CHAPTER 12: PARTIAL DIFFERENTIATION

Goals: • Two main parts of calculus are derivs (rates of change) and integrals (areas). Concentrate on first here, namely deriv of fns of several variables.

• Numerous applications; concentrate on optimization and approximation.

Will leave many of the proofs to appendices here or a higher math course. Aim is to gain proficiency in computations and understand ideas

Sections: 12.1 (short), 12.2, 12.3, 12.4, 12.5

12.6, 12.7, 12.8, 12.9

↳ will mention "existence" of results such as those from 12.10, but will leave for a linear algebra course.

Chapter 12: Partial Differentiation

Section 12.1: Introduction

Essentially, says many functions in the world depend on multiple variables

↳ $PV = nRT$

↳ movie theater's revenue

↳ expect value of baseball player / expected team run production

Goal: want to understand what happens as params vary.

Section 12.2: Functions of Several Variables

<u>Defn</u>: Domain D a nice subset of $\mathbb{R}^n$, function f assigns a unique real number to every point in D, write $f: D \to \mathbb{R}$ to denote this.

<u>Ex</u>: D = circle of radius 1, $f(x,y) = \sqrt{1-x^2-y^2}$

↳ note this is defined on D but not outside

↳ note geometric interpretation: hemisphere height.

SECTION 12.2 (CONT)

Key is to find where function is defined.

↳ Need denominators to be non-zero.

↳ ex: $f(x,y) = 11x^2 + 2xy - \frac{3}{x} + \frac{4}{y^2}$

is defined except on coordinate axes.

↳ ex: $f(x) = \frac{x^2-4}{x-2}$ is definable at $x=2$

as equals $\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = x+2$.

The <u>graph</u> of a function $f: D \to \mathbb{R}$ (for a domain $D \subset \mathbb{R}^2$) associates a height to each point in D. Think mountain ranges and valleys.

The <u>level set of value c</u> is all points in D such that $f(x,y) = c$. Level sets are convenient ways to plot / convey info (in a 3dim image, could have data hidden). Think temperature maps, elevation maps, ...

SECTION 12.2 (CONT)

Will not spend much time plotting in this course - see the book / webpage for Mathematica code.

Ex: Investigate $f(x,y) = 4 - x^2 - y^2$

Let's look at level sets. Say $f(x,y) = c$. Then

$4 - x^2 - y^2 = c \Rightarrow x^2 + y^2 = 4 - c$.

Note if $c > 4$ then corresponding level set is empty. If $c = 4$ the corresponding level set is $\{(0,0)\}$, while if $c < 4$ the set is a circle of radius $\sqrt{4-c}$.

Thus level sets are circles

Homework: #2, #4, #5, #27, #32

Section 12.3: Limits and Continuity

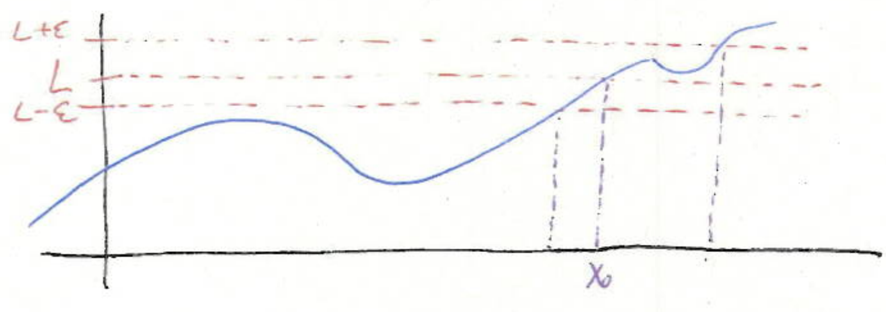
Terminology

- **Open Disk/Ball**: $D_r(\vec{x}_0) = \{\vec{x} : \|\vec{x} - \vec{x}_0\| < r\}$
- **Open Set**: $U \subset \mathbb{R}^n$ open if for all $\vec{x}_0 \in U$ there is a r (which may depend on $\vec{x}_0$) s.t. $D_r(\vec{x}_0) \subset U$

  ↳ Note: empty set $\varnothing$ considered open, $\mathbb{R}^n$ open
  ↳ use dotted lines to denote open
- **Neighborhood**: Mean any open set containing $\vec{x}_0$
- **Boundary Points**: Given $A \subset \mathbb{R}^n$, say $\vec{x}$ is a boundary point of A if every neighborhood of $\vec{x}$ contains at least one point in A and at least one point not in A.
- **Closed**: A set is closed if it contains all its ~~limit~~ boundary points.

This is "advanced" terminology that is needed in advanced courses. May safely skip it if you wish.

- **Limit**: Say the limit of f as $\vec{x}$ approaches $\vec{x}_0$ is L if $f(\vec{x})$ gets closer and closer to L as $\vec{x}$ gets close to $\vec{x}_0$. Denote $\lim_{x \to x_0} f(x) = L$.
  ↳ Can define in terms of neighborhoods
  ↳ Can do ε-δ: $\forall \varepsilon > 0\ \exists \delta$ st $|\vec{x} - \vec{x}_0| < \delta$ (and $\vec{x} \neq \vec{x}_0$) implies $|f(\vec{x}) - L| < \varepsilon$



EXAMPLE: Prove $f(x)$ is continuous at $x=3$ if $f(x)=x^2$

"Guess" $L=9$. Given ε find δ s.t. $|x-3|<\delta \Rightarrow |f(x)-9|<L$

Well, $|f(x)-9| = |x^2-9| = |x-3|\cdot|x+3| < \varepsilon$

↳ wlog, assume $\delta<1$ so $|x+3|$ is between 2 and 4

Then $|x-3|\cdot 2 < \varepsilon$ or $|x-3| < \frac{\varepsilon}{2}$

So if we take $\delta < \frac{\varepsilon}{2}$ then $|x-3|<\delta \rightarrow |f(x)-9| < \varepsilon$ ▲

• Usually won't argue so rigorously, but good to know "how"

PROPERTIES OF LIMITS

• Uniqueness: $\lim\limits_{x\to x_0} f(x)$ equals b_1 and b_2 then $b_1=b_2$

• Constant: $\lim\limits_{x\to x_0} c\,f(x) = c \lim\limits_{x\to x_0} f(x)$

• Sum/Diff: $\lim\limits_{x\to x_0}\left(f(x)+g(x)\right) = \lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x)$ if at least one exists on RHS

• Product/Quotient: $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}$ and $\lim\limits_{x\to x_0} f(x)g(x) = \lim\limits_{x\to x_0} f(x) \lim\limits_{x\to x_0} g(x)$
so long as at least one of RHS exists, and fi- quotient $\lim\limits_{x\to x_0} g(x)$ is non-zero

• Components $f(\vec{x}) = (f_1(\vec{x}),\dots, f_n(\vec{x}))$. Then $\lim\limits_{x\to x_0} f(\vec{x}) = \vec{b}$ if and only if $\lim\limits_{x\to x_0} f_i(\vec{x}) = b_i$ for $i\in\{1,2,\dots,n\}$

DANGERS: $\lim\limits_{x\to\infty}(x^2-x)$, $\lim\limits_{x\to\infty}(x^2-x^2)$, $\lim\limits_{x\to\infty}(x^2-x^3)$

Do not define $\infty-\infty$, $\pm\infty\cdot 0$; do define $\infty\cdot\infty$, $\infty+\infty$

Key fact: For $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x})$ to exist, must get the same value NO MATTER WHAT path you take to get to $\vec{x}_0$; note in the limit calculation $\vec{x}$ is never actually equal to $\vec{x}_0$ just approaches.

Think of Cam and Kayla artwork:

Ex: Consider $f(x,y) = \dfrac{xy}{x^2+y^2}$ and $\vec{x}_0 = (0,0)$

limit does not exist!

Try coord axes first

↳ along $x=0$ get $\lim_{y \to 0} \dfrac{0}{0^2+y^2} = 0$ as $y \neq 0$

↳ along $y=0$ get $\lim_{x \to 0} \dfrac{0}{x^2+0^2} = 0$ as $x \neq 0$

* Just because two paths agree does not mean limit exists!

↳ along $x=y$ get $\lim_{x \to 0} \dfrac{x^2}{x^2+x^2} = \frac{1}{2}$ as $x \neq 0$

DO MORE EXAMPLES!

SECTION 12.3 (CONT)

Common technique is to convert to polar coordinates for many limit problems.

Ex: $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

Have $x = r\cos\theta$, $y = r\sin\theta$

$(x,y)\to(0,0)$ means $r\to 0$ and θ free to do anything!

