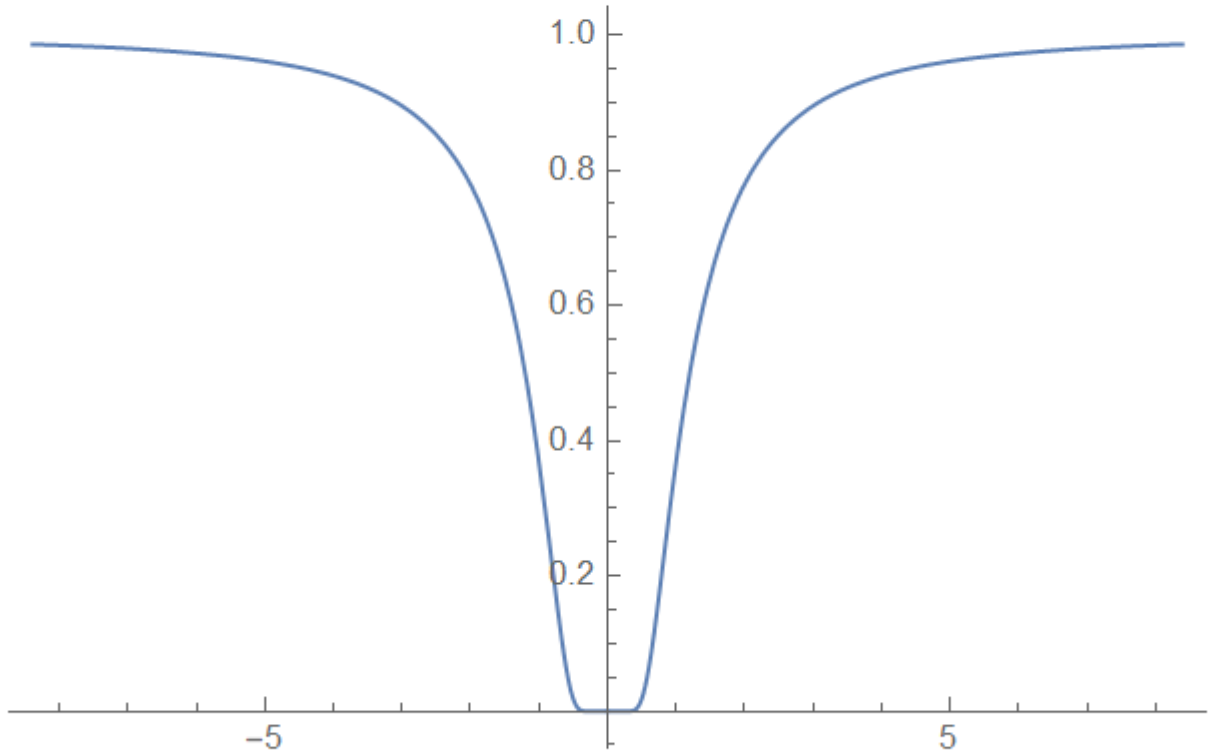
n

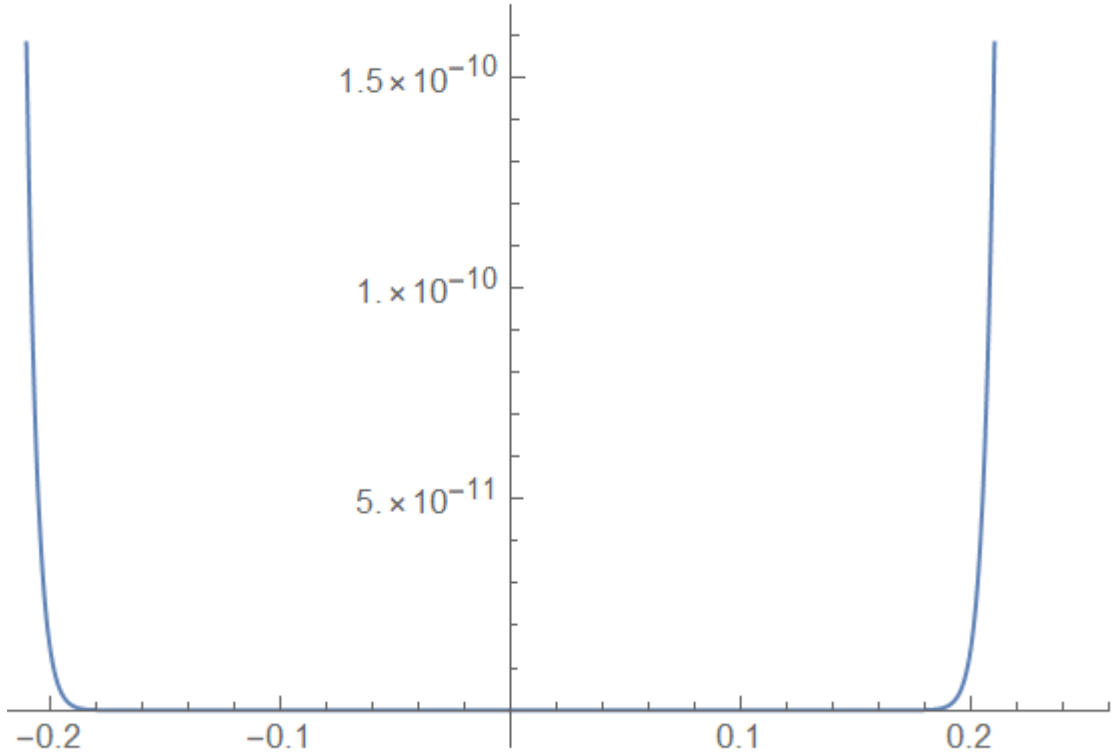
19.75

c

2.015

```
Manipulate[Plot[Exp[-1/x^2], {x, -c, c}], {c, 10, .25}]
```





Hope: as $N \to \infty$, $(T_N f)(x) \longrightarrow f(x)$

Danger: $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Then $f^{(n)}(0) = 0$

So $(T_N f)(x)$ is always zero

only agrees in the limit

with $f(x)$ at $x = 0$.

$$(T_N f)(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n$$

Big Questions:

(1) Does $(T_N f)(x)$ converge as $N \to \infty$?

(2) If yes, converge to $f(x)$?

Famous Taylor Series

$$e^x = 1 + 1\cdot x + \frac{1\cdot x^2}{2!} + \frac{1\cdot x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1\cdot x^n}{n!}$$

use $(e^x)' = e^x$

Then if $f(x) = e^x$ $\qquad f(0) = 1$

$f'(x) = e^x$ $\qquad f'(0) = 1$

$f''(x) = e^x$ $\qquad f''(0) = 1$

$\downarrow \qquad\qquad \downarrow$

$$e^{i\theta} = \cos\theta + i\sin\theta = \text{cis}(\theta)$$

$$f(x) = \cos x \qquad f(0) = 1$$

$$f'(x) = -\sin x \qquad f'(0) = 0$$

$$f''(x) = -\cos x \qquad f''(0) = -1$$

$$f'''(x) = \sin x \qquad f'''(0) = 0$$

$$f^{(iv)}(x) = \cos x \qquad f^{(iv)}(0) = 1$$

periodic

period

4

