SOLUTION KEYS FOR MATH 150 HW (SPRING 2021)

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1. HW #1

1.1. Problems: HW #1.

Problem 1: What is wrong with the following argument (from Mathematical Fallacies, Flaws, and Flimflam - by Edward Barbeau): There is no point on the parabola $16y = x^2$ closest to $(0, 5)$. This is because the distance-squared from (0,5) to a point (x, y) on the parabola is $x^2 + (y-5)^2$. As $16y = x^2$, the distance-squared is $f(y) = 16y + (y-5)^2$. As $df/dy = 2y+6$, there is only one critical point, at $y = -3$; however, there is no x such that $(x, -3)$ is on the parabola. Thus there is no shortest distance!

Problem 2: Compute the derivative of $\cos(\sin(3x^2 + 2x \ln x))$. Note that if you can do this derivative correctly, your knowledge of derivatives should be fine for the course.

Problem 3: Let $f(x) = x^2 + 8x + 16$ and $g(x) = x^2 + 2x - 8$. Compute the limits as x goes to 0, 3 and ∞ of $f(x) + g(x)$, $f(x)g(x)$ and $f(x)/g(x)$.

1.2. Solutions: HW #1.

Problem 1: What is wrong with the following argument (from Mathematical Fallacies, Flaws, and Flimflam - by Edward Barbeau): There is no point on the parabola $16y = x^2$ closest to $(0, 5)$. This is because the distance-squared from (0,5) to a point (x, y) on the parabola is $x^2 + (y-5)^2$. As $16y = x^2$, the distance-squared is $f(y) = 16y + (y-5)^2$. As $df/dy = 2y+6$, there is only one critical point, at $y = -3$; however, there is no x such that $(x, -3)$ is on the parabola. Thus there is no shortest distance!

Solution: The error in the argument is that, to find maxima and minima, it is not enough to just check the critical points; you must also check the boundary points. The boundary points here are $y = 0$ and $y = \infty$ (ok, just $y = 0$). We thus see that $y = 0$ gives the closest point, while $y \to \infty$ gives ever increasing distances, indicating that there is no maximum.

Problem 2: Compute the derivative of $\cos(\sin(3x^2 + 2x \ln x))$. Note that if you can do this derivative correctly, your knowledge of derivatives should be fine for the course.

Solution: We use the chain rule multiple times. Remember that the derivative of $f(g(x))$ is $f'(g(x)) * g'(x)$.The derivative of $\cos(\sin(3x^2 + 2x \ln x))$ is two chain rules (with a sum rule and a product rule inside):

$$- \sin(\sin(3x^2 + 2x \ln x)) \; * \; \frac{d}{dx}\left[\sin(3x^2 + 2x \ln x)\right],$$

which is

$$- \sin(\sin(3x^2 + 2x \ln x)) \; * \; \cos(3x^2 + 2x \ln x) \; * \; \frac{d}{dx}\left[3x^2 + 2x \ln x\right],$$

which is just

$$- \sin(\sin(3x^2 + 2x \ln x)) \; * \; \cos(3x^2 + 2x \ln x) \; * \; (6x + 2 \ln x + 2).$$

Problem 3: Let $f(x) = x^2 + 8x + 16$ and $g(x) = x^2 + 2x - 8$. Compute the limits as x goes to 0, 3 and 8 of $f(x) + g(x)$, $f(x)g(x)$ and $f(x)/g(x)$.

