CHAPTER 13: MULTIPLE INTEGRALS

Goals: • To review the Theory of the Riemann Integral in one-dimension, and discuss generalization to higher dimensions

• To learn how to compute iterated integrals, switch orders of integration, and change variables.

Will frequently prove results in one-dim and sketch argument in higher dimensions, or refer to book, appendices or advanced, future classes.

Sections: 13.1, 13.2, 13.3, 13.4, 13.7, 13.9

CHAPTER 13: DOUBLE AND TRIPLE INTEGRALS

Generalize integrating on a line

Easiest is integrating over a box, or cube, ...

In many cases, double and triple integrals become iterated integrals

We'll review the proof of the Fund Thm of Calc (Riemann Sums) in one variable and then briefly discuss generalizations. Won't prove for widest class of functions to avoid appealing to Real Analysis. Key ingredient in proof is the Mean Value Theorem:

If g is cont and differentiable on $[a,b]$

Then there is a c st $\frac{g(b)-g(a)}{b-a} = g'(c)$

with $c \in [a,b]$.

The next three pages are notes of mine on proving FTC.

ELEMENTARY AREA COMPUTATIONS

Know area of rectangle, right triangle , general triangle 

↳ area under curve: approx with rectangles

inscribe + circumscribe: squeeze limitive value between

Ex: $f(x) = x$, $0 \le x \le 1$

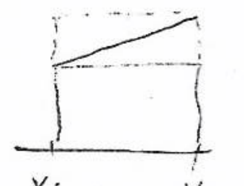
↳ clear area just $1/2$ 

Divide $[0,1]$ into n equal pieces, $0 = x_0 \le x_1 \le \cdots \le x_n = 1$

so $x_i = i/n$ and $x_{i+1} - x_i = \frac{1}{n}$

Let $I_i = [x_i, x_{i+1}]$

on I_i, $f(x)$ satisfies $f(x_i) \le f(x) \le f(x_{i+1})$



so $\sum_{i=0}^{n-1} f(x_i) \cdot \frac{1}{n} \le \text{Area} \le \sum_{i=0}^{n-1} f(x_{i+1}) \frac{1}{n}$

$$\sum_{i=0}^{n-1} \frac{i}{n}\frac{1}{n} \le \text{Area} \le \sum_{i=0}^{n-1} \frac{i+1}{n}\frac{1}{n} = \sum_{i=0}^{n-1} \frac{i}{n^2} + \frac{n}{n^2}$$

Claim: $\sum_{i=0}^{m} i = \frac{m(m+1)}{2}$ (induction, match largest + smallest, ...)

Thus $\frac{1}{n^2}\frac{(n-1)n}{2} \le \text{Area} \le \frac{1}{n^2}\frac{(n-1)n}{2} + \frac{n}{n^2}$

take $\lim_{n\to\infty}$, get $\frac{1}{2} \le \text{Area} \le \frac{1}{2}$ so Area is $1/2$

Ex: $f(x) = x^2$, $0 \le x \le 1$

↳ Before $f(x_i) = i/n$, now $f(x_i) = (i/n)^2$

Get $\sum_{i=0}^{n-1} \frac{i^2}{n^2}\frac{1}{n} \le \text{Area} \le \sum_{i=0}^{n-1} \frac{i^2}{n^2}\frac{1}{n} + \frac{n^2}{n^3}$

Claim: $\sum_{i=0}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}$

↳ yields $\frac{1}{3} \le \text{Area} \le \frac{1}{3}$ after taking limits HW: 3, 9, 19, 22, 39

Generalizations $\sum_{i=0}^{m} i^3 = \frac{m^2(m+1)^2}{4}$ Suggested: 52, 53

↳ $f(x) = x^3$: need

↳ have to prove "hard" sums each time

↳ what of more general f, say $f(x) = x^2 \sin x$?