Get $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r}$

$= \lim_{r\to 0} r\cos\theta\sin\theta = 0$

Ex: $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Did this already. Polar change gives this limit equals $\lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r^2} = \lim_{r\to 0} \cos\theta\sin\theta$

which can be anything depending on path θ takes.

For definiteness, take $\theta = 0$ and $\theta = \pi/4$.

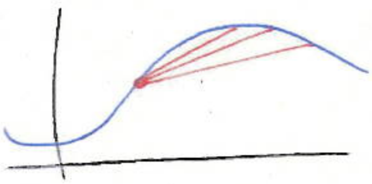
Homework: Pg 917: #1, #8, #10, #24, #38, #54
(hint: $\lim_{t\to 0} \frac{\sin t}{t} = 1$)

Suggested: #41, #51, #55

Section 12.4: Partial Differentiation

Defn of the deriv (one variable)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{or} \quad f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$



Interpretation: average speed

Partial Deriv

$$\frac{\partial f}{\partial x_j}(x_1,\dots,x_n) = \lim_{h\to 0} \frac{f(x_1,x_2,\dots,x_j+h,\dots,x_n) - f(x_1,\dots,x_n)}{h}$$

$$= \lim_{h\to 0} \frac{f(\vec{x}+h\vec{e}_j) - f(\vec{x})}{h}$$

↳ Just treat all other variables as constants

Example: Find partials of $f(x,y) = x\cos(xy)$

Outstanding Example: $f(x,y) = (xy)^{1/3}$

↳ $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0 = \frac{\partial f}{\partial y}(0,0)$

Let $g: \mathbb{R}\to\mathbb{R}^2$ be given by $g(x) = (x,x)$

Let $A(x) = f(g(x)) = (f\circ g)(x) = x^{2/3}$

↳ f and g differentiable

$A'(x) = \lim_{h\to 0} \frac{A(h)-A(0)}{h} = \lim_{h\to 0} \frac{h^{2/3}}{h}$ = undefined!

↳ Composition of diff is not necc diff

↳ Chain rule will be even harder

↳ Problem here: no good tangent plane at (0,0)

Section 12.4: Partial Derivatives

Ex: $f(x,y,z) = x^2 y + y\cos(xz)$

$$\frac{\partial f}{\partial x} = 2xy + y\left[-\sin(xz)\cdot z\right]$$

$$\frac{\partial f}{\partial y} = x^2 + \cos(xz)$$

$$\frac{\partial f}{\partial z} = y\left[-\sin(xz)\cdot x\right]$$

Notation

$\frac{\partial f}{\partial x} = D_x f$. If $f: \mathbb{R}^n \to \mathbb{R}$, so have $f(x_1, \ldots, x_n)$, then

$Df = \langle D_{x_1} f, \ldots, D_{x_n} f\rangle = \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right\rangle$. Call

Df the gradient of f, often write $\mathrm{grad}(f)$ or ∇f.

Right now allowing f to have multiple inputs but only one output. Can generalize to several outputs....

Linear Approximations

↳ Very important: locally complex fns well approx with simple fns

↳ tangent line: $y = f(a) + f'(a)(x-a)$

↳ Size of error?

↳ MVT: $f(x) = f(a) + f'(c)(x-a)$

Thus $|f(x) - y| = |f'(c) - f'(a)| \cdot |x-a|$

$$\le \left| 2 \max_{w \in [a,c]} f'(w) \right| \cdot |x-a|$$

↳ better estimate if f'' exists

↳ Generalize to higher dimensions (see book for proof)

Tangent Plane:

$$\boxed{z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)}$$

Deriv in one-var: $\displaystyle \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0$

ADVANCED DEFN OF DERIV: FOR COMPLETENESS BUT SHOULD SKIP!

Defn of the deriv: Open $U \subset \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff at $\vec{x}_0$ if partial derivs exist at $\vec{x}_0$, and if $T = Df(\vec{x}_0)$ is the $m \times n$ matrix with elements $\partial f_i / \partial x_j (\vec{x}_0)$ satisfies

$$\lim_{\vec{x} \to \vec{x}_0} \frac{\| f(\vec{x}) - f(\vec{x}_0) - T(\vec{x} - \vec{x}_0) \|}{\| \vec{x} - \vec{x}_0 \|} = 0,$$

with $\vec{x} - \vec{x}_0$ a column vector. Call T the deriv of f at x_0 or the matrix of partial derivs of f at $\vec{x}_0$

This is the definition you'll see in advanced courses in later years

Ex: $f(x,y) = x^2 + y^4 + e^{xy}$. Calculate the eq of the tangent plane at point $(1, 0, 2)$, where we are writing it as $z = f(x,y)$.

As $z = f(x,y)$, $z_0 = f(x_0, y_0)$ or $2 = f(1,0)$, which is true as $f(1,0) = 1^2 + 0^4 + e^0 = 1 + 1 = 2$. ✓

Tangent plane eq

$$z = z_0 + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

here $(x_0, y_0) = (1, 0)$

$$z_0 = f(x_0, y_0) = 2$$

$$\frac{\partial f}{\partial x} = 2x + ye^{xy} \Rightarrow \frac{\partial f}{\partial x}(1,0) = 2$$

$$\frac{\partial f}{\partial y} = 4y^3 + xe^{xy} \Rightarrow \frac{\partial f}{\partial y}(1,0) = 1$$

$\Rightarrow$ tangent plane is $z = 2 + 2(x - 1) + 1(y - 0)$.

Higher Derivatives

f_{xy} means $(f_x)_y = \dfrac{\partial f_x}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$

Similarly f_{yx} means $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$

Key Result: Equality of Mixed Partial Derivatives: If f is twice continuously differentiable Then $f_{xy} = f_{yx}$. Condition often met (especially in first course). Proof in Appendix.

The order of the partial derivative is the number of differentiations.

Ex: $f(x,y) = x^2 y \sin x$. Then

$$\partial f/\partial x = 2xy \sin x + x^2 y \cos x$$
$$\partial f/\partial y = x^2 \sin x$$
$$\partial^2 f/\partial x \partial y = 2x \sin x + x^2 \cos x = \partial^2 f/\partial y \partial x$$

Note one way, f_{yx}, easier than other, f_{xy}.

Will now state some properties of derivatives. Remember if have $f(x,y,z)$ then Df means $\langle D_x f, D_y f, D_z f\rangle$ which is $\left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z} \right\rangle$.

SECTION 2.~~4~~: ~~PROPERTIES OF THE~~ PARTIAL DERIVATIVES

THM: PROPERTIES OF THE DERIVATIVE:

Assume $f, g: U \subset \mathbb{R}^n \to \mathbb{R}^m$ diff at $\vec{x}_0$, let c be a constant.

- Const Rule: $h(\vec{x}) = c f(\vec{x})$ Then $Dh(\vec{x}_0) = c Df(\vec{x}_0)$
- Sum Rule: $h(\vec{x}) = f(\vec{x}) + g(\vec{x})$ Then $Dh(\vec{x}_0) = Df(\vec{x}_0) + Dg(\vec{x}_0)$
- Product Rule: $m=1$, $h(\vec{x}) = f(\vec{x})g(\vec{x})$ Then $Dh(\vec{x}_0) = Df(\vec{x}_0)\,\cancel{g(\vec{x}_0)}\,g(\vec{x}_0) + f(\vec{x}_0)Dg(\vec{x}_0)$
- Quotient Rule: $m=1$, $h(\vec{x}) = f(\vec{x})/g(\vec{x})$ Then $Dh(\vec{x}_0) = \dfrac{Df(\vec{x}_0)g(\vec{x}_0) - f(\vec{x}_0)Dg(\vec{x}_0)}{g(\vec{x}_0)^2}$

<u>Example</u>: $f(x,y,z) = x^2+y^2+z^2 \quad g(x) = x^3+y^3$

$$h(x,y,z) = f(x,y,z)\,g(x,y,z) = x^5 + x^3y^2 + x^3z^2 + x^2y^3 + y^5 + z^2y^3$$

$$Dh(x,y,z) = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}\right) = \big(5x^4 + 3x^2y^2 + 3x^2z^2 + 2xy^3,\ 2x^3y + 3x^2y^2 + 5y^4 + 3z^2y^2,\ 2x^3z + 2zy^3\big)$$

$$Df(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 2y, 2z) \quad f(x,y,z) = x^2+y^2+z^2$$

$$Dg(x,y,z) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = (3x^2, 3y^2, 0)$$

$$Df(x,y,z)g(x,y,z) + f(x,y,z)Dg(x,y,z) = (2x,2y,2z) * (x^3+y^3\cancel{+z^3}) + (x^2+y^2+z^2) * (3x^2, 3y^2, 0)$$

$$\cancel{= 5x^4 + 2x^3y + 2x^2z}$$

$$= \big(5x^4 + 3x^2y^2 + 3x^2z^2 + 2xy^3,\ 5y^4 + 2x^3y + 3x^2y^2 + 3z^2y^2,\ 2x^3z + 2y^3z\big)$$

↳ Note agree. Faster is to note symmetry in $x \to y$ and $y \to x$

We end with the homework problems, followed by five pages of advanced,

OPTIONAL

appendices. This material

WILL NOT

be on any exam, and is solely for personal edification.

