Calculus II: Integration

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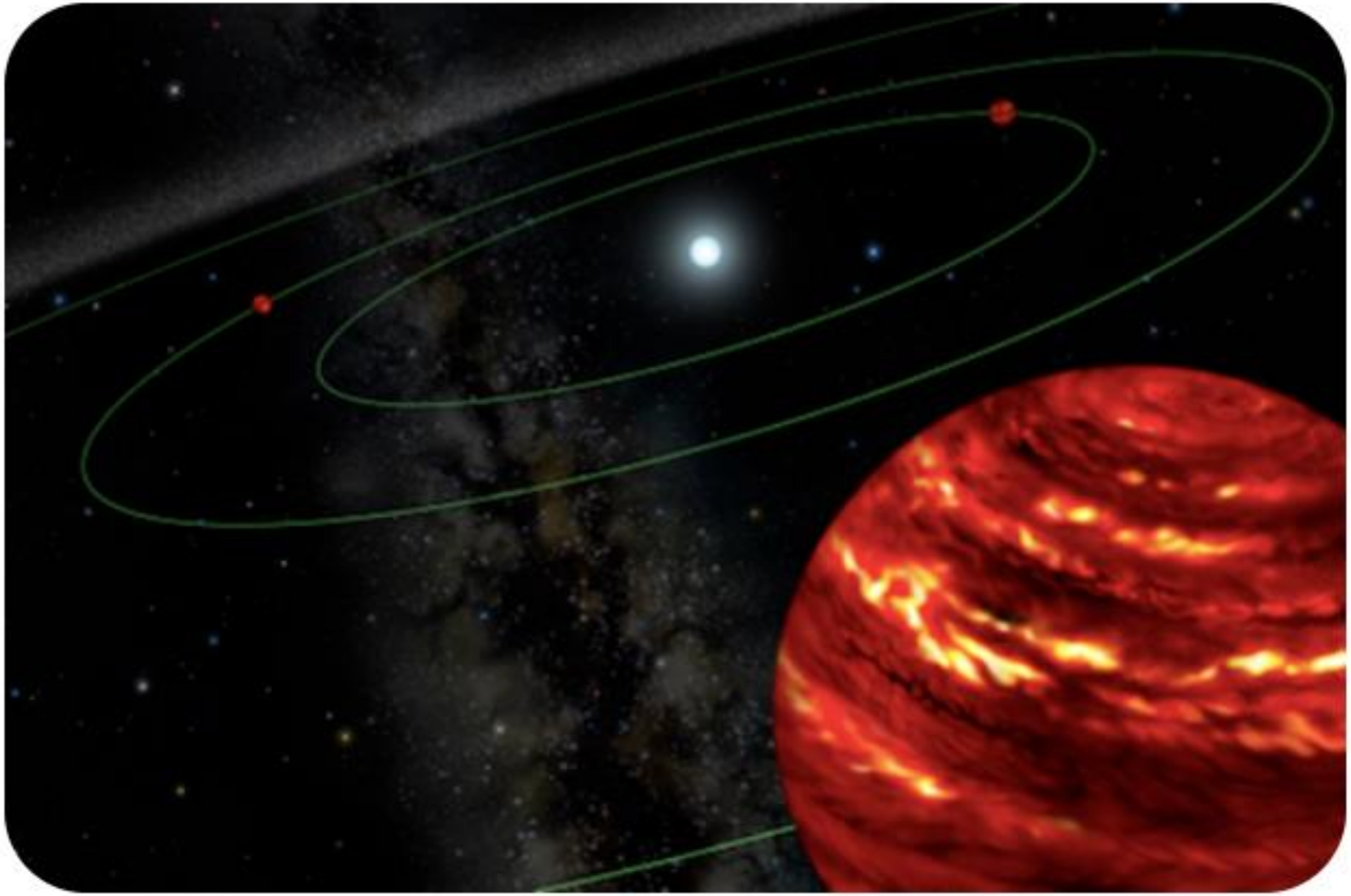
Lecture 1: 2-4-2022:

Motivation: Newton's Integral

Plan for the day: Lecture 1: December 19, 2021:

- Discuss motivation of calculus
- Motivate integration: passing to the limit of a sum
- Dangers of extrapolating and what happens when you ass|u|me.