↳ $[a,b]$ set $x_i = a + \frac{b-a}{n}$ ~~NEXT~~

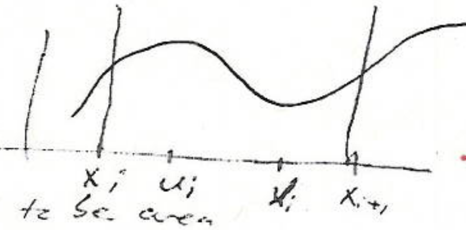
~~SAM~~ RIEMANN SUMS AND THE INTEGRAL 5.6 [illegible]

Will not prove in greatest generality

↳ each n partition $[a,b]$ into n pieces, find upper + lower sums, limit

↳ will assume all pieces equal

↳ will assume $[a,b]=[0,1]$



FTC: Let f be cont on $[a,b]$. Set $\int_a^b f(x)\,dx$ to be area under f from a to b. If F is any anti-derivative of f then $\int_a^b f(x)\,dx = F(b) - F(a)$ (Fund Thm of Calculus)

↳ Note: G any other anti-deriv: $F(x) = G(x) + C$, so $F(b) - F(a) = G(b) - G(a)$

↳ will prove assuming f' ~~cont~~ bounded: simplifies proof Talk about bounded

Proof (The One With The MVT)

↳ $I_i := [x_i, x_{i+1}]$: let max be at u_i, min at l_i

so $\forall x \in I_i$, $f(l_i) \le f(x) \le f(u_i)$

so $L(n) = \sum_{i=0}^{n-1} f(l_i)\frac{1}{n} \le \text{Area} \le \sum_{i=0}^{n-1} f(u_i)\frac{1}{n} = U(n)$

Claim: $\lim_{n\to\infty} U(n) - L(n) = 0$

Proof: Equals ~~lim~~ $U(n) - L(n) = \frac{1}{n}\sum_{i=0}^{n-1}\left[f(u_i) - f(l_i)\right]$

By MVT, $= \frac{1}{n}\sum_{i=0}^{n-1} |f'(c_i)|\,|u_i - l_i|$

$\le \frac{1}{n}\sum_{i=0} B \cdot \frac{1}{n} = \frac{B}{n} \longrightarrow 0$

Let $x_i^* \in I_i$ be any seq of points

↳ $L(n) \le \sum_{i=0}^{n-1} f(x_i^*)\frac{1}{n} \le U(n)$: choose clever seq

Apply MVT to F on I_i: Choose x_i^* st $F(x_{i+1}) - F(x_i) = \underset{f(x_i^*)}{F'(x_i^*)}(x_{i+1} - x_i)$

so $L(n) \le \sum_{i=0}^{n-1} f(x_i^*)\frac{1}{n} = \sum_{i=0}^{n-1} F(x_{i+1}) - F(x_i) \le U(n)$

└ telescoping: is $F(1) - F(0)$ or $F(b) - F(a)$

so $L(n) \le F(1) - F(0) \le U(n)$

$\downarrow$ Area, thus ? goes to area, $\downarrow$ Area

[redacted]

Call f <u>Riemann Integrable</u> if above method works

Not all f are RI: $f(x) = 0$ if $x \in \mathbb{Q}$, 1 if $x \notin \mathbb{Q}$

↳ <u>Lebesgue Integral</u>: agrees when RI exists, generalizes

↳ throw down coins: RI: sum in order, LI: group pennies, nickels...

<u>Average Values</u>

If f integrable on $[a,b]$, ave value is $\frac{1}{b-a} \int_a^b f(x)\,dx$

↳ if f cont, $\exists\, \bar{x} \in [a,b]$ st $f(\bar{x}) =$ Ave Value

[redacted]

*<u>FTC</u>: Define $F(x) = \int_a^x f(x)\,dx$ Then $F'(x) = f(x)$

<u>EXTRA CREDIT</u>: Say $G(x) = \int_0^{x^3} f(x)\,dx$. Express $G'(x)$ in terms of nice fns related to the problem

<u>5.5 EVALUATION OF INTEGRALS</u>

① Constant: $\int_a^b c\,dx = c(b-a)$

② Multiple: $\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$

③ Sum/Diff: $\int_a^b (f(x) \pm g(x))\,dx =$

④ Interval Union: $a < c < b$: $\int_a^b f = \int_a^c f + \int_c^b f$

⑤ Comparison: $f \le g$ on $[a,b]$ Then $\int_a^b f \le \int_a^b g$

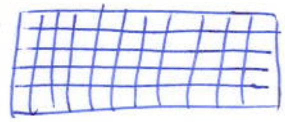
↳ special case: $m \le f(x) \le M \rightarrow m(b-a) \le \int_a^b f \le M(b-a)$

<u>[illegible] EXAMPLES</u>

[redacted]

Section 13.1: Double Integrals (Cont)

Start with a rectangle. Divide into subrectangles. Take limit, show answer is indep of how sub-divide so long as each subrectangle's size tends to zero. Going through the Riemann Sum Calculations we find