Note first order partials mean one derivative.

Homework: pg 928: #1, #4, #5, #22, #25, #33, #36, #63 (is this surprising?)

Suggested: #55, #57, #58, #68

APPENDIX 1: OPTIONAL

THM: $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ diff at $\vec{x}_0 \Rightarrow f$ is cont at $\vec{x}_0$

Proof: $f(\vec{x}) = f(\vec{x}_0) + Df(\vec{x}_0)(\vec{x}-\vec{x}_0) + E_{\vec{x}_0}(\vec{x})$

with $\lim\limits_{x \to \vec{x}_0} \|E_{\vec{x}_0}(\vec{x})\| / \|\vec{x}-\vec{x}_0\| = 0$

THM: $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ st all partial derivs $\partial f_i / \partial x_j$ exist and continuous in a nbhood of $\vec{x} \in U$. Then f is differentiable at $\vec{x}$.

↳ Recall $f(x,y) = (xy)^{1/3}$

↳ partial derivs not continuous at $(0,0)$: $\frac{\partial f}{\partial x} = \frac{1}{3} x^{-\frac{2}{3}} y^{\frac{1}{3}}$

and if $y=x$ then $\frac{\partial f}{\partial x}(x,x) = \frac{1}{3} x^{-1/3}$

Following valid: Continuous Partial Derivatives $\Longrightarrow$ Function is Differentiable $\Longrightarrow$ Partial Derivatives Exist

Each converse statement, obtained by reversing arrows, can fail.

Say a function is C^1 if it is continuously differentiable, C^2 if twice continuously differentiable, and so on.

APPENDIX ~~2~~ 1 : OPTIONAL

PROOF PARTIALS EXIST AND CONT $\Rightarrow$ FN IS DIFFERENTIABLE

- Will do $n=2$, generalization possible
- Highlights an important technique: adding zero.
- Natural guess for derivative: try it:

$$f(x,y) - f(0,0) - \left[\frac{\partial f}{\partial x}(0,0)\right]x - \left[\frac{\partial f}{\partial y}(0,0)\right]y$$

$$= \Big\{\underbrace{f(x,y) - f(0,y)}_{MVT} - \left[\frac{\partial f}{\partial x}(0,0)\right]x\Big\} + \Big\{\underbrace{f(0,y) - f(0,0)}_{MVT} - \left[\frac{\partial f}{\partial y}(0,0)\right]y\Big\}$$

$$= \left\{\left[\frac{\partial f}{\partial x}(c,y)\right]x - \left[\frac{\partial f}{\partial x}(0,0)\right]x\right\} + \left\{\left[\frac{\partial f}{\partial y}(0,\tilde{c})\right]y - \left[\frac{\partial f}{\partial y}(0,0)\right]y\right\}$$

$$= \underbrace{\left[\frac{\partial f}{\partial x}(c,y) - \frac{\partial f}{\partial x}(0,0)\right]}_{\to\, 0 \text{ (by continuity)}} x + \underbrace{\left[\frac{\partial f}{\partial y}(0,\tilde{c}) - \frac{\partial f}{\partial y}(0,0)\right]}_{\to\, 0 \text{ (by continuity)}} y$$

As $\frac{\|x\|}{\|(x,y)\|}$ and $\frac{\|y\|}{\|(x,y)\|}$ bounded, above tends to zero even after dividing by $\|(x,y)\| = \|(x,y) - (0,0)\|$. ∎

~~[illegible]~~ APPENDIX 2: OPTIONAL

Proof of product rule

↳ Use powerful technique of adding zero!

$$\lim_{\vec{x}\to\vec{x}_0} \frac{\| f(\vec{x})g(\vec{x}) - f(\vec{x}_0)g(\vec{x}_0) - [Df(\vec{x}_0)g(\vec{x}_0) + f(\vec{x}_0)Dg(\vec{x}_0)](\vec{x}-\vec{x}_0)\|}{\|\vec{x} - \vec{x}_0\|}$$

↳ add $-f(\vec{x}_0)g(\vec{x}) + f(\vec{x}_0)g(\vec{x})$

use triangle inequality: $\|\vec{P} + \vec{Q}\| \leq \|\vec{P}\| + \|\vec{Q}\|$

$$\leq \lim_{\vec{x}\to\vec{x}_0} \frac{\|[f(\vec{x}) - f(\vec{x}_0) - Df(\vec{x}_0)(\vec{x}-\vec{x}_0)]\|\,|g(\vec{x})|}{\|\vec{x} - \vec{x}_0\|}$$

$$+ \lim_{\vec{x}\to\vec{x}_0} \frac{\| Df(\vec{x}_0)(g(\vec{x}) - g(\vec{x}_0))(\vec{x}-\vec{x}_0)\|}{\|\vec{x}-\vec{x}_0\|}$$

$$+ \lim_{\vec{x}\to\vec{x}_0} \frac{\|f(\vec{x}_0)\|\,\|g(\vec{x}) - g(\vec{x}_0) - Dg(\vec{x}_0)(\vec{x}-\vec{x}_0)\|}{\|\vec{x}-\vec{x}_0\|}$$

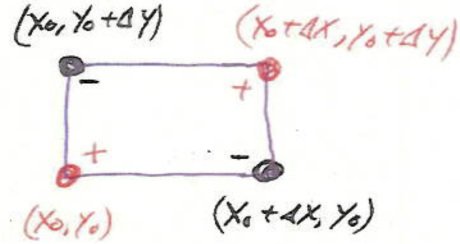
and each piece tends to zero. ∎

APPENDIX 3: OPTIONAL

~~Section 3.4 (2017)~~

Proof of Equality of Mixed Partials Thm

$$S(\Delta x, \Delta y) = f(x_0+\Delta x,\, y_0+\Delta y) - f(x_0+\Delta x,\, y_0) - f(x_0,\, y_0+\Delta y) + f(x_0, y_0)$$



Holding $y_0, \Delta y$ fixed: $g(x) := f(x, y_0+\Delta y) - f(x, y_0)$

$$\Rightarrow S(\Delta x, \Delta y) = g(x_0+\Delta x) - g(x_0)$$

$$= g'(\tilde{x})\,\Delta x \qquad \tilde{x} \text{ b/w } x_0 \text{ and } x_0+\Delta x \ (\underline{MVT})$$

$$= \left[\frac{\partial f}{\partial x}(\tilde{x}, y_0+\Delta y) - \frac{\partial f}{\partial x}(\tilde{x}, y_0)\right]\Delta x$$

$$= \frac{\partial^2 f}{\partial y\, \partial x}(\tilde{x}, \tilde{y})\, \Delta x \Delta y \quad \text{by MVT}$$

As $\partial^2 f/\partial y \partial x$ is cont

$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \lim_{(\Delta x, \Delta y)\to(0,0)} \frac{S(\Delta x, \Delta y)}{\Delta x \Delta y}$$

As $S(\Delta x, \Delta y)$ is symmetric, similar calculation shows above is also $\partial^2 f/\partial x \partial y$, completing the proof. ∎

See book for many famous partial diff eqs

~~Homework: #5, #4~~

~~Suggested: #16, #19~~

APPENDIX B: OPTIONAL

~~[illegible]~~t: ITERATED PARTIAL DERIVS

GOAL: When does $\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$ equal $\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$?

Notation: $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$, $\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right)$

Computing easy: $f(x,y) = x^2 y \sin x$ Then

$$\partial f/\partial x = 2xy\sin x + x^2 y \cos x$$

$$\partial f/\partial y = x^2 \sin x$$

$$\partial^2 f/\partial x \partial y = 2x\sin x + x^2\cos x = \partial^2 f/\partial y\partial x$$

Defn: C^2: f is of class C^2 if all partial derivs $\partial f/\partial x_i$ exist and further each of these have continuous partial derivatives.