Solution: We have $f(0) = 16$, $f(3) = 49$, and $f(\infty) = \infty$, while $g(0) = -8$, $g(3) = 7$ and $g(\infty) = \infty$. Using the limit of a sum (product, quotient) is the sum (product, quotient) of the limit (so long as everything is defined), we see there is no problem at 0 or 3. For the first, $f(x)+g(x)$ goes to 16-8 = 8 as x goes to 0, 49 + 7 = 56 as x goes to 3, and $\infty + \infty$ as x goes to ∞ (note that while $\infty - \infty$ is not defined, $\infty + \infty$ is and just equals ∞). For $f(x)g(x)$, this tends to 16 * (-8) = -128 as x goes to 0, to 49 * 7 = 343 as x goes to 3, and $\infty * \infty = \infty$ as x goes to ∞. For the quotient, it is important that we do not have 0/0 or ∞/∞. Thus we can immediately do the first two cases, and see $f(x)/g(x)$ goes to $16/(-8) = -2$ as x tends to 0 and 49/7 = 7 as x tends to 3. For the last, as we have ∞/∞ we need to work a bit harder. As $f(x) = x^2 + 8x + 16$ and $g(x) = x^2 + 2x - 8$, $f(x)/g(x) = (1 + 8/x + 16/x^2)/(1 + 2/x - 8/x^2)$ (from pulling out an x^2 from the numerator and denominator). Now each piece has a well-defined and finite limit as x tends to ∞, and we see that $f(x)/g(x)$ tends to 1 as x tends to ∞.

Note you could also do Problem 3 by expanding out the expressions, but that is much harder. For example, $f(x)g(x)$ is a polynomial of degree 4 that can be analyzed directly. Also, for $f(x)/g(x)$ one could proceed by L'Hopital's rule. That said, the point of this exercise was to remind you that the limit of a sum is the sum of the limits, and so on.

2. HW #2

2.1. Problems: HW #2.

Page 823: #9:: Find $|\overrightarrow{a}|$, $|-2\overrightarrow{b}|$, $|\overrightarrow{a}-\overrightarrow{b}|$, $\overrightarrow{a}+\overrightarrow{b}$ and $3\overrightarrow{a}-2\overrightarrow{b}$ for $\overrightarrow{a}=\langle 1,-2\rangle$ and $\overrightarrow{b}=\langle -3,2\rangle$.

Page 823: #18:: Find a unit vector $\overrightarrow{u}$ in the same direction as $\overrightarrow{a}=\langle 5,-12\rangle$. Express $\overrightarrow{u}$ in terms of $\overrightarrow{i}$ and $\overrightarrow{j}$, and find a vector $\overrightarrow{v}$ in the opposite direction as that of $\overrightarrow{a}$.

Page 823: #38:: Given three points $A(2,3)$, $B(-5,7)$ and $C(1,-5)$, verify by direct computation that $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}$ is the zero vector.

Page 824: #42:: Let $\overrightarrow{a}=\langle a_1,a_2\rangle$ and $\overrightarrow{b}=\langle b_1,b_2\rangle$. Prove by componentwise arguments that if $\overrightarrow{a}+\overrightarrow{b}=\overrightarrow{a}$ then $\overrightarrow{b}=\overrightarrow{0}$.

Page 833: #1:: Let $\overrightarrow{a}=\langle 2,5,-4\rangle$ and $\overrightarrow{b}=\langle 1,-2,-3\rangle$. Find $2\overrightarrow{a}+\overrightarrow{b}$, $3\overrightarrow{a}-4\overrightarrow{b}$, $\overrightarrow{a}\cdot\overrightarrow{b}$, $|\overrightarrow{a}-\overrightarrow{b}|$ and $\overrightarrow{a}/|\overrightarrow{a}|$.

Page 834: 39: Two vectors are parallel provided that one is a scalar multiple of the other. Determine whether the vectors $\overrightarrow{a}=\langle 4,-2,6\rangle$ and $\overrightarrow{b}=\langle 6,-3,9\rangle$ are parallel, perpendicular or neither.

Additional Problem: Find the cosine of the angle between $\overrightarrow{a}=\langle 2,5,-4\rangle$ and $\overrightarrow{b}=\langle 1,-2,-3\rangle$.

2.2. **HW #2.**

Page 823: #9:: Find $|\overrightarrow{a}|$, $|-2\overrightarrow{b}|$, $|\overrightarrow{a}-\overrightarrow{b}|$, $\overrightarrow{a}+\overrightarrow{b}$ and $3\overrightarrow{a}-2\overrightarrow{b}$ for $\overrightarrow{a}=\langle 1,-2\rangle$ and $\overrightarrow{b}=\langle -3,2\rangle$.