Thm: Double Integrals as Iterated Single Integrals.
If f is a continuous fn on rectangle $R=[a,b]\times[c,d]$

Then $$\iint_R f(x,y)\,dA = \int_{x=a}^{b}\left[\int_{y=c}^{d} f(x,y)\,dy\right]dx = \int_{y=c}^{d}\left[\int_{x=a}^{b} f(x,y)\,dx\right]dy$$

Ex: $$\int_{x=0}^{1}\left[\int_{y=0}^{2}(3xy+2x^2)\,dy\right]dx$$

$$= \int_{x=0}^{1}\left[\tfrac{3}{2}xy^2\Big|_0^2 + 2x^2y\Big|_0^2\right]dx$$

$$= \int_{x=0}^{1}\left[6x+4x^2\right]dx$$

$$= 3x^2\Big|_0^1 + \tfrac{4}{3}x^3\Big|_0^1$$

$$= 3+\tfrac{4}{3} = \tfrac{13}{3}$$

Note: don't *need* to write the variable in the bounds, but I find it helps prevent confusion.

SECTION 13.1 (CONT)

Can do volumes as double integral

Let $f(x,y)$ be height at x,y

Volume is $\iint_R f(x,y)\,dx\,dy$

If use planes perpendicular to x-axis:

$$\iint_R f(x,y)\,dA = \int_{x=a}^{b}\left[\int_{y=c}^{d} f(x,y)\,dy\right]dx$$

If use planes perpendicular to y-axis:

$$\iint_R f(x,y)\,dA = \int_{y=c}^{d}\left[\int_{x=a}^{b} f(x,y)\,dx\right]dy$$

For nice regions, these must be equal.

What if not a volume: can we still change order?

Ex: $\int_0^1\int_0^1 x e^{xy}\,dy\,dx$ $\qquad$ $\int_0^1\int_0^1 x e^{xy}\,dx\,dy$

$= \int_0^1\left[\int_0^1 e^{xy}\,x\,dy\right]dx$ $\qquad$ ↳ ugh: integrate by parts

$= \int_0^1 e^{xy}\Big|_0^1\,dx$

$= \int_0^1 (e^y - 1)\,dx = e^y - 2$

$\boxed{\text{Changing orders can lead to a simpler computation!}}$

Homework: Pg 1004: #15, #24, #25, #37

Suggested: #33

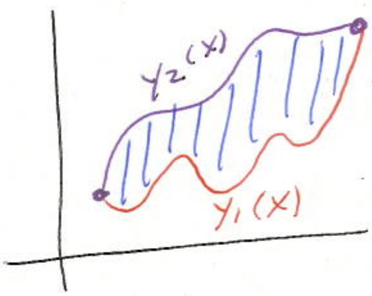
SECTION 13.2: Double Integrals over More General Regions

Main difficulty here is not calculus but rather algebra.

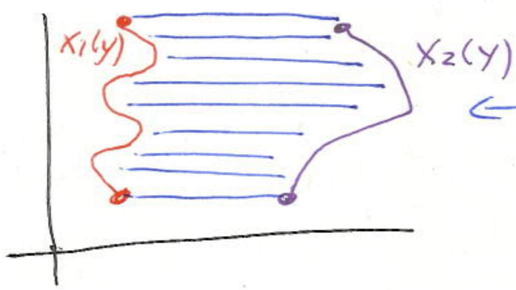
Definitions: A region is

(1) vertically simple if described by all (x,y) such that $a \le x \le y$ and $y_1(x) \le y \le y_2(x)$ for continuous fns y_1, y_2 on $[a,b]$

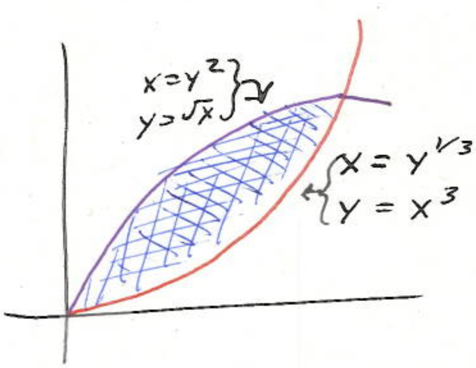
(2) horizontally simple if described by all (x,y) such that $c \le y \le d$ and $x_1(y) \le x \le x_2(y)$ for continuous fns x_1, x_2 on $[c,d]$.