THM: EQUALITY OF MIXED PARTIAL DERIVS
If f is of class C^2 Then $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$

↳ Not always equal

↳ Exercise 24: $f(x,y) = \begin{cases} xy(x^2-y^2)/(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

↳ Show $\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$ but $\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1$

SECTION 12.5: MULTIVARIABLE OPTIMIZATION PROBLEMS

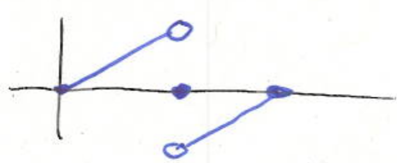
Say f attains its <u>global (or absolute) maximum</u> at $\vec{x}_0$ if for all $\vec{x}$, $f(\vec{x}) \leq f(\vec{x}_0)$; similarly <u>global (or absolute) minimum</u> means $f(\vec{x}_0) \leq f(\vec{x})$.

We have a <u>local maximum</u> if for all $\vec{x}$ close to $\vec{x}_0$, $f(\vec{x}) \leq f(\vec{x}_0)$.

KEY RESULT: For any nice, closed region (containing boundary and points inside), a continuous fn attains its max and min

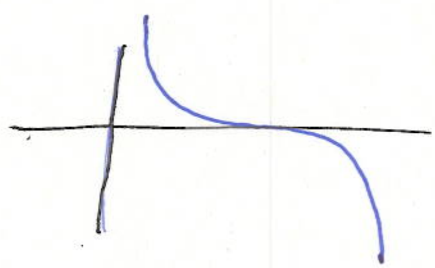
Check conditions: all important:

- Violate continuity:



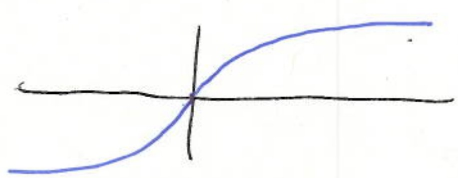
no max
no min

- Endpoints missing:
  $f(x) = \frac{1}{x} + \frac{1}{x-1}$



no max
no min

- Unbounded region



no max
no min
$x>0$: $f(x) = \frac{x^2}{1+x^2}$

Key Question: How to find local extrema (max/min)?

Good idea to revisit one dimension.

One-dim: (1) look for places $f'(x)=0$ (critical points) for interior extrema

(2) Check boundary.

↳ not bad in one-dim as at most two points to check. In higher dims have infinitely many and much harder; leads to Lagrange multipliers of Section 12.9.

Higher Dim: Assume function is continuously differentiable, so $\langle \partial f/\partial x_1, \dots, \partial f/\partial x_n \rangle$ exist; This is grad(f) or ∇f or Df. At an interior extremum have grad$(f) = \vec{0}$.

Will give a sketch of the proof. A full proof requires a bit more about the interplay between the partial derivatives, directional derivatives and differentiability.

We see that $\vec{x}_0$ is an extremum, and want to show grad$(f) = \vec{0}$; This will show that finding all points where grad(f) vanishes gives us a list of candidates for extrema. As have extrema, if move parallel to a coordinate axis must always be less if have a max or always more if have a min. Regarding as a function of one variable, we see all the partial derivs must vanish!

Can interpret as tangent plane is horizontal.

Difficulty in finding is solving systems of simultaneous equations. If linear, learn how to do in a linear algebra class.

Call $\vec{x}_0$ such that $(Df)(\vec{x}_0) = \vec{0}$ a critical point.

Ex: Find critical points of $f(x,y) = x^2+y^2-6x+2y+5$

Have $\frac{\partial f}{\partial x} = 2x-6 = 0 \implies x = 3$

$\frac{\partial f}{\partial y} = 2y+2 = 0 \implies y = -1$

Only critical point is $(3,-1)$.

Ex: Find critical points of $f(x,y) = x^2y + 3xy^2$

Have $\frac{\partial f}{\partial x} = 2xy + \cancel{6}\, 3y^2 = 0$

$\frac{\partial f}{\partial y} = x^2 + 6xy = 0$

lots of ways to do algebra

Get $xy = -\frac{1}{6}x^2 = -\frac{3}{2}y^2$

so $x^2 = 9y^2$ or $x = \pm 3y$

If $x=3y$ get $6y^2+3y^2=0 \to y=0$

$9y^2+18y^2=0 \to y=0$

If $x=-3y$ get $-6y^2+3y^2=0 \to y=0$

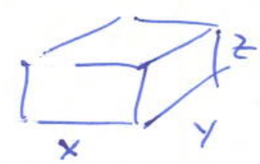
$9y^2-18y^2=0 \to y=0$

So only soln is $(0,0)$.

SECTION 12.5 (CONT)

Not yet able to do "Farmer Brown" problem as a 2-dim problem: 40 meters of fence, what is largest area enclosable by a rectangle? Equiv to maximize xy subject to $2x+2y=40$.

<u>Ex</u>: Box must have volume of $48\,ft^3$. Front costs \$1/ft^2, as does back; top and bottom cost \$2/ft^2, and ends \$3/ft^2. Find cheapest box. (Example 7, pg 937).



Volume is $xyz = 48$ so $z = 48/xy$

Cost is $1 \cdot 2xz + 2 \cdot 2xy + 3 \cdot 2yz$

$$C(x,y) = \frac{96}{y} + 4xy + \frac{288}{x}$$

$$\frac{\partial C}{\partial x} = 4y - \frac{288}{x^2} = 0 \Rightarrow 4xy = \frac{288}{x}$$

$$\frac{\partial C}{\partial y} = 4x - \frac{96}{y^2} = 0 \Rightarrow 4xy = \frac{96}{y}$$

$\Rightarrow$ We have $\frac{3}{x} = \frac{1}{y}$ or $x = 3y$

Thus $4(3y) - \frac{96}{y^2} = 0 \Rightarrow 12y^3 = 96 \Rightarrow y = 2,\ x = 6$

Cost is \$144.

Homework: Pg 940: #5, #11, #29, #61

Do Method of Least Squares: See handout

Suggested: #10, #17, #46

CHAPTER 12: SEC 6: INCREMENTS AND LINEAR APPROX

Goal: Replace complicated fns with simple ones. Allows us to estimate how function changes. Even just doing a linear approx suffices for many applications. Leads to Taylor Series.

KEY RESULT: Let f be a nice, continuously differentiable function. Then $f(x+\Delta x, y+\Delta y) \approx f(x,y)$. This is a zeroth order approximation. First order is

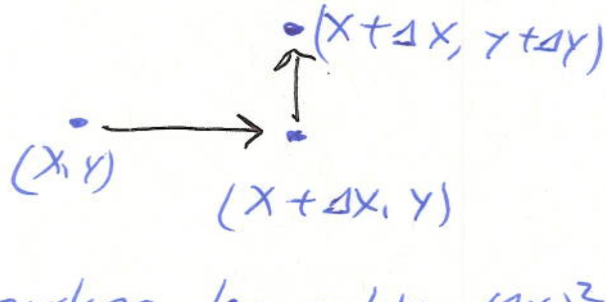
$$f(x+\Delta x, y+\Delta y) = f(x,y) + \frac{\partial f}{\partial x}(x,y)\cdot\Delta x + \frac{\partial f}{\partial y}(x,y)\,\Delta y$$

This generalizes tangent line from one-dim.

↳ if $\Delta y = 0$, clear as reduce to one-var

↳ if $\Delta x = 0$, clear as reduce to one-var.

take path



The error includes terms like $(\Delta x)^2$ or $\Delta x \Delta y$, much smaller than Δx or Δy, and can be ignored.

For example, say $f(x,y) = \sqrt{x^2+y^2}$. What is $f(3.1, 4.9)$?

Try $(x_0, y_0) = (3,4)$ as $f(3,4) = 5$ is nice and $(3,4)$ is close to our point $(3.1, 4.9)$. We have $\Delta x = .1$ and $\Delta y = -.1$, and thus

$$f(3.1, 4.9) \approx f(3,4) + \frac{\partial f}{\partial x}(3,4)(.1) + \frac{\partial f}{\partial x}(3,4)(-.1)$$

$$\text{as} \quad \frac{\partial f}{\partial x} = \frac{1}{\cancel{2}}(x^2+y^2)^{-\frac{1}{2}}(\cancel{2}x)$$

$$\frac{\partial f}{\partial y} = \frac{1}{\cancel{2}}(x^2+y^2)^{-\frac{1}{2}}(\cancel{2}y)$$

$$\text{we see} \quad \frac{\partial f}{\partial x}(3,4) = \frac{3}{5}, \quad \frac{\partial f}{\partial y}(3,4) = \frac{4}{5}$$

$$f(3.1, 4.9) \approx 5 + \frac{3}{5}(.1) + \frac{4}{5}(-.1) = 4.98$$

actual answer is $f(3.1, 4.9) = 4.98197$

WOW: $\Delta x, \Delta y$ of size $\frac{1}{10}$ and answer is only off by $\frac{2}{100}$ - not a coincidence: error should be on the order of $(\Delta x)^2 + \Delta x \Delta y + (\Delta y)^2$

SEC 12.6 (CONT)

Book (finally!) introduces notion of the gradient vector.

$$(\nabla f)(\vec{x}) = \langle (D_1 f)(\vec{x}), \dots, (D_n f)(\vec{x}) \rangle = \left\langle \frac{\partial f}{\partial x_1}(\vec{x}), \dots, \frac{\partial f}{\partial x_n}(\vec{x}) \right\rangle$$

Have the approximation

$$f(\vec{x} + \vec{h}) \simeq f(\vec{x}) + \underbrace{(\nabla f)(\vec{x})}_{\text{vector}} \cdot \vec{h}$$

Ignore rest of section: will discuss if have time for Taylor Series.