In this illustration, you can see three young planets tracing orbits around a star called HR 8799 that lies about 130 light-years from Earth. Image credit: Gemini Observatory Artwork by Lynette Cook https://spaceplace.nasa.gov/other-solar-systems/en/

Newton's Law of Gravity

$$F = G\frac{m_1 m_2}{r^2}$$

F = force

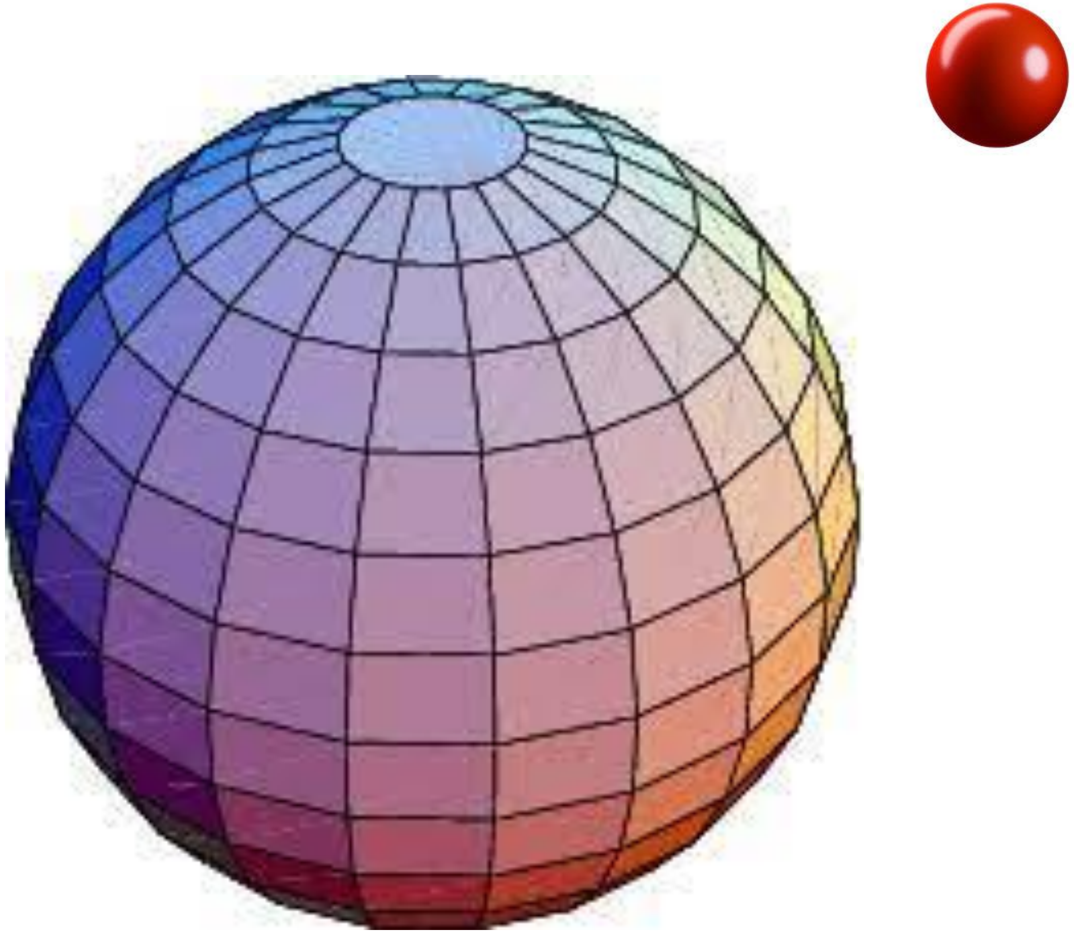
G = gravitational constant

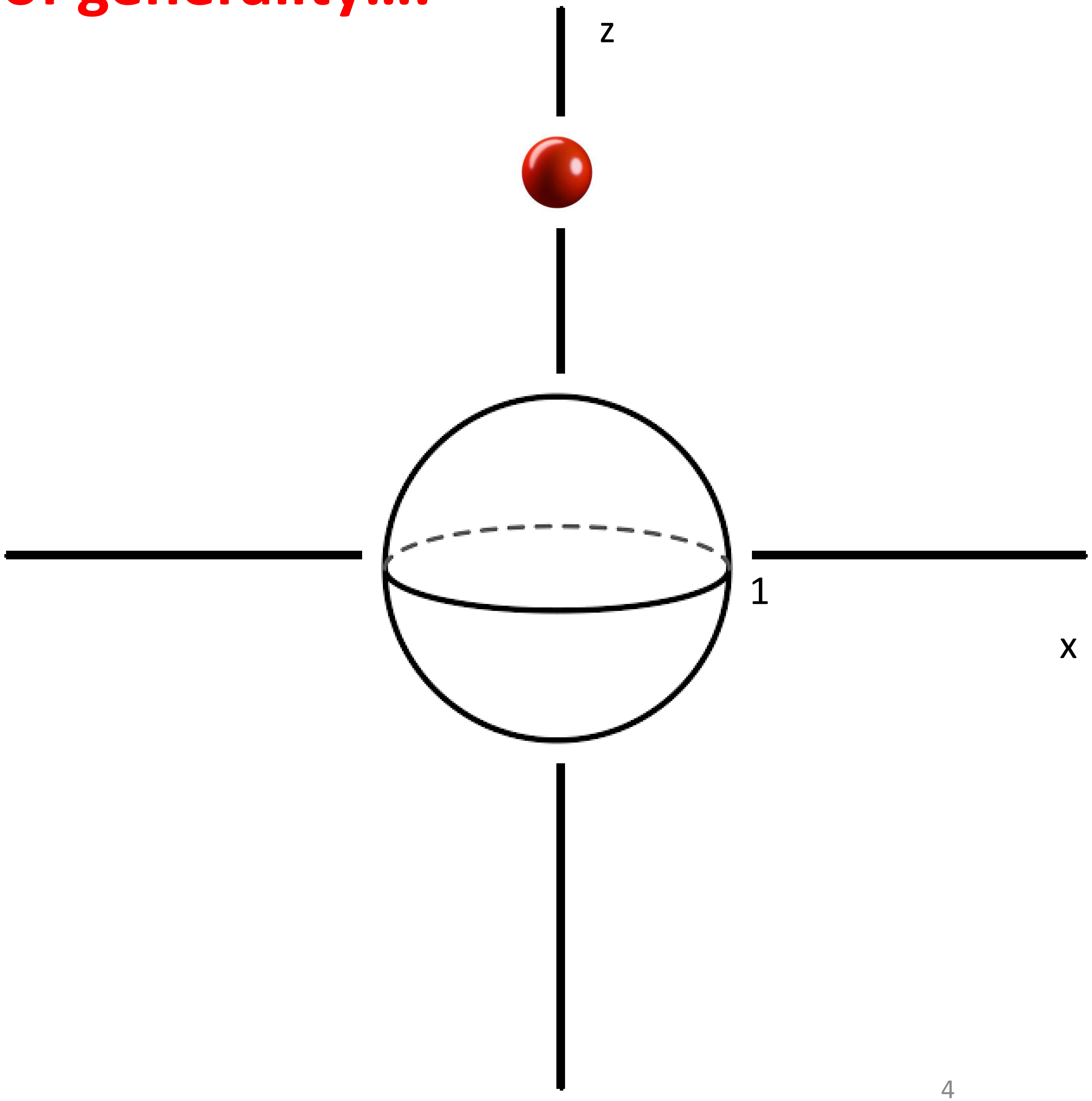
m_1 = mass of object 1

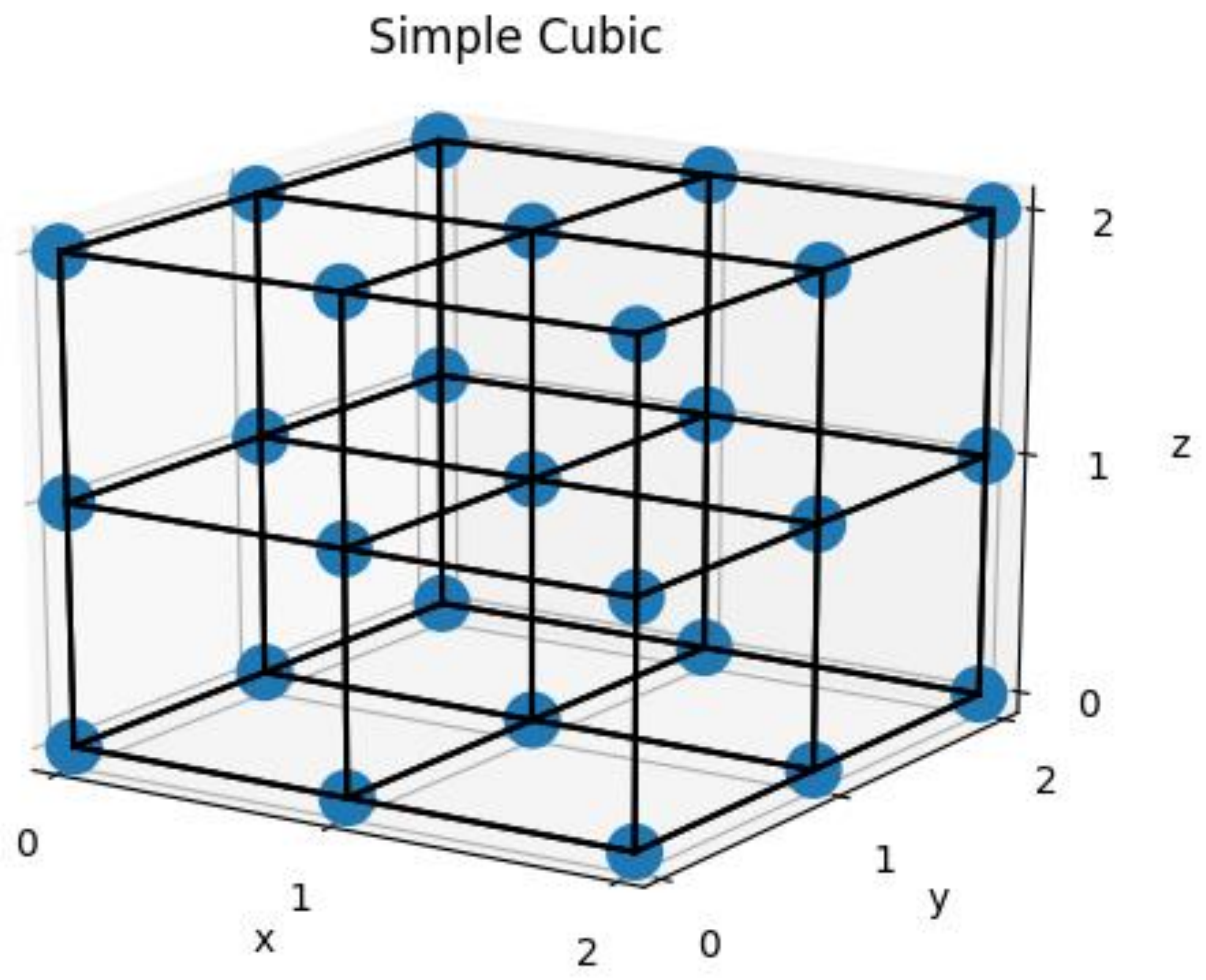
m_2 = mass of object 2

r = distance between centers of the masses

Symmetry Arguments: Without loss of generality….







https://www.juliabloggers.com/computationally-visualizing-crystals/

```
symmgravityapprox[range_, height_, printme_] := Module[{},