← vertically simple region
Go up until hit $y_1(x)$ and then enter region, stay in region until $y_2(x)$ and then leave and do not return



← horizontally simple region
Note $x_1(c)$ need not equal $x_2(c)$



← A region both vertically and horizontally simple.

SECTION 13.2 (CONT)

KEY Thm: Evaluation of Double Integrals

Let $f(x,y)$ be continuous on region R. Then:

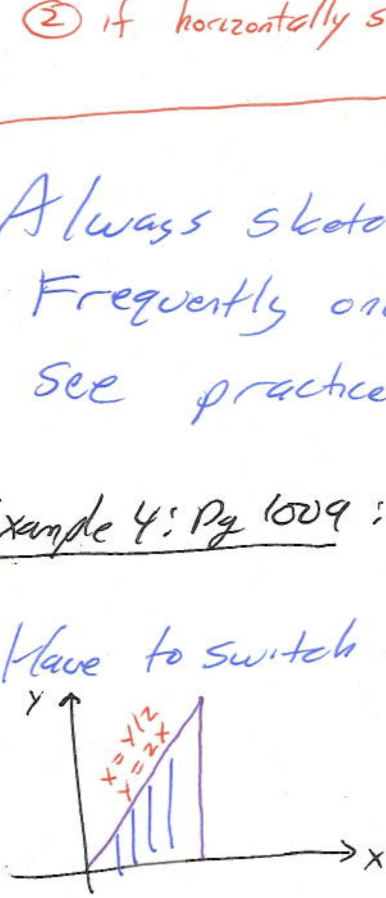
① if vertically simple: $\iint_R f(x,y)\,dA = \int_{x=a}^{b}\left[\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\right]dx$

② if horizontally simple: $\iint_R f(x,y)\,dA = \int_{y=c}^{d}\left[\int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx\right]dy$

- Always sketch region first!
- Frequently one way is <u>much</u> easier
- See practice regions on line

<u>Example 4: Pg 1009</u>: Evaluate $\int_{y=0}^{2}\int_{x=y/2}^{1} y e^{x^3}\,dx\,dy$

Have to switch order! e^{x^3} has no nice anti-derivative



Before: Fix y, then $\frac{y}{2} \le x \le 1$

Now: Fix x: What are bounds?

Now have y goes from 0 to $2x$

So $y_1(x) = 0$ and $y_2(x) = 2x$

Integral is $\int_{x=0}^{1}\left[\int_{y=0}^{2x} y e^{x^3}\,dy\right]dx = \int_{x=0}^{1} \frac{y^2}{2}\Big|_0^{2x} e^{x^3}\,dx$

$= \int_{x=0}^{1} 2x^2 e^{x^3}\,dx = \frac{2}{3} e^{x^3}\Big|_0^1 = \frac{2}{3}(e-1)$

5.2. The Double Integral over a Rectangle

- Standard props
  - ↳ linearity
  - ↳ homogeneity (const mult)
  - ↳ monotonic: $f \le g$
  - ↳ additivity

- Interchanging orders!

$$\sum_m \sum_n a_{mn} \ne \sum_n \sum_m a_{mn}$$

[illegible]

↳ (1) infinite range

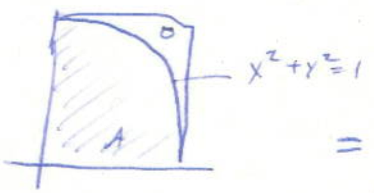
(2) absolute value not integrable

Thm (Fubini): If f integrable on finite R then can interchange orders

↳ can generalize to bounded fns + discont

↳ see book for proof

Ex:



$$\iint_A \sqrt{1-x^2}\, dA$$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1-x^2}} \sqrt{1-x^2}\, dy\right]dx = \int_{y=0}^{1}\left[\int_{x=0}^{\sqrt{1-y^2}} \sqrt{1-x^2}\, dx\right]dy$$

one easy, one hard

$$\left(\frac{\sin(2\arcsin x)}{4} + \frac{\arcsin x}{2}\right)' = \sqrt{1-x^2}\ !$$

What is $\iint_A \sqrt{1-x^2-y^2}\, dA$? Vol sphere, neither is easy!

↳ will get with change of variable later.