Solution: We have $|\overrightarrow{a}|=\sqrt{1^2+(-2)^2}=\sqrt{5}$. As $-2\overrightarrow{b}=\langle 6,-4\rangle$, $|-2\overrightarrow{b}|=\sqrt{6^2+(-4)^2}=\sqrt{52}$. Since $\overrightarrow{a}-\overrightarrow{b}=\langle 4,-4\rangle$, $|\overrightarrow{a}-\overrightarrow{b}|=\sqrt{4^2+(-4)^2}=\sqrt{32}$. Finally, $\overrightarrow{a}+\overrightarrow{b}=\langle -2,0\rangle$ and

$$3\overrightarrow{a}-2\overrightarrow{b} \;=\; \langle 3,-6\rangle-\langle -6,4\rangle \;=\; \langle 9,-10\rangle.$$

Page 823: #18:: Find a unit vector $\overrightarrow{u}$ in the same direction as $\overrightarrow{a}=\langle 5,-12\rangle$. Express $\overrightarrow{u}$ in terms of $\overrightarrow{i}$ and $\overrightarrow{j}$, and find a vector $\overrightarrow{v}$ in the opposite direction as that of $\overrightarrow{a}$.

Solution: We have $|\overrightarrow{a}|=\sqrt{5^2+(-12)^2}=\sqrt{169}=13$. A unit vector is $\overrightarrow{u}=\overrightarrow{a}/|\overrightarrow{a}|$, or $\overrightarrow{u}=\langle 5/13,-12/13\rangle$. As $\overrightarrow{i}=\langle 1,0\rangle$ and $\overrightarrow{j}=\langle 0,1\rangle$, we have $\overrightarrow{u}=\frac{5}{13}\overrightarrow{i}-\frac{12}{13}\overrightarrow{j}$. As $-\overrightarrow{a}$ has the opposite direction as $\overrightarrow{a}$, we see we may take $\overrightarrow{v}=-\overrightarrow{a}=\langle -5,12\rangle$. Of course, there are multiple answers. We could also take $\overrightarrow{v}=-\overrightarrow{u}$, as $\overrightarrow{u}$ and $\overrightarrow{a}$ are in the same direction.

Page 823: #38:: Given three points $A(2,3)$, $B(-5,7)$ and $C(1,-5)$, verify by direct computation that $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}$ is the zero vector.

Solution: Given two points $P=(p_1,p_2)$ and $Q=(q_1,q_2)$, by $\overrightarrow{PQ}$ we mean the vector from P to Q, which is $\langle q_1-p_1,q_2-p_2\rangle$. We thus have

$$\begin{aligned}\overrightarrow{AB} &= \langle -5,7\rangle-\langle 2,3\rangle \;=\; \langle -7,4\rangle\\ \overrightarrow{BC} &= \langle 1,-5\rangle-\langle -5,7\rangle \;=\; \langle 6,-12\rangle\\ \overrightarrow{CA} &= \langle 2,3\rangle-\langle 1,-5\rangle \;=\; \langle 1,8\rangle,\end{aligned}$$

which implies

$$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} \;=\; \langle -7,4\rangle+\langle 6,-12\rangle+\langle 1,8\rangle \;=\; \langle 0,0\rangle.$$

Why is this true? We are traveling in a directed way along the three edges of a triangle, and we return to where we started.

Page 824: #42:: Let $\overrightarrow{a}=\langle a_1,a_2\rangle$ and $\overrightarrow{b}=\langle b_1,b_2\rangle$. Prove by componentwise arguments that if $\overrightarrow{a}+\overrightarrow{b}=\overrightarrow{a}$ then $\overrightarrow{b}=\overrightarrow{0}$.