  (* we assume the object is "height" units above the center of a sphere of radius 1 *)
  force = 0;
  numpoints = 0;
  For[x = 0, x ≤ range, x++,
   {
    If[printme == 1, If[Mod[x, range / 10] == 0, Print["Have done ", x, " of ", range, "."]]];
    For[y = 0, y ≤ range, y++,
     For[z = -range, z ≤ range, z++,
      {
       (* only find contribution if point in sphere *)
       If[x^2 + y^2 + z^2 ≤ range^2,
        {
         distsquared = (x / range) ^2 + (y / range) ^2 + (z / range - height) ^2 ;
         (* two vectors: (0,0,-height) and (x/range,y/range,z/range-height) *)
         (* dot product is product of lengths times cos(angle) *)
         (* we take the force and multiply by cos(angle) *)
         contribution = (z / range - height) * (-height) / (distsquared * height * Sqrt[distsquared]);
         multiplier = (Sign[x] ^2 + 1) * (Sign[y] ^2 + 1);
         force = force + contribution * multiplier;
         numpoints = numpoints + multiplier;
        }]; (* end of if loop *)
      }]; (* end of z *)
    ]; (* end of y *)
   }]; (* end of x *)
  If[printme == 1,
   {
    Print["Discrete approx: ", 1.0 force / numpoints];
    Print["Theory: ", 1.0 / height^2];
    Print["Numpoints inside = ", numpoints];
    Print["Predicted numpoints inside = ", 1.0 ((4 Pi / 3) / 8) * (2 range + 1) ^3];
   }];
  Return[{1.0 force / numpoints, 1.0 / height^2}];
 ]
```

Notice ranges:

x, y from 0 to range
z from –range to range.