~~[illegible]~~

~~[illegible]~~

Homework: Pg 1011: #4, #11, #13, #25, #30

Suggested: #41, #44, #49

Section 13.3: Area and Vol by Double Integration

Key Theorem: f continuous and bounded on nice planar region R. Then volume of solid above R and below $z = f(x,y)$ is $\iint_R f(x,y)\,dA$

- Can use our methods to evaluate
- can have negative volume if below plane (signed volume)
- If want region between two surfaces becomes
  $$\iint_R \left[f_{top}(x,y) - f_{bottom}(x,y)\right] dA.$$

Homework: Pg 1018: #13, #42 (give hint)

Suggested: #29

SECTION 13.4: Double Integrals in Polar Coords

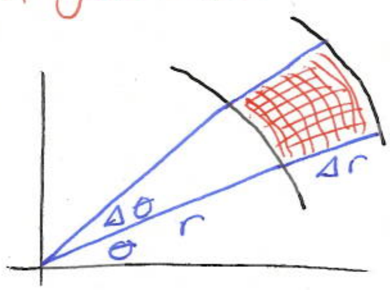
<u>Goal</u>: Given $\iint_R f(x,y)\,dx\,dy$, want to transform to $\iint_{R^*} g(r,\theta)\,dr\,d\theta$. Questions: (1) what is g? (2) what about bounds of integration?

Unit Analysis

- Recall power of unit analysis (proof of Pythagoras).
- If x, y are in meters so too is r; θ is dimensionless

↳ Thus $dx\,dy$ cannot just go to $dr\,d\theta$! wrong units. Simplest fix is $r\,dr\,d\theta$.

Full justification comes from the Change of Variables formula.



What is this area?

Whole annular ring is

$$\pi(r+\Delta r)^2 - \pi r^2 = 2\pi r\Delta r + \pi(\Delta r)^2$$

But only have $\frac{\Delta\theta}{2\pi}$ of it, thus the area is just $r\Delta r\Delta\theta + \pi(\Delta r)^2\Delta\theta$

As $\Delta r, \Delta\theta \to 0$ the cube term is so much smaller than the square term that $(\Delta r)^2\Delta\theta$ becomes negligible and the new area element is $r\,dr\,d\theta$.

Do exercise 3, pg 1026: Find area bounded by cardioid $r = 1+\cos\theta$ 

Bounds: $0 \le \theta \le 2\pi$, $0 \le r \le 1+\cos\theta$, $f(x,y)=1$

Ans: $\displaystyle\int_{\theta=0}^{2\pi}\left[\int_{r=0}^{1+\cos\theta} 1\,dr\right]d\theta = \int_{\theta=0}^{2\pi}(1+\cos\theta)\,d\theta = 2\pi$

SECTION 13.4 (CONT)

Ex: $\iint_R f(x,y)\,dx\,dy$: take $f(x,y)=1$ and R the unit circle.

The region $0 \le x^2+y^2 \le 1$ becomes the region $0 \le \theta \le 2\pi$ and $0 \le r \le 1$. We find

$$\iint_R dx\,dy = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r\,dr\,d\theta = 2\pi \int_0^1 r\,dr = \pi r^2 \Big|_0^1 = \pi$$

Note how quickly can compute area of circle!

Ex: $\iint_R f(x,y)\,dx\,dy$: take $f(x,y)=\sqrt{1-x^2-y^2}$ and R circle radius 1

The region $0 \le x^2+y^2 \le 1$ becomes the region $0 \le \theta \le 2\pi$ and $0 \le r \le 1$. We find

$$\iint_R \sqrt{1-x^2-y^2}\,dx\,dy = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \sqrt{1-r^2}\; r\,dr\,d\theta$$

$$= 2\pi \int_{r=0}^{1} (1-r^2)^{1/2}\, r\,dr$$

$$= 2\pi\, \frac{2}{3} (1-r^2)^{3/2} \Big|_1^0$$

$$= \frac{4}{3}\pi$$

We've just computed the vol of the unit sphere!

