Math 140: Calculus II: Spring ‘22 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/

Lecture 31: 5-2-22: https://youtu.be/r8xr1gcwZb4

https://web.williams.edu/Mathematics/sjmiller/public_html/140Sp22/talks2022/140Sp22_lecture31.pdf

Plan for the day: Lecture 31: May 2, 2022:

Topics: Difference Equations

- **Fibonacci Numbers**
- **Generating Function for Fibonacci Numbers**
- **Application: Double plus one: Roulette and Fibonaccis**

Exercise 1.1 (Recurrence Relations). *Let $\alpha_0, \dots, \alpha_{k-1}$ be fixed integers and consider the recurrence relation of order k*

$$x_{n+k} \;=\; \alpha_{k-1}x_{n+k-1} + \alpha_{k-2}x_{n+k-2} + \cdots + \alpha_1 x_{n+1} + \alpha_0 x_n. \tag{1.1}$$

Show once k values of x_m are specified, all values of x_n are determined. Let

$$f(r) \;=\; r^k - \alpha_{k-1}r^{k-1} - \cdots - \alpha_0; \tag{1.2}$$

we call this the characteristic polynomial of the recurrence relation. Show if $f(\rho) = 0$ then $x_n = c\rho^n$ satisfies the recurrence relation for any $c \in \mathbb{C}$.

Exercise 1.2. *Notation as in the previous problem, if $f(r)$ has k distinct roots $r_1, \dots, r_k$, show that any solution of the recurrence equation can be represented as*

$$x_n \;=\; c_1 r_1^n + \cdots + c_k r_k^n \tag{1.3}$$

for some $c_i \in \mathbb{C}$. The Initial Value Problem is when k values of x_n are specified; using linear algebra, this determines the values of $c_1, \dots, c_k$. Investigate the cases where the characteristic polynomial has repeated roots. For more on recursive relations, see [GKP], *§7.3.*

Exercise 1.3. *Solve the Fibonacci recurrence relation $F_{n+2} = F_{n+1} + F_n$, given $F_0 = F_1 = 1$. Show F_n grows exponentially, i.e., F_n is of size r^n for some $r > 1$. What is r? Let $r_n = \frac{F_{n+1}}{F_n}$. Show that the even terms r_{2m} are increasing and the odd terms r_{2m+1} are decreasing. Investigate* $\lim_{n\to\infty} r_n$ *for the Fibonacci numbers. Show r_n converges to the golden mean, $\frac{1+\sqrt{5}}{2}$. See* [PS2] *for a continued fraction involving Fibonacci numbers.*

Exercise 1.4 (Binet's Formula). *For F_n as in the previous exercise, prove*

$$F_{n-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{1.4}$$

This formula should be surprising at first: F_n is an integer, but the expression on the right involves irrational numbers and division by 2.

Exercise 1.5. *Notation as in the previous problem, more generally for which positive integers m is*

$$\frac{1}{\sqrt{m}}\left[\left(\frac{1+\sqrt{m}}{2}\right)^n - \left(\frac{1-\sqrt{m}}{2}\right)^n\right] \tag{1.5}$$

an integer for any positive integer n?

Exercise[(h)] **1.6** (Zeckendorf's Theorem)**.** Consider the set of distinct Fibonacci numbers: $\{1, 2, 3,$ 5, 8, 13, $\dots\}$. Show every positive integer can be written uniquely as a sum of distinct Fibonacci numbers where we do not allow two consecutive Fibonacci numbers to occur in the decomposition. Equivalently, for any n there are choices of $\epsilon_i(n) \in \{0, 1\}$ such that

$$n \ = \ \sum_{i=2}^{\ell(n)} \epsilon_i(n) F_i, \quad \epsilon_i(n)\epsilon_{i+1}(n) = 0 \text{ for } i \in \{2, \dots, \ell(n) - 1\}. \tag{1.6}$$

Does a similar result hold for all recurrence relations? If not, can you find another recurrence relation where such a result holds?

Exercise[(hr)] **1.7.** Assume all the roots of the characteristic polynomial are distinct, and let λ_1 be the largest root in absolute value. Show for almost all initial conditions that the coefficient of λ_1 is non-zero.