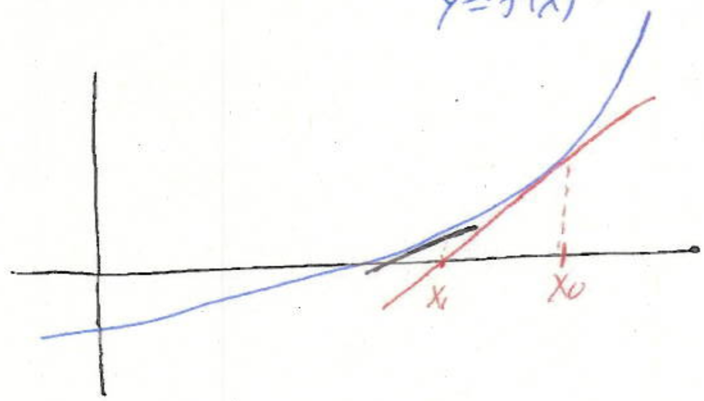
<u>Homework</u>: Pg 949: #18, #23

Next page is on a wonderful application of linearization, Newton's Method.

NEWTON'S METHOD

Find root of $f(x)=0$

Assume f is "nice" and differentiable



Step 1: Guess x_0 for root. If $f(x_0)=0$ done else continue

Step 2: Consider point $(x_0, f(x_0))$ on graph. Tangent line has slope $f'(x_0)$. Approx fn by tangent line, see where it crosses the x-axis, call that x_1.

↳ tangent line: $y - f(x_0) = f'(x_0)(x-x_0)$

intercept: $0 - f(x_0) = f'(x_0)(x_1 - x_0) \rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Lather, rinse, repeat: incredible fast

↳ Numerous applications

↳ Fractals

If $f(x)=x^2-3$ then $x_{n+1} = x_n - \frac{x_n^2-3}{2x_n} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right)$

↳ sequence converges really fast!

Much better than Divide and Conquer: Halve error each iteration

SECTION 12.7: MULTIVARIATE CHAIN RULE

To do the subject full justice helps to know linear algebra and matrices. For simplicity will assume afunctions are $f: \mathbb{R}^n \to \mathbb{R}$ and not $f: \mathbb{R}^n \to \mathbb{R}^m$.

Review: One-Variable

$$A(x) = f(g(x)) \text{ then } \frac{dA}{dx} = \cancel{\frac{d}{dt}}\, f'(g(x)) \cdot g'(x)$$

Often write $f(u)$ with $u = g(x)$

Then $A'(x) = \dfrac{df}{du} \cdot \dfrac{du}{dx}$

Chain Rule

Suppose $x = g(t)$ and $y = h(t)$ and $w(t) = f(x, y)$ so $w(t) = f(g(t), h(t))$. Then $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$

Not a big fan of this notation — I think it is close to maximizing confusion!

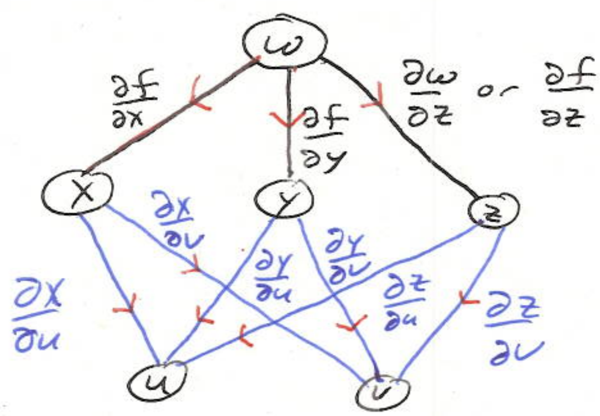
Chain Rule (Take Two)

Assume x, y, z functions of u and v

↳ Either write
$$x = g(u,v) \quad \text{or} \quad x(u,v)$$
$$y = h(u,v) \quad \text{or} \quad y(u,v)$$
$$z = k(u,v) \quad \text{or} \quad z(u,v)$$

Consider $w(u,v) = f(x(u,v), y(u,v), z(u,v))$

Then
$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$



If we want $\frac{\partial w}{\partial u}$ we just go down the graph along all paths ending in u

Note: big error is to write $\frac{df}{dx}$ or $\frac{dx}{du}$ instead of $\frac{\partial f}{\partial x}$ and $\frac{\partial x}{\partial u}$; d means full derivative.

Ex: Consider $w = x\cos(x^2+y^2)$ with $x = r\cos\theta$ and $y = r\sin\theta$. Find $\frac{\partial w}{\partial \theta}$ and $\frac{\partial w}{\partial r}$.

$f(x,y) = x\cos(x^2+y^2)$

$w(r,\theta) = f(x(r,\theta), y(r,\theta))$

$= r\cos\theta \cdot \cos(r^2)$

Note can find partials DIRECTLY!

$\frac{\partial w}{\partial r} = \cos\theta \cdot \cos(r^2) - 2r^2\cos\theta\cos(r^2)$

$\frac{\partial w}{\partial \theta} = -r\sin\theta \cdot \cos(r^2)$

Using chain rule!

$\frac{\partial w}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$

$\boxed{\text{Important: Evaluate } \frac{\partial f}{\partial x} \text{ at } (x(r,\theta), y(r,\theta)) \text{ and } \frac{\partial x}{\partial \theta} \text{ at } (r,\theta)}$

$= \cancel{r\cos(x^2+y^2)}$

$\frac{\partial f}{\partial x} = \cos(x^2+y^2) - 2x^2\cos(x^2+y^2)$

$\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) = \cos(r^2) - 2r^2\cos^2\theta \cdot \cos(r^2)$

$\frac{\partial f}{\partial y} = -2xy\cos(x^2+y^2)$

$\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta) = -2r^2\cos\theta\sin\theta \cdot \cos(r^2)$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial w}{\partial \theta} = \left[\cos(r^2) - 2r^2\cos^2\theta\cos(r^2)\right]\left[-r\sin\theta\right]$$

$$+ \left[-2r^2\cos\theta\sin\theta\cos(r^2)\right]\left[r\cos\theta\right]$$

$$= \cos(r^2)\left[-r\sin\theta\right] + 0$$

$$= -r\sin\theta\cos(r^2)$$

In this case it's easier to substitute directly!

A mostly complete proof is given in the typed Appendix. We'll just do the 1-dim case in class

Homework: Pg 960: #2, ~~#5~~ #8, #34, #41

Suggested: #38, #53

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Math 350: The Chain Rule

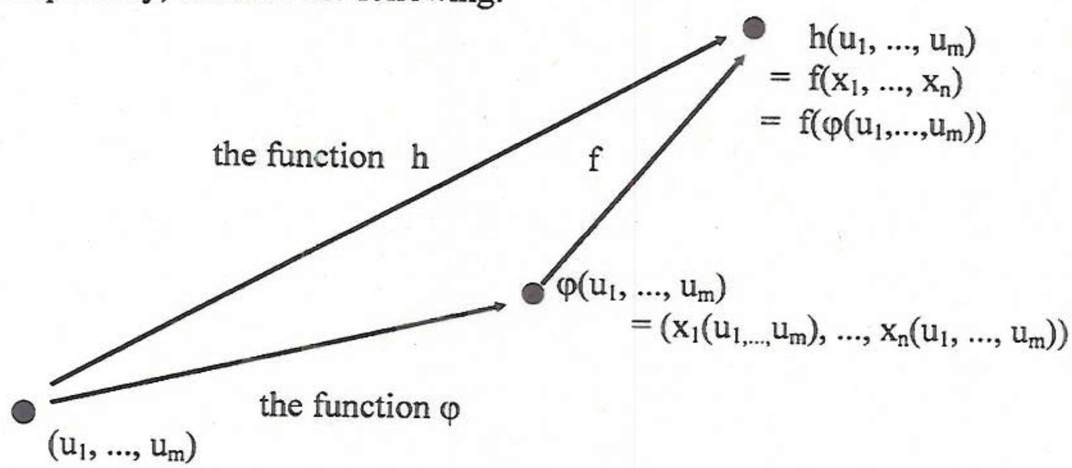
The Chain Rule is a very useful tool for analyzing the following: Say you have a function f of $(x_1, x_2, ..., x_n)$, and these variables are themselves functions of $(u_1, u_2, ..., u_m)$. How does our function f change as we vary u_1 thru u_m??? We'll state and explain the Chain Rule, and then give a DIFFERENT PROOF FROM THE BOOK, using *only* the definition of the derivative. This is a slight modification of notes I wrote years ago for a similar class at Princeton.