$\boxed{\text{Summary: } f(x,y) \text{ becomes } f(r\cos\theta, r\sin\theta),\ dx\,dy \text{ becomes } r\,dr\,d\theta, \text{ and adjust bounds to reflect the new region.}}$

Homework: Pg 1026: #4, #13

Suggested: #7, #34

$\boxed{\text{Do the Gaussian Integral: } \int_{-\infty}^{\infty} e^{-x^2}\,dx}$

Section 13.7: Integration in Cylindrical and Spherical Coords

KEY THM: CYLINDRICAL: f a nice function, nice region T mapped to T^*, find

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_{T^*} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

KEY THM: SPHERICAL: f a nice function, T mapped to T^*, find

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_{T^*} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) * \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

- Cylindrical fairly straight forward
- Can build intuition for spherical similar to polar intuition
  - ↳ note units of $\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$ are meters^3: good
  - ↳ note as $0 \le \varphi \le \pi$ that $\rho^2 \sin\varphi \ge 0$: good for a volume element!

Exercise: integrate $(x^2+y^2+z^2)^3$ over unit sphere

Homework: Pg 1056: #37

Suggested: Very Important: #47, #48 : Make Extra Credit

APPENDIX 4: OPTIONAL

SUPPLEMENTAL TOPIC: MONTE CARLO INTEGRATION

History: Most $\int \cdots \int$ too hard to do in closed form (Finance: 360+ dim)

↳ ex. $\int_a^b e^{-x^2/2}\,dx$

↳ using prob can approx very well with high confidence

Prob Review

- density $f(x) \geq 0$, $\int_{-\infty}^{\infty} f(x)\,dx = 1$
- mean $\mu = \int_{-\infty}^{\infty} x f(x)\,dx$ if exists, ave value, write $E[X]$
- variance $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$: measures how spread out $= E[(X-\mu)^2]$

  ↳ std dev σ, same units as μ

  ↳ aside k^{th} centered moment $\mu_k = \int_{-\infty}^{\infty} (x-\mu)^k f(x)\,dx$

  ↳ like Taylor coeff: know moments, know fn (if nice)

- Chebyshev: $\text{Prob}(|X-\mu| \leq k\sigma) \geq 1 - \frac{1}{k^2}$

  ↳ Proof. $\text{Prob}(|X-\mu| > k\sigma) = \int_{|X-\mu|>k\sigma} f(x)\,dx$

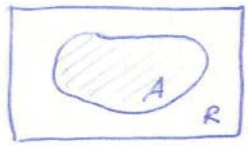
$$\leq \int_{|X-\mu|>k\sigma} \frac{(x-\mu)^2}{k^2\sigma^2} f(x)\,dx$$

$$\leq \frac{1}{k^2\sigma^2} \int (x-\mu)^2 f(x)\,dx = \frac{1}{k^2}$$

↳ in general have stronger results, but holds $\forall$ distr. with finite mean and variance

APPENDIX 4: OPTIONAL

SUPPLEMENTAL TOPIC: MONTE CARLO INTEGRATION

Simple case: 

DARTS!

"Nice" region A inside rectangle R ($= [0,1]^n$, say)

Assume easy to tell if $\vec{x} \in A$ or $\vec{x} \notin A$

Choose N points unif in R ($= [0,1]^n$), each choice indep of others

↳ Prob a point is in A is $\text{Vol}(A) / \text{Vol}(R) = \text{Vol}(A) \equiv p$.

$$X_i = \begin{cases} 1 & \text{if inside } A \quad \text{happens with prob } p \\ 0 & \text{if } i^{th} \text{ point } \notin A \quad \text{" " " } 1-p. \end{cases}$$

$X = X_1 + \cdots + X_N$

What is expected value (mean) of X? What is its std dev?

<u>Lemma</u>: $\mathbb{E}[\Sigma X_i] = \Sigma \mathbb{E}[X_i]$

$\text{Var}(\Sigma X_i) = \Sigma \text{Var}(X_i)$ if indep

Each X_i same distr: mean is $1 \cdot p + 0 \cdot (1-p) = p$

Var is $(1-p)^2 \cdot p + (0-p)^2 (1-p) = p(1-p) \cdot 1$

so $\mathbb{E}[X] = Np$

$\text{Var}(X) = p(1-p)N$ or $\text{Std Dev}(X) = \sqrt{p(1-p)}\sqrt{N}$

APPENDIX Y: OPTIONAL

SUPPLEMENTAL TOPIC: MONTE CARLO INTEGRATION

Let N be enormous

Chebyshev: "high" prob that X very close to Np

↳ Prob at least $1 - \frac{1}{B^2}$ that $|X - Np| \le B \cdot \sqrt{p(1-p)}\,\sqrt{N}$

Estimate of volume is $\frac{X}{N}$, true vol is P

Thus $\left|\frac{X}{N} - p\right| \le \frac{B\sqrt{p(1-p)}}{\sqrt{N}}$ with prob at least $1-\frac{1}{B^2}$

↳ max error at $p = 1/2$

↳ "trouble" getting accurate answer if p or $1-p$ very small

<u>Exercises</u>! All supplemental

① Write a program to estimate area of quarter-circle.