Solution: Assume $\overrightarrow{a}+\overrightarrow{b}=\overrightarrow{a}$. Substituting for these vectors yields

$$\langle a_1,a_2\rangle+\langle b_1,b_2\rangle \;=\; \langle a_1,a_2\rangle,$$

or equivalently

$$\langle a_1+b_1,a_2+b_2\rangle \;=\; \langle a_1,a_2\rangle.$$

This is a pair of equations:

$$a_1+b_1 \;=\; a_1, \quad a_2+b_2 \;=\; a_2.$$

We now have simple equations of numbers and not vectors. For the first, subtracting a_1 from both sides gives $b_1=0$, while for the second subtracting a_2 from both sides gives $b_2=0$. Thus our vector $\overrightarrow{b}=\langle 0,0\rangle$. The key observation here is that we can reduce a vector question to a system of equations about numbers, and we know how to handle / analyze numbers.

Page 833: #1:: Let $\overrightarrow{a}=\langle 2,5,-4\rangle$ and $\overrightarrow{b}=\langle 1,-2,-3\rangle$. Find $2\overrightarrow{a}+\overrightarrow{b}$, $3\overrightarrow{a}-4\overrightarrow{b}$, $\overrightarrow{a}\cdot\overrightarrow{b}$, $|\overrightarrow{a}-\overrightarrow{b}|$ and $\overrightarrow{a}/|\overrightarrow{a}|$.

Solution: First,

$$\begin{aligned}2\overrightarrow{a}+\overrightarrow{b} &= \langle 4,10,-8\rangle+\langle 1,-2,-3\rangle \;=\; \langle 5,8,-11\rangle\\ 3\overrightarrow{a}-4\overrightarrow{b} &= \langle 6,15,-12\rangle-\langle 4,-8,-12\rangle \;=\; \langle 2,23,0\rangle.\end{aligned}$$

Next,

$$\overrightarrow{a} \cdot \overrightarrow{b} \ = \ 2 \cdot 1 + 5 \cdot (-2) + (-4) \cdot (-3) \ = \ 2 - 10 + 12 \ = \ 4.$$

As $\overrightarrow{a} - \overrightarrow{b} = \langle 1, 7, -1 \rangle$,

$$|\overrightarrow{a} - \overrightarrow{b}| \ = \ \sqrt{1^2 + 7^2 + (-1)^2} \ = \ \sqrt{51}.$$

Finally, as $|\overrightarrow{a}| = \sqrt{2^2 + 5^2 + (-4)^2} = \sqrt{45}$, we see $\overrightarrow{a}/|\overrightarrow{a}| = \langle 2/\sqrt{45}, 5/\sqrt{45}, -4/\sqrt{45} \rangle$.

Page 834: 39: Two vectors are parallel provided that one is a scalar multiple of the other. Determine whether the vectors $\overrightarrow{a} = \langle 4, -2, 6 \rangle$ and $\overrightarrow{b} = \langle 6, -3, 9 \rangle$ are parallel, perpendicular or neither.

Solution: We have

$$\overrightarrow{a} \cdot \overrightarrow{b} \ = \ 4 \cdot 6 + (-2) \cdot (-3) + 6 \cdot 9 \ = \ 24 + 6 + 54 \ = \ 84.$$

If the two vectors were perpendicular, the dot product should be zero. As it isn't zero, we know the vectors are not perpendicular. We now check to see if they are parallel; that means the cosine of the angle should be 1 or -1. To compute this, we need the lengths of the two vectors. We have

$$|\overrightarrow{a}| \ = \ \sqrt{4^2 + (-2)^2 + 6^2} \ = \ \sqrt{16 + 4 + 36} \ = \ \sqrt{56}$$

and

$$|\overrightarrow{b}| \ = \ \sqrt{6^2 + (-3)^2 + 9^2} \ = \ \sqrt{36 + 9 + 81} \ = \ \sqrt{126}.$$

Thus if θ is the angle between the two vectors,

$$\cos\theta \ = \ \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|\,|\overrightarrow{b}|} \ = \ \frac{94}{\sqrt{56}\sqrt{126}} \ = \ \frac{84}{84} \ = \ 1,$$

so the two vectors are indeed parallel.