(I). Statement:

We'll state the Chain Rule. First, some notation:

Let $h: \mathbb{R}^m \to \mathbb{R}$	say h is a function of $(u_1, u_2, ..., u_m)$
$f: \mathbb{R}^n \to \mathbb{R}$	say f is a function of $(x_1, x_2, ..., x_n)$
$\varphi: \mathbb{R}^m \to \mathbb{R}^n$	say φ is a function of $(u_1, u_2, ..., u_m)$

Graphically, we have the following:



Our function h lives on $\mathbb{R}^m$. So, you give it an m-tuple, $(u_1, ..., u_m)$, and it will give you a real number back. The function f lives on $\mathbb{R}^n$. If you give it an n-tuple, $(x_1, ..., x_n)$, it will give you back a number. And what of the variables x_1 thru x_n? Well, they can be thought of as functions on $\mathbb{R}^m$: you give them an m-tuple, $(u_1, ..., u_m)$, and they'll return a number.

We cannot look at $f(x_1(u_1, ..., u_m))$, for f composed with x_1 doesn't make sense: x_1 gives us just ONE number; f needs n numbers.

What do we do? Remember, we're trying to understand the beast:

$$h(u_1, ..., u_m) = f(x_1(u_1, ..., u_m), ..., x_n(u_1, ..., u_m))$$

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We define an auxiliary function, φ, to help us. What will $\varphi(u_1, \ldots, u_m)$ be? Whatever we want. We now look for something useful. Look at the Right Hand Side above—wouldn't it be nice if we could choose a φ that would give us this? We can! Just let:

$$\varphi(u_1, \ldots, u_m) = (x_1(u_1, \ldots, u_m), x_2(u_1, \ldots, u_m), \ldots, x_n(u_1, \ldots, u_m))$$

Now we can write $h = f\circ\varphi$, f composed with φ. The advantage of this is that we know that often compositions of nice functions are nice: if we compose two continuous functions, we get a continuous function. In one dimension, we have the 1-dimensional chain rule for compositions. We hope to be able to do something similar here. Anyway, here is the long awaited statement of:

The Chain Rule:

$$\begin{aligned}(Dh)(u_1, \ldots, u_m) &= (Df)(\varphi\,(u_1, \ldots, u_m))\,(D\varphi)(u_1, \ldots, u_m)\\ &= (Df)(x_1, \ldots, x_n)\,(D\varphi)(u_1, \ldots, u_m)\end{aligned}$$

Let's write out what this is: for the sake of space, I will not explicitly write WHERE the functions are being evaluated—we always evaluate h at $(u_1, \ldots, u_m)$, f at $\varphi(u_1, \ldots, u_m) = (x_1(u_1, \ldots, u_m), \ldots, x_n(u_1, \ldots, u_m))$, and φ at $(u_1, \ldots, u_m)$.

The Chain Rule:

$$\mathbf{Dh} = \left(\frac{\partial h}{\partial u_1}, \frac{\partial h}{\partial u_2}, \ldots, \frac{\partial h}{\partial u_m}\right) \quad \mathbf{Df} = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

$D\varphi$ is more complicated: Unlike Df and Dh, which are vectors, $D\varphi$ is a matrix quantity. This is because φ is really a collection of m functions,

$$\begin{aligned}\varphi(u_1, \ldots, u_m) &= (\varphi_1(u_1, \ldots, u_m), \ldots, \varphi_n(u_1, \ldots, u_m))\\ &= (x_1(u_1, \ldots, u_m), \ldots, x_n(u_1, \ldots, u_m))\end{aligned}$$

We obtain:

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$$(D\varphi) = \begin{pmatrix} \frac{\partial x_1}{\partial u_1}, & \frac{\partial x_1}{\partial u_2}, & \dots, & \frac{\partial x_1}{\partial u_m} \\ \frac{\partial x_2}{\partial u_1}, & \frac{\partial x_2}{\partial u_2}, & \dots, & \frac{\partial x_2}{\partial u_m} \\ & & & \\ \frac{\partial x_n}{\partial u_1}, & \frac{\partial x_n}{\partial u_2}, & \dots, & \frac{\partial x_n}{\partial u_m} \end{pmatrix}$$

Combining the above expressions for Dh, Df, and $D\varphi$ yields:

Chain Rule:

$$\frac{\partial h}{\partial u_1} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial u_1} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial u_1}$$

$$\frac{\partial h}{\partial u_2} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_2} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial u_2} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial u_2}$$

and so on till

$$\frac{\partial h}{\partial u_m} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_m} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial u_m} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial u_m}$$

(II). One Dimensional Case:

OK. We now have the above formula, but WHERE DID IT COME FROM? Let's go back to one-dimension, and take a look at what is happening:

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Translating from our language to what we spoke in High School:

$$h(u) = f(\varphi(u)) \rightarrow h'(u) = f'(\varphi(u))\,\varphi'(u)$$

How do we go about proving this? Always go back to what you know: here we're trying to find the derivative. Okay, so, let's recall the definition of the derivative. We know that. The derivative is defined by:

$$\begin{aligned} h'(u) &= \lim_{y \to u} \{h(y) - h(u)\} / \{y-u\} \\ &= \lim_{y \to u} \{f(\varphi(y)) - f(\varphi(u))\} / \{y-u\} \\ &= \lim_{y \to u} \frac{f(\varphi(y)) - f(\varphi(u))}{\varphi(y) - \varphi(u)} * \frac{\varphi(y) - \varphi(u)}{y - u} \end{aligned}$$

All we did was multiply by 1 in a very clever way. Why did we do this? Our function f is a function of one variable. The second term looks like $\varphi'(u)$ in the limit, and the first term looks like f' evaluated at $\varphi(u)$. As the two limits exist, the limit of the product is the product of the limits, so we can conclude:

$$h'(u) = f'(\varphi(u))\,\varphi'(u)$$

Why isn't this proof rigorous? The definition of $f'(z)$ is the following:

$$f'(z) = \lim_{w \to z} \{f(w) - f(z)\} / \{w - z\}$$

We cheated in the above: this limit has to hold *FOR ALL* paths where w heads to z. We didn't consider *all* paths, only a special path. But maybe this isn't too bad: if the limit exists, then it doesn't matter *WHICH* path we take. In better words: look, I know $f'(z)$ exists, and I know the value is *INDEPENDENT* of the path I take. So why don't I just make life easy on myself and take this nice path? What a great idea! We leave for the interested, rigorous reader what to do if $\varphi(y)$ equals $\varphi(u)$ infinitely often (this cannot happen if $\varphi'(u) \neq 0$). Hint: go back to the definition of $\partial h/\partial u$ and calculate it directly, going along points where $\varphi(y) = \varphi(u)$.

(III). Higher Dimensions:

We now argue as in above, but in higher dimensions. To make things easier to view, let's just look at $n = 3$, $m = 2$, so we have (x_1, x_2, x_3), which we denote by (x, y, z) for convenience, and (u_1, u_2), which we denote by (u, w).