Exercise[(hr)] **1.8.** Consider 100 tosses of a fair coin. What is the probability that at least three consecutive tosses are heads? What about at least five consecutive tosses? More generally, for a fixed k what can you say about the probability of getting at least k consecutive heads in N tosses as $N \to \infty$?

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;

$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$.

Cookie Monster Meets the Fibonacci Numbers. Mmmmmm -- Theorems!: http://youtu.be/5e6HsfxqVSE
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/CookiesToCLTtoGaps_Yale2014.pdf

Other choices: $F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3, \ldots$

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

$F_{1,000,000} = ?$

Base 2: $a_n \cdot 2^n + a_{n-1} 2^{n-1} + \cdots + a_1 2 + a_0$

each $a_i \in \{0, 1\}$

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$F_{n+1} = F_n + F_{n-1}$ $\quad F_0 = 0 \quad F_1 = 1$

$F_{n+1} \leq 2F_n$ means grows slower than 2^n

$F_{n+1} \geq 2F_{n-1}$ so every 2 grow at least 2, so faster than $\sqrt{2}^n$

Believe $\sqrt{2}^n \leq F_n \leq 2^n$ (at least of n bits)

Divine Inspiration: Try $F_n = r^n$

Gives $r^{n+1} = r^n + r^{n-1}$ characteristic polynomial

$r^{n-1}(r^2 - r - 1) = 0$ roots are 0 and $\frac{1 \pm \sqrt{5}}{2}$

Show if r_1 and r_2 are roots and solve the recurrence, so

does $c_1 r_1^n + c_2 r_2^n$ for any c_1, c_2

$n=0$ $\quad c_1 + c_2 = F_0 = 0 \Rightarrow c_2 = -c_1$

$n=1$ $\quad c_1 r_1 + c_2 r_2 = F_1 = 1 \Rightarrow c_1(r_1 - r_2) = 1$ or $c_1 = \frac{1}{r_1 - r_2} = \frac{1}{\sqrt{5}}$

Get Binet! $F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$

```
lower = 0;
upper = 1;
max = 10000000;
Timing[For[n = 2, n <= max, n++,
  {
   new = lower + upper;
   lower = upper;
   upper = new;
   }];]
Print[upper];
{356.063,Null}
```

```
Log[10.,Fibonacci[1000000]]
Log[10.,Fibonacci[500000]]
208987.
104493.
```

Estimate on how many digit operations base 10 to get to the millionth Fibonacci number. The 500,000th has 104,493 digits, so have at least
100000 * 500000 = 50,000,000,000.

How many seconds in a year?
3600*24*365.25*4 = $1.2623*10^8$ or approximately 100,000,000.

So if do 100 digits a second get to 10,000,000,000.

We're off by AT LEAST a factor of 5, and this is doing 100 digits a second!

Binet's Formula

$$F_1 = F_2 = 1; \; F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

Binet's Formula

$$F_1 = F_2 = 1;\ F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

$$(1) \quad \Rightarrow \quad \underbrace{\sum_{n\geq 2} F_{n+1}x^{n+1}}_{g(x) - F_1 x - F_2 x^2} = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 2} F_{n-1}x^{n+1}$$

$x^n x$

pull out x

x^{n-1} x^2

pull out x^2

Binet's Formula

$$F_1 = F_2 = 1;\; F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

$$(1) \;\Rightarrow\; \sum_{n\geq 2} F_{n+1}x^{n+1} = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 2} F_{n-1}x^{n+1}$$

$$\Rightarrow\; \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2}$$

Binet's Formula

$$F_1 = F_2 = 1;\; F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

$$\begin{aligned}
(1) \quad &\Rightarrow \sum_{n\geq 2} F_{n+1}x^{n+1} = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 2} F_{n-1}x^{n+1} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = x\sum_{n\geq 2} F_n x^n + x^2\sum_{n\geq 1} F_n x^n
\end{aligned}$$