Additional Problem: Find the cosine of the angle between $\overrightarrow{a} = \langle 2, 5, -4 \rangle$ and $\overrightarrow{b} = \langle 1, -2, -3 \rangle$.

Solution: If θ denotes the angle, then

$$\cos\theta \ = \ \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|\,|\overrightarrow{b}|}.$$

We worked with these two vectors in Problem 1, and saw $\overrightarrow{a} \cdot \overrightarrow{b} = 4$ and $|\overrightarrow{a}| = \sqrt{45}$. A similar calculation gives $|\overrightarrow{b}| = \sqrt{1^2 + (-2)^2 + (-3)^2} = \sqrt{14}$. Thus

$$\cos\theta \ = \ \frac{4}{\sqrt{45}\sqrt{14}}.$$

3. HW #3

3.1. Problems: HW #3.

Section 11.2: Question 1: The corollary on page 830 states two vectors are perpendicular if and only if their dot product is zero. Find a non-zero vector, say $\overrightarrow{u}$, that is perpendicular to $\langle 1, 1, 1 \rangle$. (Extra credit: find another vector perpendicular to $\langle 1, 1, 1 \rangle$ and the vector $\overrightarrow{u}$ that you just found. This extra credit should be written right after this problem, or as part of this problem.)

Question 2: Consider a triangle with sides of length 4, 5 and 6. Which two sides surround the largest angle, and what is the cosine of that angle?

Section 11.3: Question 3: Find the determinant of the 2×2 matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; in other words, we filled in the entries with the numbers 1, 2, 3 and 4 in that order, row by row. Similarly, find the determinant of the 3×3 matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$; in other words, we fill in the numbers by 1, 2, 3, 4, 5, 6, 7, 8, 9.(Extra credit: find a nice formula for the determinant of the $n \times n$ matrix where the entries are 1, 2, ..., n^2 filled as above, and prove your claim. This extra credit should be turned in on a separate sheet of paper.)

Question 4: Find the area of the parallelogram with vertices (0,0), (2,4), (1,6), (3,10).

3.2. **HW #3.**

Section 11.2: Question 1: The corollary on page 830 states two vectors are perpendicular if and only if their dot product is zero. Find a non-zero vector, say $\overrightarrow{u}$, that is perpendicular to $\langle 1,1,1\rangle$. (Extra credit: find another vector perpendicular to $\langle 1,1,1\rangle$ and the vector $\overrightarrow{u}$ that you just found. This extra credit should be written right after this problem, or as part of this problem.)

Solution: Let's say $\overrightarrow{u} = \langle x,y,z\rangle$. Then $\overrightarrow{u}\cdot\langle 1,1,1\rangle = 0$ means

$$x\cdot 1 + y\cdot 1 + z\cdot 1 \ = \ 0.$$

If we take $z = -(x+y)$, we see the dot product is zero. There are thus many possibilities, such as $\overrightarrow{u} = \langle 1,1,-2\rangle$. Another possibility is to take $z=0$ and then $y=-x$, giving us $\langle 1,-1,0\rangle$. Notice the solution space is two-dimensional; we'll see later it's a plane. There are three dimensions initially; we lose one in the direction $\langle 1,1,1\rangle$ and thus two dimensions remain.

Let's say now we want to find a vector $\overrightarrow{w} = \langle x,y,z\rangle$ perpendicular to $\langle 1,1,1\rangle$ and $\langle 1,-1,0\rangle$. We then have

$$x\cdot 1 + y\cdot 1 + z\cdot 1 \ = \ 0 \quad \text{and} \quad x\cdot 1 + y\cdot(-1) + z\cdot 0 \ = \ 0.$$

The first gives us $x+y+z=0$, while the second gives us $x-y=0$ or $x=y$. Substituting this into the first gives $2x+z=0$ so $z=-2x$. Taking $x=1$ we see $y=1$ and $z=-2$, for the vector $\langle 1,1,-2\rangle$ is perpendicular to both $\langle 1,1,1\rangle$ and $\langle 1,-1,0\rangle$.