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$$h(u,w) = f(x(u,w), y(u,w), z(u,w))$$

We calculate $\partial h/\partial u$, at the point (u,w), and compare with $\partial h/\partial u_1$ from page 3.

$$\partial h/\partial u = \lim_{v \to u} \{ h(v, w) - h(u, w) \} / \{ v - u \}$$

$$= \lim_{v \to u} \frac{f(x(v,w), y(v,w), z(v,w)) - f(x(u,w), y(u,w), z(u,w))}{v - u}$$

So, we start at the point $(x(u,w), y(u,w), z(u,w))$ and we finish at the point $(x(v,w), y(v,w), z(v,w))$. We cannot directly mimic the 1-dimensional case, but what if our starting point were $(x(u,w), y(v,w), z(v,w))$? Then all we would've done is change the x-coordinate of the 3-tuple, and we could multiply and divide by $x(v,w) - x(u,w)$. We would then have:

$$\partial f/\partial x \quad \partial x/\partial u$$

Sadly, life isn't quite that simple: we don't have that as our starting point. But, what if we added and subtracted $f(x(u,w), y(v,w), z(v,w))$ in the numerator? Then we would get:

$$\frac{\partial h}{\partial u} = \lim_{v \to u} \frac{f(x(v,w),y(v,w),z(v,w)) - f(x(u,w),y(v,w),z(v,w))}{v - u} +$$

$$\lim_{v \to u} \frac{f(x(u,w),y(v,w),z(v,w)) - f(x(u,w),y(u,w),z(u,w))}{v - u}$$

We now multiply the first term by 1:

$$\frac{\partial h}{\partial u} = \lim_{v \to u} \frac{f(x(v,w),y(v,w),z(v,w)) - f(x(u,w),y(v,w),z(v,w))}{x(v,w) - x(u,w)} * \frac{x(v,w) - x(u,w)}{v - u}$$

$$+ \lim_{v \to u} \frac{f(x(u,w),y(v,w),z(v,w)) - f(x(u,w),y(u,w),z(u,w))}{v - u}$$

APPENDIX 3: OPTIONAL

$$\frac{\partial h}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \lim_{v \to u} \frac{f(x(u,w),y(v,w),z(v,w)) - f(x(u,w),y(u,w),z(u,w))}{v - u}$$

Now we just repeat what we did before! We've got two points, start at (x(u,w),y(u,w),z(u,w)), end at (x(u,w),y(v,w),z(v,w)). Again, what if our first point were (x(u,w),y(u,w),z(v,w))? Then all we would've done is change the y-coordinate of the 3-tuple, and we could multiply and divide by y(v,w) - y(u,w). We would then (in the limit) get $\partial f/\partial y\ \partial y/\partial u$, plus another term, the difference of the point we added and our *true* first point. Let's do it!

$$\frac{\partial h}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \lim_{v \to u} \frac{f(x(u,w),y(v,w),z(v,w)) - f(x(u,w),y(u,w),z(v,w))}{v - u}$$
$$+ \lim_{v \to u} \frac{f(x(u,w),y(u,w),z(v,w)) - f(x(u,w),y(u,w),z(u,w))}{v - u}$$

Multiplying the first limit by {y(v,w) - y(u,w)} / {y(v,w) - y(u,w)} we get:

$$\frac{\partial h}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \lim_{v \to u} \frac{f(x(u,w),y(u,w),z(v,w))-f(x(u,w),y(u,w),z(u,w))}{v - u}$$

Multiplying the last term by {z(v,w) - z(v,w)} / {z(v,w) - z(v,w)}, we get that this term, in the limit, is just $\partial f/\partial z\ \partial z/\partial u$.

Hence we get:

$$\boxed{\frac{\partial h}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} \quad \textbf{which is The Chain Rule!}}$$

SECTION 12.8: DIRECTIONAL DERIVATIVES AND THE GRADIENT VECTOR

~~SECTION 2.6: GRADIENTS AND DIRECTIONAL DERIVATIVES~~

- Gradient: $\text{grad}(f) = \nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

  ↳ This is derivative Df written as a vector

- Directional Deriv: Directional derivative of f at $\vec{x}$ along vector $\vec{v}$ (usually unit length) is $\frac{d}{dt} f(\vec{x} + t\vec{v})$. If $\|\vec{v}\| = 1$ say the directional derivative in the direction of $\vec{v}$.

  ↳ if $\|\vec{v}\| \neq 1$, changing scale

  ↳ Equivalent to $\lim\limits_{h \to 0} \frac{f(\vec{x} + h\vec{v}) - f(\vec{x})}{h}$

THM: $f: \mathbb{R}^n \to \mathbb{R}$ diff. Then all derivs exist and the dir deriv in the dir of $\vec{v}$ is $D_{\vec{v}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{v} = \frac{\partial f}{\partial x_1}(\vec{x}) v_1 + \dots + \frac{\partial f}{\partial x_n}(\vec{x}) v_n$

Proof: Let $c(t) = \vec{x} + t\vec{v}$ and use chain rule

Note $c'(0) = \vec{v}$, $c(0) = \vec{x}$

Thus $\frac{d}{dt} f(c(t)) = Df(c(0))\, c'(t) = \nabla f(c(0)) \cdot \vec{v}$ ∎

THM: GEOMETRIC INTERPRETATION:

If $\nabla f(\vec{x}_0) \neq \vec{0}$ then $\nabla f(\vec{x}_0)$ points in dir of fastest increase of f

Proof: Rate of change of f in unit dir $\vec{n}$ is $\nabla f(\vec{x}_0) \cdot \vec{n}$

Have magnitude $\|\nabla f(\vec{x}_0)\| \cdot \|\vec{n}\| \cdot |\cos\theta|$, largest when $\theta = 0, \pi$

so parallel ($\theta = 0$ gives max, $\theta = \pi$ gives min)

Do not want to resort to taking limits to compute.

While the defn of the directional deriv involves a limit, for computations use the gradient formulation.

EX: $f(x,y,z) = x^2 + 2y^2 + 3z^2$. Find how fast the function is increasing in the direction $\vec{u}$, where $\vec{u}$ is a unit vector in the direction $\langle 3, 4, 12 \rangle$.

First: $\vec{u} = \dfrac{\langle 3,4,12\rangle}{|\langle 3,4,12\rangle|} = \dfrac{\langle 3,4,12\rangle}{\sqrt{3^2+4^2+5^2}} = \left\langle \dfrac{3}{13}, \dfrac{4}{13}, \dfrac{5}{13} \right\rangle$

$\nabla f = \langle 2x, 4y, 6z \rangle$

Not enough information: need a point too!

Let's say the point is $P = (1,1,1)$.

$$
\begin{aligned}
(D_{\vec{u}} f)(P) &= (\nabla f)(P) \cdot \vec{u} \\
&= \langle 2, 4, 6 \rangle \cdot \left\langle \tfrac{3}{13}, \tfrac{4}{13}, \tfrac{5}{13} \right\rangle \\
&= \frac{6 + 16 + 30}{13} \\
&= 4
\end{aligned}
$$

THM: Gradient is normal to level surfaces. Specifically, $f: \mathbb{R}^n \to \mathbb{R}$ a C^1 map, $\vec{x}_0$ in level surface S defined by $f(\vec{x}) = k$. If curve $c(t)$ in S with $c(0) = \vec{x}_0$ and $\vec{v} = c'(t)$ is the tangent vector at $t=0$, then $\nabla f(\vec{x}_0) \cdot \vec{v} = \vec{0}$

Proof: Chain rule again!

Apply to $h(t) = f(c(t)) = k$ ∎

Note: These results will be <u>VERY</u> useful for max/min problems

<u>Defn: Tangent Plane</u>: S be surface $f(\vec{x}) = k$. The tangent plane at $\vec{x}_0$ is defined by $\{\vec{x} : \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0\}$

Often call ∇f the gradient vector field, means at point $\vec{P}$ draw vector $\nabla f(\vec{P})$.

↳ Example: Gravity: $\vec{F}(x,y,z) = -G \frac{m_1 m_2}{r^2} \vec{n} = \nabla\left(\frac{G m_1 m_2}{r}\right)$

where $\vec{n} = \vec{r}/r$, $\vec{r} = (x,y,z)$

Homework: Pg 971: #3, #10, #11, #19, #21, #29

Suggested: #40, #41, #60

~~Homework: #2ab, #4a, #6a, #16 (Ralph), #18~~

~~Suggested: #5a, #12, #17, #21, #23~~

~~Review Problems~~

~~HW: Pg 176: #23 homog~~
~~#47 chemistry~~

~~Suggested: Pg 176: #26, #41~~
~~#42~~

Section 12.9: Constrained Extrema + Lagrange Multipliers

Subtle, important point: one thing to find candidates for max/min; another to prove that the function attains a max/min. These proofs are sketched in the previous pages.

Method of Lagrange Multipliers

$f, g: U \subset \mathbb{R}^n \to \mathbb{R}$ are $\mathcal{C}^1$, $\vec{x}_0 \in U$ st $g(\vec{x}_0) = c$, $S = \{\vec{x} \in U : g(\vec{x}) = c\}$. Assume $\nabla g(\vec{x}_0) \neq \vec{0}$. Let $f|_S$ be f restricted to S. Then $f|_S$ has extremum at $\vec{x}_0$ if and only if there is a λ st $\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0)$.

Proof: S level set: $\frac{d}{dt}\left(g(c(t))\right) = \nabla g(\vec{x}_0) \cdot c'(0) = 0$

where $c'(0)$ is any vector tangent to S at $\vec{x}_0$.