Record errors for various N.

② f prob distr with μ, σ finite. Prove $\exists\, a, b$ st

↳ $g(x) = f(x+a)$ has mean 0 and std dev σ

↳ $h(x) = g(bx) = f(b(x+a))$ has mean 0 and var 1

Important: normalizes: easier to compare different distr

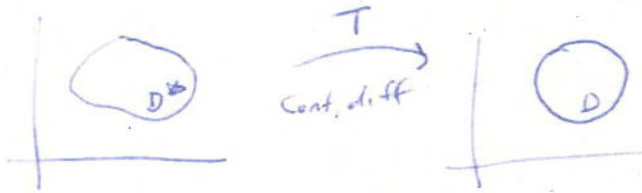
↳ "nice" distr: don't see shape until 3rd/4th moment

↳ universality ⟹ CLT (Math 161-162)

SECTION 13.9: JUST KNOW STATEMENT

: THE CHANGE OF VARIABLES FORMULA + APPS TO $\int$

. THE GEOMETRY OF MAPS FROM $\mathbb{R}^2$ TO $\mathbb{R}^2$



ex: $T: [0,1] \times [0, 2\pi] \to$ unit circle : $T(r,\theta) = (r\cos\theta, r\sin\theta)$

↳ not 1-1 / inj: $\{0\} \times [0, 2\pi] \to (0,0)$

is onto (surj): all points in D hit

Thm: A 2x2 matrix, $\operatorname{Det} A \neq 0$, maps ▱ to ▱, vertices → vertices

Thm (Adv): D^*, D elemental, $T: D^* \to D$ st $DT(u,v) \neq \vec{0}$ $\forall (u,v) \in D^*$,
if $T: \partial D^* \xrightarrow[\text{onto}]{\text{1-1}} \partial D$ Then T is 1-1, onto (bij)

Thm: $\operatorname{Det} A \neq 0 \Leftrightarrow A$ is 1-1, onto

~~HW: pg 375: #2, #7, #12~~
~~Suggested: #3, #8~~

THE CHANGE OF VARIABLE FORMULA

$T: D^* \to D$ with $T(u,v) = (x,y) = (x(u,v), y(u,v))$

Can't be $\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v))\,du\,dv$

↳ in general D, D^* diff areas, take $f \equiv 1$

<u>Defn: Jacobian Det</u>

$$\text{"}JT\text{"} = DT(u_0,v_0) = \begin{vmatrix} \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial x}{\partial v}(u_0,v_0) \\ \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) \end{vmatrix} \overset{\text{def}}{=} \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

Yields $\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

Think of as $\frac{dx\,dy}{du\,dv}\,du\,dv$

↳ assuming T is C^1, 1-1

<u>Justification</u>

Pg 134: Differentiability $g(\vec{v}) = g(\vec{v}_0) + Dg(\vec{v}_0)\cdot(\vec{v} - \vec{v}_0) + \text{small}$

wlog $(u_0, v_0) = (x_0, y_0) = (0,0)$, $T(u_0, v_0) = (x_0, y_0)$

$$T(\Delta u, \Delta v) \approx \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} + \text{small}$$

so Area $\Delta x \Delta y \approx |\text{Det } DT| \cdot \Delta u \Delta v$

<u>Change of vars: 1-var</u>

$\int_a^b f(x(u))\cdot x'(u)\,du = \int_{x(a)}^{x(b)} f(x)\,dx$ Chain Rule

SECTION 13.9 (CONT)

Exercise #1, Page 1071: Solve for x and y in terms of u and v, compute the Jacobian $\partial(x,y)/\partial(u,v)$, with $u = x+y$, $v = x-y$

Sol: $\left.\begin{array}{l} u = x+y \\ v = x-y \end{array}\right\}$ add and find $u+v = 2x$
subtract and see $u-v = 2y$

Thus $x = x(u,v) = \frac{u}{2} + \frac{v}{2}$

$y = y(u,v) = \frac{u}{2} - \frac{v}{2}$

Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

where we used $\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$

Homework: Pg 1070: #2, #3, #14

Suggested: #10, #28 and #29