Question 2: Consider a triangle with sides of length 4, 5 and 6. Which two sides surround the largest angle, and what is the cosine of that angle?

Solution: Let θ_{ij} denote the angle between the sides of length i and j. By the law of cosines, if $c^2 = a^2+b^2-2ab\cos\theta_{ab}$, then $\cos\theta_{ab} = (a^2+b^2-c^2)/2ab$, so the cosines are

$$\cos\theta_{45} \ = \ \frac{4^2+5^2-6^2}{2\cdot 4\cdot 5} \ = \ \frac{1}{8} \ = \ \frac{2}{16} \qquad \cos\theta_{46} \ = \ \frac{4^2+6^2-5^2}{2\cdot 4\cdot 6} \ = \ \frac{9}{16} \qquad \cos\theta_{56} \ = \ \frac{5^2+6^2-4^2}{2\cdot 5\cdot 6} \ = \ \frac{3}{4} \ = \ \frac{12}{16}.$$

Note all the angles are between 0 and 90 degrees (i.e., all angles are acute). The larger the angle, the smaller the cosine. Thus the largest angle has the *smallest* cosine, so the largest angle is the one between the sides of length 4 and 5.

Section 11.3: Question 3: Find the determinant of the 2×2 matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; in other words, we filled in the entries with the numbers 1, 2, 3 and 4 in that order, row by row. Similarly, find the determinant of the 3×3 matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$;in other words, we fill in the numbers by 1, 2, 3, 4, 5, 6, 7, 8, 9.(Extra credit: find a nice formula for the determinant of the $n\times n$ matrix where the entries are 1, 2, ..., n^2 filled as above, and prove your claim. This extra credit should be turned in on a separate sheet of paper.)

Solution: For the 2×2 matrix, the determinant is just $1\cdot 4 - 2\cdot 3 = -2$. For the 3×3 matrix, we write the first two columns again and find the determinant is

$$1\cdot 5\cdot 9 + 2\cdot 6\cdot 7 + 3\cdot 4\cdot 8 - 7\cdot 5\cdot 3 - 8\cdot 6\cdot 1 - 9\cdot 4\cdot 2 \ = \ 0.$$

A little inspection illustrates why this is zero. Note that twice the second row is the sum of the first and third row. Thus the three vectors *do not* really form a 3-dimensional parallelpiped, but rather just a 2-dimensional parallelogram, and the volume of a 2-dimensional parallelogram in 3-dimensional space is just zero. Similarly, the determinant for the $n\times n$ matrix is zero if $n\geq 3$ as twice the second row is always the first row plus the third.

Building on this observation, we can show the determinant of the $n\times n$ matrix with entries from 1 to n^2 is zero for $n\geq 3$, *even though we don't have a formula to compute these determinants for* $n\geq 4$*!* The reason is we have the geometric definition of the determinant, namely that it gives the n-dimensional volume of the region spanned by the rows. Notice that when $n\geq 3$, the sum of the first and third rows equals twice the sum of the second row. Thus these three vectors all lie in a plane, and we have lost at least one dimension. This implies the n-dimensional volume is zero.

Question 4: Find the area of the parallelogram with vertices (0,0), (2,4), (1,6), (3,10).
Solution: The parallelogram is generated by the vectors $\overrightarrow{v} = \langle 2,4\rangle$ and $\overrightarrow{w} = \langle 1,6\rangle$; we find these by looking at $\langle 2,4\rangle - \langle 0,0\rangle$ and $\langle 1,6\rangle - \langle 0,0\rangle$; note that $\langle 3,10\rangle = \langle 2,4\rangle + \langle 1,6\rangle$. We know the area is equal to the determinant of the matrix with first row $\overrightarrow{v}$ and second row $\overrightarrow{w}$. Thus we need the determinant of the matrix $\begin{pmatrix} 2 & 4 \\ 1 & 6 \end{pmatrix}$,which is $2\cdot 6 - 1\cdot 4 = 8$. Note that if we wrote the vectors in the other order we would have the matrix $A' = \begin{pmatrix} 1 & 6 \\ 2 & 4 \end{pmatrix}$,which has determinant $1\cdot 4 - 6\cdot 2 = -8$. What went wrong? We have to remember it is the absolute value of the determinant that is the area.