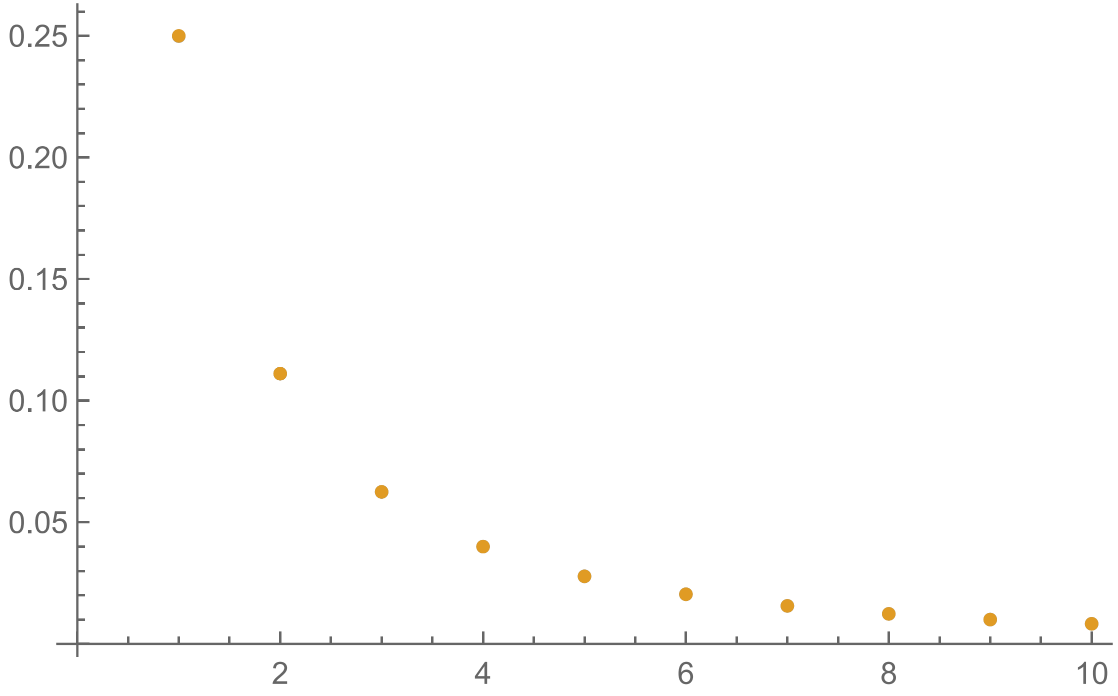
Reason is the force is down.

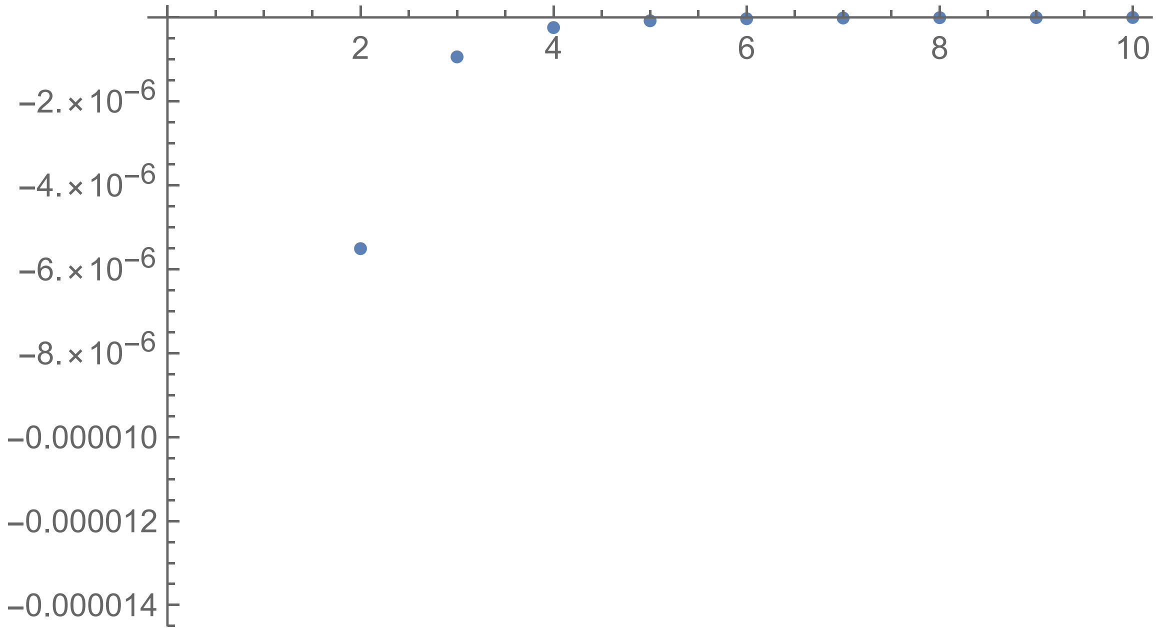
The following four points have the same contribution:

- (x, y, z)
- (-x, y, z)
- (x, -y, z)
- (-x, -y, z)

Thus saves a factor of four if compute contribution of one of these and multiply by 4.

Note if x or y is zero would multiply by 2 (if both are zero multiply by 1).





Comparing the gravitational force on an object at height h above the north pole of a unit sphere two ways:

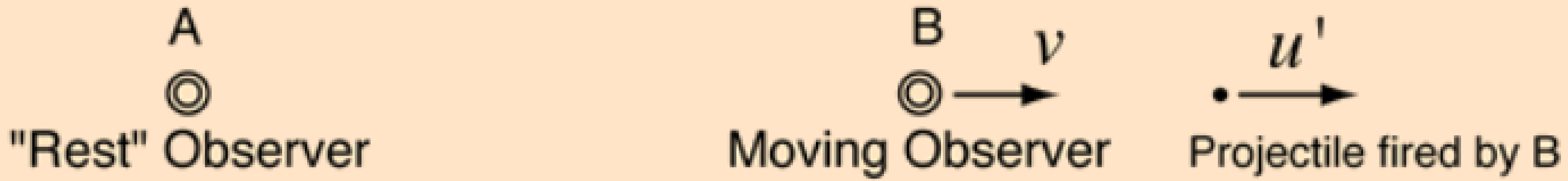
(1) all the mass is at the center,

(2) compute the force from points at (x/40, y/40, z/40) for x, y and z integers.

The left is the plot of both, the right is the difference between the two.

Einstein Velocity Addition

The relative velocity of any two objects never exceeds the velocity of light. Applying the Lorentz transformation to the velocities, expressions are obtained for the relative velocities as seen by the different observers. They are called the Einstein velocity addition relationships.



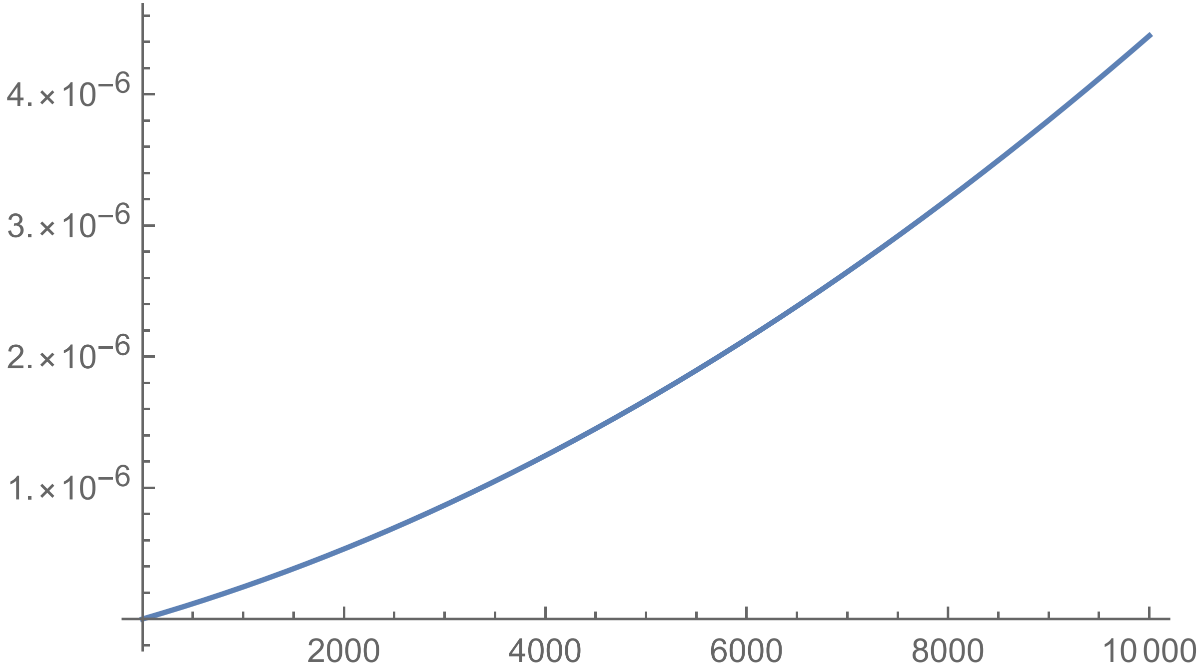
$$u = \frac{v + u'}{1 + \frac{vu'}{c^2}}$$