Binet's Formula

$$F_1 = F_2 = 1;\ F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

$$
\begin{aligned}
(1) \quad &\Rightarrow \sum_{n\geq 2} F_{n+1}x^{n+1} = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 2} F_{n-1}x^{n+1} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = x\sum_{n\geq 2} F_n x^n + x^2\sum_{n\geq 1} F_n x^n \\
&\Rightarrow g(x) - F_1x - F_2x^2 = x(g(x) - F_1x) + x^2 g(x)
\end{aligned}
$$

Binet's Formula

$$F_1 = F_2 = 1;\ F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

$$(1) \Rightarrow \sum_{n\geq 2} F_{n+1}x^{n+1} = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 2} F_{n-1}x^{n+1}$$

$$\Rightarrow \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2}$$

$$\Rightarrow \sum_{n\geq 3} F_n x^n = x\sum_{n\geq 2} F_n x^n + x^2\sum_{n\geq 1} F_n x^n$$

$$\Rightarrow g(x) - F_1 x - F_2 x^2 = x(g(x) - F_1 x) + x^2 g(x)$$

$$\Rightarrow g(x) = x/(1-x-x^2).$$

- Generating function: $g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2}$. $= x \frac{1}{1-r}$

with $r = x+x^2$

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$$

$$= 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \cdots$$

$$= 1 + (x+x^2) + (x^2+2x^3+x^4) + (x^3+3x^4+\cdots)$$

- Generating function: $g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2}$.
- Partial fraction expansion:

$$\Rightarrow\ g(x) = \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \left(\frac{\frac{1+\sqrt{5}}{2} x}{1 - \frac{1+\sqrt{5}}{2} x} - \frac{\frac{-1+\sqrt{5}}{2} x}{1 - \frac{-1+\sqrt{5}}{2} x} \right).$$

- Generating function: $g(x) = \sum_{n>0} \boldsymbol{F}_n x^n = \frac{x}{1-x-x^2}$.
- Partial fraction expansion:

$$\Rightarrow \; g(x) \; = \; \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}} \left(\frac{\frac{1+\sqrt{5}}{2}x}{1-\frac{1+\sqrt{5}}{2}x} - \frac{\frac{-1+\sqrt{5}}{2}x}{1-\frac{-1+\sqrt{5}}{2}x} \right).$$

Coefficient of x^n (power series expansion):

$$\boldsymbol{F}_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{-1+\sqrt{5}}{2} \right)^n \right]$$ - Binet's Formula!

(using geometric series: $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$).

We consider the following simplified model for the number of pairs of whales alive at a given moment in time. We make the following simplifying assumptions:

(1) Time moves in discrete steps of 1 year.
(2) The number of whale pairs that are 0, 1, 2 and 3 years old in year n are denoted by a_n, b_n, c_n and d_n respectively; all whales die when they turn 4.
(3) If a whale pair is 1 year old it gives birth to two new pairs of whales, if a whale pair is 2 years old it gives birth to one new pair of whales, and no other pair of whales give birth.

zero $a_{n+1} = 0\cdot a_n + 2\cdot b_n + 1\cdot c_n + 0\cdot d_n$

one $b_{n+1} = 1\cdot a_n + 0\cdot b_n + 0\cdot c_n + 0\cdot d_n$

two $c_{n+1} = 0\cdot a_n + 1\cdot b_n + 0\cdot c_n + 0\cdot d_n$

three $d_{n+1} = 0\cdot a_n + 0\cdot b_n + 1\cdot c_n + 0\cdot d_n$

dead

Letting

$$v_n = \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix}, \tag{1}$$

we see that

$$v_{n+1} = Av_n, \tag{2}$$

where

$$A = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

Thus

$$v_{n+1} = A^{n+1}v_0, \tag{4}$$

where v_0 is the initial populations at time 0. As discussed before, it is one thing to write down a solution and another to have be able to numerically work with it. This matrix is fortunately easily diagonalizable.

$$\vec{v}_{n+1} = A\vec{v}_n$$

but $\vec{v}_n = A\vec{v}_{n-1}$

So $\vec{v}_{n+1} = A^2\vec{v}_{n-1}$

$$= A^3\vec{v}_{n-2}$$

$$\vdots$$

$$\vec{v}_{n+1} = A^{n+1}\vec{v}_0$$