4. HW #4

4.1. **HW #4.**

Page 842: #1: Find $\overrightarrow{a} \times \overrightarrow{b}$ with $\overrightarrow{a} = \langle 5, -1, -2\rangle$ and $\overrightarrow{b} = \langle -3, 2, 4\rangle$

Page 842: #5: Find the cross product of the $\overrightarrow{a} = \langle 2, -3\rangle$ and $\overrightarrow{b} = \langle 4, 5\rangle$ by extending them to 3-dimensional vectors $\overrightarrow{a} = \langle 2, -3, 0\rangle$ and $\overrightarrow{b} = \langle 4, 5, 0\rangle$.

Page 842: #11: Prove that the vector product is not associative by comparing $\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c})$ with $(\overrightarrow{a} \times \overrightarrow{b}) \times \overrightarrow{c}$ in the case $\overrightarrow{a} = \overrightarrow{i}$, $\overrightarrow{b} = \overrightarrow{i} + \overrightarrow{j}$, and $\overrightarrow{c} = \overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$.

Page 842: #12: Find nonzero vectors $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ such that $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{a} \times \overrightarrow{c}$, but $\overrightarrow{b} \neq \overrightarrow{c}$.

Section 11.4: Question 1: Write parametric equations of the straight line that passes through the point P and is parallel to the vector $\overrightarrow{v}$, with P the point (0,0,0) and $\overrightarrow{v}$ the vector **i** + 2**j** + 3**k** $\langle 1, 2, 3\rangle$.

Section 11.4: Question 2: Write parametric equations of the straight line that passes through the point P and is parallel to the vector $\overrightarrow{v}$, with P equal to (3,-4,5) and $\overrightarrow{v}$= -2**i** + 7**j** + 3**k** $= \langle -2, 7, 3\rangle$.

Section 11.4: Question 3: Write parametric equations of the straight line that passes through the point P and is parallel to the vector $\overrightarrow{v}$, with P equal to (4,13,-3) and $\overrightarrow{v}$= 2**i** - 3**k** $= \langle 2, 0, -3\rangle$.

Section 11.4: Question 22: Write an equation of the plane with normal vector $\overrightarrow{n} = \langle -2, 7, 3\rangle$ that passes through the point $P = (3, -4, 5)$.

4.2. **HW #4.**

Page 842: #1: Find $\overrightarrow{a} \times \overrightarrow{b}$ with $\overrightarrow{a} = \langle 5, -1, -2\rangle$ and $\overrightarrow{b} = \langle -3, 2, 4\rangle$

Solution: We have $\overrightarrow{a} \times \overrightarrow{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$, which in this case is

$$\langle (-1) \cdot 4 - (-2) \cdot 2, (-2) \cdot (-3) - 5 \cdot 4, 5 \cdot 2 - (-1) \cdot (-3)\rangle \;=\; \langle 0, -14, 7\rangle.$$

We could also do the determinant approach, and write the first two columns again:

$$\left|\begin{array}{ccc|cc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} & \overrightarrow{i} & \overrightarrow{j} \\ 5 & -1 & -2 & 5 & -1 \\ -3 & 2 & 4 & -3 & 2 \end{array}\right.$$

and then do the three diagonals (from upper left to bottom right) with positive signs, and then the three diagonals (from bottom left to upper right) with negative signs.

Page 842: #5: Find the cross product of the $\overrightarrow{a} = \langle 2, -3\rangle$ and $\overrightarrow{b} = \langle 4, 5\rangle$ by extending them to 3-dimensional vectors $\overrightarrow{a} = \langle 2, -3, 0\rangle$ and $\overrightarrow{b} = \langle 4, 5, 0\rangle$.