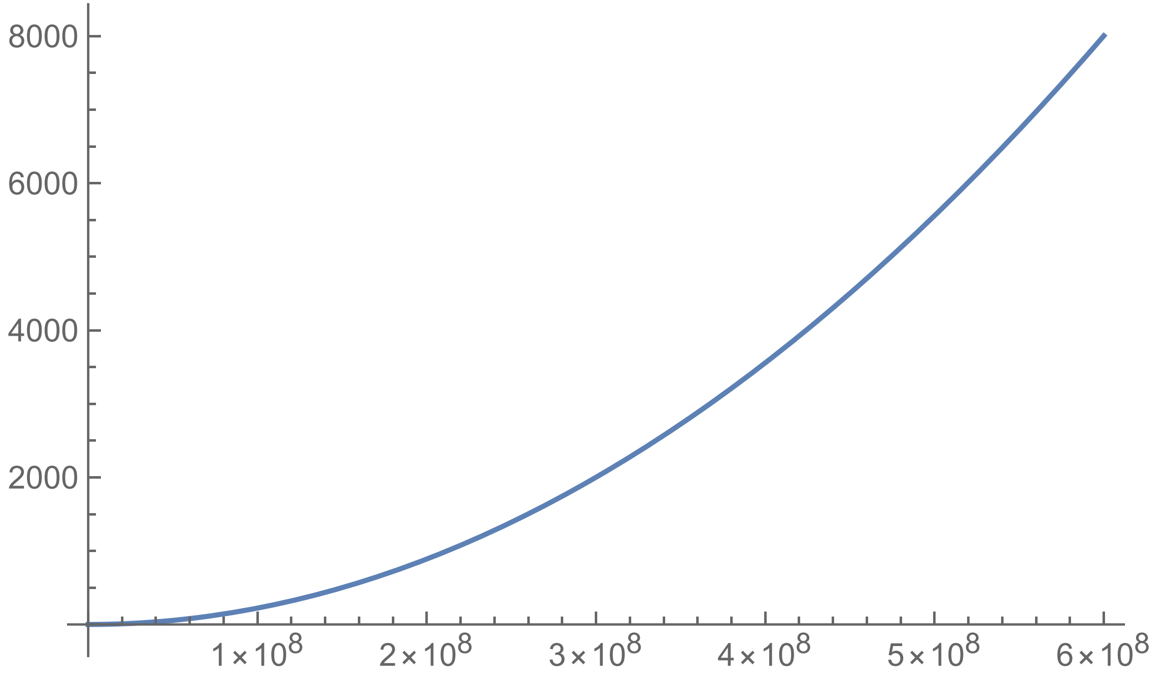
u' = velocity of projectile as seen by B

u = velocity of projectile as seen by A

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

http://hyperphysics.phy-astr.gsu.edu/hbase/Relativ/einvel.html





Plotting the difference between the Einstein correction and the classical prediction for adding two speeds.

We throw a projectile forward on a train (or rocket ship) traveling in the same direction at 10,000 mph.

The x-axis is the speed of the thrown object, the y-axis is the difference between the relativistic correction and the classical prediction. Note the order of the error for speeds up to 10,000 mph is on the same order as our integration approximation!