↳ If f has max/min at $\vec{x}_0$ then $\frac{d}{dt}\left(f(c(t))\right) = \nabla f(\vec{x}_0) \cdot c'(0) = 0$

↳ Thus $\nabla f(\vec{x}_0)$ and $\nabla g(\vec{x}_0)$ perpendicular to all tangent dirs. Only one dir left, so $\nabla f(\vec{x}_0)$ and $\nabla g(\vec{x}_0)$ in that dir, and hence parallel! ∎

Interpretation: $\nabla g(\vec{x}_0)$ is normal to surface, says max/min means $\nabla f(\vec{x}_0)$ normal to surface: if not, flow in proper direction and increase.

Ex: "Farmer Brown" Problem: 40m of fence, love rectangles, what is largest enclosable area?

Soln: $A(x,y) = xy \qquad g(x,y) = 2x+2y-40 = 0$

$\nabla A = \lambda \nabla g$ and $g(x,y)=0$

$\nabla A = \left\langle \frac{\partial A}{\partial x}, \frac{\partial A}{\partial y} \right\rangle = \langle y, x \rangle$

$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle = \langle 2, 2 \rangle$

so: system of equations

$y = 2\lambda$

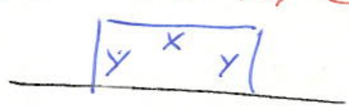
$x = 2\lambda$

$2x+2y-40=0$

↳ as know x and y in terms of λ: substitute

↳ find $2(2\lambda)+2(2\lambda)-40=0 \Rightarrow 8\lambda-40=0 \Rightarrow \lambda=5$

↳ Thus $x=y=10$

Note even though only care about x and y easy if first find λ, can do related problems: Farmer Bob: 

Now constraint is $x+2y=40$.

Can solve this by "symmetry": Aquaman reflection.

APPLICATION: $\frac{\partial f}{\partial x_i}(\vec{x}_0) = \lambda \frac{\partial g}{\partial x_i}(\vec{x}_0)$ and $g(\vec{x}_0) = c$

↳ have $n+1$ equations in $n+1$ variables: should be solvable

Example: $f(x,y) = 3x+2y \quad g(x,y) = 2x^2+3y^2 = 3$ (ellipse)

$\nabla f(x,y) = (3,2) \quad \nabla g(x,y) = (4x, 6y)$

↳ $(3,2) = \lambda(4x, 6y)$ and $2x^2+3y^2=3$

↳ $\left.\begin{array}{l} 4\lambda x = 3 \\ 6\lambda y = 2 \end{array}\right\} \xrightarrow{\text{ratio}} \frac{4x}{6y} = \frac{3}{2}$ or $y = \frac{4}{9}x$

$2x^2+3y^2=3$

↳ Thus $2x^2 + 3\left(\frac{4}{9}x\right)^2 = 3$

so $2x^2 + \frac{16}{27}x^2 = 3 \Longrightarrow 18x^2 = 81$

↳ Thus $x = \pm\frac{3\sqrt{2}}{2}, \quad y = \pm\frac{2\sqrt{2}}{3}$

Have four candidate points: check

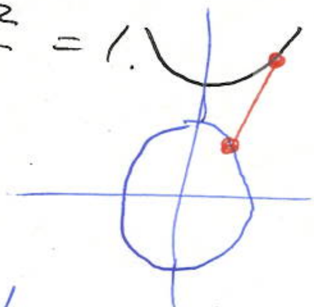
~~Note: will not do Section 3.5 (Inverse + Implicit Fn Thms). This important result with reading.~~

~~Homework: #8, #10~~

~~Suggested: #20, #27~~

In many problems you can make life <u>much</u> easier by looking at an alternate but equivalent function. For example, for distance look at distance squared.

<u>Ex</u>: Find the point on the parabola $y = x^2 + 4$ that is closest to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.



Drawing picture, answer is clear.

This can be a <u>very</u> hard problem: we have to look at general points on both the parabola <u>and</u> the ellipse.

We can fix a point (a,b) on parabola and then constrain (x,y) to lie on the ellipse. We then vary (a,b) and see which gives smallest value. Arg!

$d(x,y) = (x-a)^2 + (y-b)^2$: <u>Equiv to minimize dist squared</u>

subject to $g(x,y) = \frac{x^2}{4} + \frac{y^2}{9} = 1$

get $\nabla d = \lambda \nabla g, \quad g(x,y) = 0$

$$\nabla d = \langle 2(x-a),\ 2(y-b) \rangle$$
$$\nabla g = \left\langle \frac{x}{2},\ \frac{2y}{9} \right\rangle$$
$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Have $2(x-a) = \lambda \frac{x}{2}$, $2(y-b) = \lambda \frac{y}{2}$ } take ratio: $\frac{x-a}{y-b} = \frac{x}{y}$ unless $y=0, b$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

$xy - ay = xy - bx$

$bx = ay$

$x = \frac{a}{b} y$ unless $b=0$

feed back

$\left(\frac{a^2}{4b^2} + 9\right) y^2 = 1$

$y^2 = \frac{4b^2}{a^2 + 36b^2}$

$y = \pm \frac{2b}{\sqrt{a^2+36b^2}}$

Need to deal with these special cases

now need to find x- coord
then distance function
then minimize that...

This is becoming a nightmare!

$$\boxed{\begin{array}{l}\text{LAGRANGE MULTIPLIERS: TWO CONSTRAINTS:}\\ \text{To find extrema for } f(x_1,\dots,x_n) \text{ subject to}\\ g(x_1,\dots,x_n) \text{ and } h(x_1,\dots,x_n) \text{ must have real numbers } \lambda_1, \lambda_2\\ \text{such that } \nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h \text{ and } g(x_1,\dots,x_n)=0\\ \text{and } h(x_1,\dots,x_n)=0.\end{array}}$$

Linear Algebra greatly aids the interpretation.

Consider again parabola $v = u^2 + 4$ and ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Want to find the two closest points:

$$f(u,v,x,y) = (x-u)^2 + (y-v)^2 \quad \text{(distance squared)}$$

$$g(u,v,x,y) = u^2 - v + 4 = 0$$

$$h(u,v,x,y) = \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0$$

$$\nabla f = \langle -2(x-u), -2(y-v), 2(x-u), 2(y-v) \rangle$$

$$\nabla g = \langle 2u, -1, 0, 0 \rangle$$

$$\nabla h = \langle 0, 0, \frac{x}{2}, \frac{2y}{9} \rangle$$

$$\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h, \quad g(u,v,x,y) = h(u,v,x,y) = 0$$

① $-2(x-u) = \lambda_1 2u$

② $-2(y-v) = -\lambda_1$

③ $2(x-u) = \lambda_2 \frac{x}{2}$

④ $2(y-v) = \lambda_2 \frac{2y}{9}$

① and ③ give $\frac{\lambda_2}{2} x = -2\lambda_1 u$

② and ④ give $\frac{2\lambda_2}{9} y = \lambda_1$

So $\frac{\lambda_2 x}{2} = -2 \frac{2\lambda_2}{9} yu$

So either $\lambda_2 = 0$ (which is impossible as then $x = u$, $y = v$ and parabola and ellipse disjoint) or $x = -\frac{8}{9} yu$

Have $-2(x-u) = 2\lambda_1 u$ and $x = -\frac{8}{9}yu$

$$\cancel{2}\left(\frac{8}{9}yu + u\right) = 2\lambda_1 u$$

$$\left(\frac{8}{9}y + 1 - 2\lambda_1\right)u = 0$$

Either $u=0$ or $\lambda_1 = \dfrac{8y+9}{16}$

↳ if $u=0$ then $x=0 \Rightarrow y = \pm 3$ and $v=4$

We found our solution!

Will leave the other cases....
Clear there is no max distance...

<u>Homework</u>: Pg 981: #1, #14 (note the symmetry), #19, #35, #51

<u>Suggested</u>: #36, #37, #47, #49, #62 (important)

SECTION 12.10: CRITICAL POINTS OF FNS OF TWO VARS

Read thm 1 page 984 and be aware that such statements exist. Without linear algebra or multidimensional Taylor series, the formula is unsatisfying.

CHAPTER 13: MULTIPLE INTEGRALS

Goals:
- To review the theory of the Riemann Integral in one-dimension, and discuss generalization to higher dimensions
- To learn how to compute iterated integrals, switch orders of integration, and change variables.

Will frequently prove results in one-dim and sketch argument in higher dimensions, or refer to book, appendices or advanced, future classes.

Sections: 13.1, 13.2, 13.3, 13.4, 13.7, 13.9