$$\cos(x) = 1 - \frac{1 \cdot x^2}{2!} + \frac{1 \cdot x^4}{4!} - \frac{1 \cdot x^6}{6!} + \frac{1 \cdot x^8}{8!} - \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$1,\ i = \sqrt{-1},\ i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i$$

periodic with period 4

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{\substack{n \text{ even} \\ n=2m}} \frac{(ix)^n}{n!} + \sum_{\substack{n \text{ odd} \\ n=2k+1}} \frac{(ix)^n}{n!}$$

$$= \sum_{m=0}^{\infty} \frac{i^{2m} x^{2m}}{(2m)!} + \sum_{k=0}^{\infty} \frac{i^{2k+1} x^{2k+1}}{(2k+1)!}$$

$$i^{2m} = (i^2)^m = (-1)^m$$

$$= \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}}_{\cos(x)} + \underbrace{i \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}}_{i \sin(x)}$$

$$e^{ix} e^{-ix} = e^{0} = 1$$

$$[\cos x + i\sin x][\cos x - i\sin x] = 1$$

$$\left(\left(e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin x\right)\right)$$

$$\cos^2 x + \sin^2 x = 1$$

$$e^{ix}\, e^{iy} = e^{i(x+y)}$$

$$[\cos x + i\sin x][\cos y + i\sin y] = \cos(x+y) + i\sin(x+y)$$

$$\underbrace{[\cos x \cos y - \sin x \sin y]}_{\cos(x+y)} + i\underbrace{[\sin x \cos y + \cos x \sin y]}_{\sin(x+y)}$$

deg 8 Taylor Series of e^{x^2} at $x=0$

Know: $e^u = 1 + u + u^2/2! + u^3/3! + u^4/4! + \cdots$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \text{Higher order terms}$$