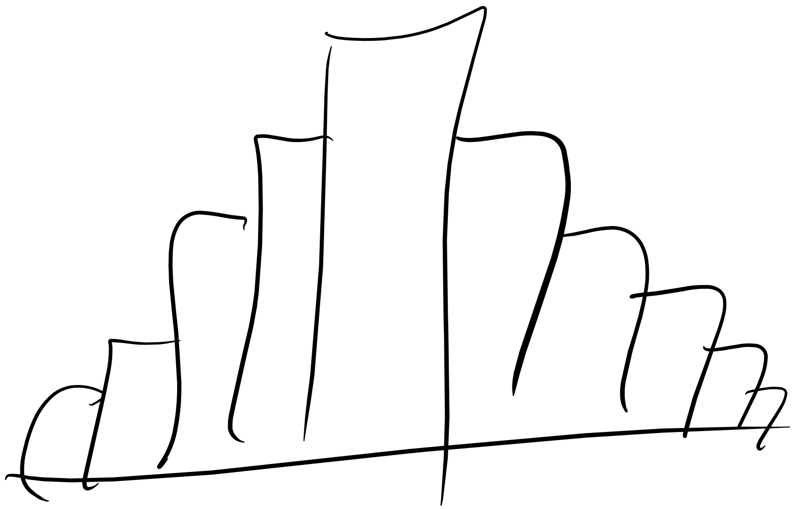
Note the Apollo 11's fastest speed was about 25,000 miles per hour! Lightspeed is about 6.7 x 10^8 mph.

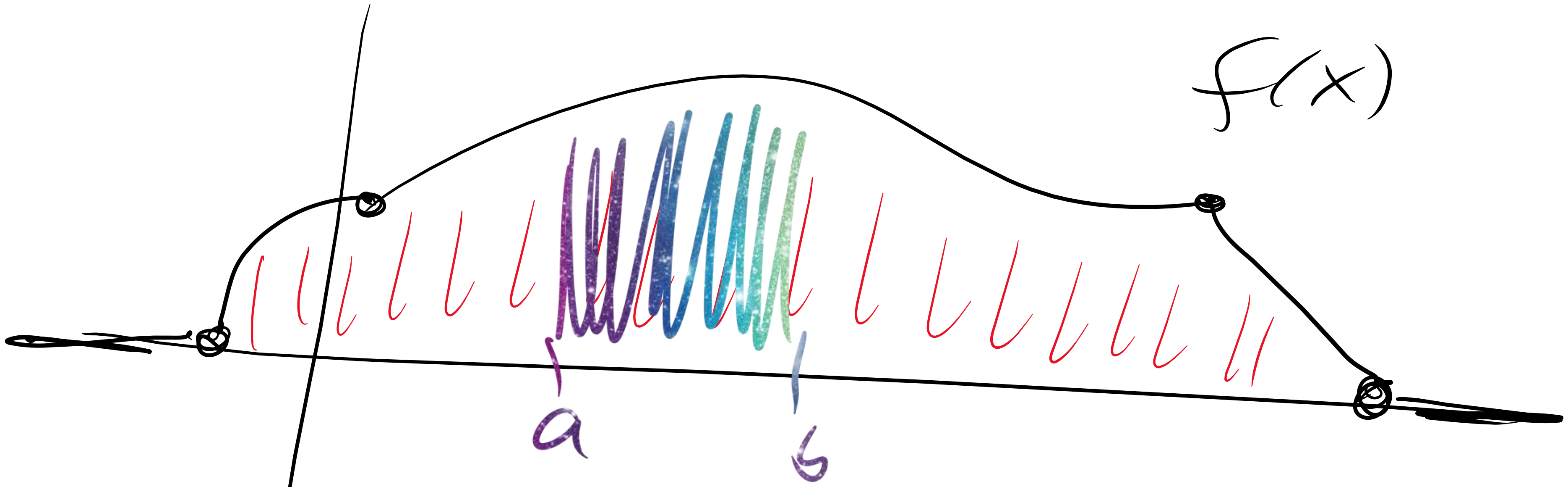
Two fair die sum can be 2, 3, ..., 12

$\text{Prob}(2) = \text{Prob}(12) = \frac{1}{36}$

$\text{Prob}(3) = \text{Prob}(11) = \frac{2}{36}$

$\text{Prob}(7) = \frac{6}{36}$





If $f(x) \geq 0$ and area under $y = f(x)$ is 1

Then f is a probability density

Fundamental Thm of Calc

Let $f(x)$ be a nice function (bounded, differentiable) and F such that $F' = f$. Then the area under $y = f(x)$ from $x = a$ to $x = b$, denoted by $\int_a^b f(x)\,dx$, is $F(b) - F(a)$.