Solution: Again we have $\overrightarrow{a} \times \overrightarrow{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$, which in this case is

$$\langle (-3) \cdot 0 - 0 \cdot 5, 0 \cdot 4 - 2 \cdot 0, 2 \cdot 5 - (-3) \cdot 4\rangle \;=\; \langle 0, 0, 22\rangle.$$

This is a very powerful technique, and allows us to use the cross product, initially defined only in three dimensions, in two dimensions.

Page 842: #11: Prove that the vector product is not associative by comparing $\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c})$ with $(\overrightarrow{a} \times \overrightarrow{b}) \times \overrightarrow{c}$ in the case $\overrightarrow{a} = \overrightarrow{i}$, $\overrightarrow{b} = \overrightarrow{i} + \overrightarrow{j}$, and $\overrightarrow{c} = \overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$.

Solution: Rewriting $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ in terms of their components we have

$$\overrightarrow{a} \;=\; \langle 1, 0, 0\rangle, \quad \overrightarrow{b} = \langle 1, 1, 0\rangle, \quad \overrightarrow{c} = \langle 1, 1, 1\rangle.$$

Using the definition of the cross product, we find that $\overrightarrow{b} \times \overrightarrow{c} = \langle 1, -1, 0\rangle$ and therefore $\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = \langle 0, 0, -1\rangle$.

Similarly, we see that $\overrightarrow{a} \times \overrightarrow{b} = \langle 0, 0, 1\rangle$, which gives $(\overrightarrow{a} \times \overrightarrow{b}) \times \overrightarrow{c} = \langle -1, 1, 0\rangle$. Since $\langle -1, 1, 0\rangle \neq \langle 0, 0, -1\rangle$, we see that the cross product is not associative.

Page 842: #12: Find nonzero vectors $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ such that $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{a} \times \overrightarrow{c}$, but $\overrightarrow{b} \neq \overrightarrow{c}$.

Solution: Here's one solution. Let's start with a specific $\overrightarrow{a}$ and see what happens. The simplest $\overrightarrow{a}$ to take would be $\overrightarrow{a} = \langle 1, 0, 0\rangle$ (we can't take the zero vector, so let's have two components zero). This is a great way to build intuition. Then for any vector $\overrightarrow{b} = \langle b_1, b_2, b_3\rangle$, we have $\overrightarrow{a} \times \overrightarrow{b} = \langle 0, -b_3, b_2\rangle$. Notice that $\overrightarrow{a} \times \overrightarrow{b}$ does not depend on b_1! Therefore let $\overrightarrow{b} = \langle 1, 1, 1\rangle$ and $\overrightarrow{c} = \langle 2011, 1, 1\rangle$. We see that $\overrightarrow{a} \times \overrightarrow{b} = \langle 0, -1, 1\rangle = \overrightarrow{a} \times \overrightarrow{c}$, but $\overrightarrow{b} \neq \overrightarrow{c}$.

For another solution, recall that $\overrightarrow{a} \times \overrightarrow{a} = \overrightarrow{0}$. Thus, if $\overrightarrow{a}$ is *any* vector, we always have

$$\overrightarrow{a} \times \overrightarrow{b} \;=\; \overrightarrow{a} \times \left(\overrightarrow{b} + \overrightarrow{a}\right),$$

so we can take $\overrightarrow{a}$ and $\overrightarrow{b}$ arbitrary, and set $\overrightarrow{c} = \overrightarrow{b} + \overrightarrow{a}$. (Okay, we can't take the zero vector for $\langle a\rangle$.) *An interesting question becomes: given $\overrightarrow{a}$, describe all vectors $\overrightarrow{b}$ and $\overrightarrow{c}$ such that $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{a} \times \overrightarrow{c}$. If you want, you may do this for extra credit.*

Section 11.4: Question 1: Write parametric equations of the straight line that passes through the point P and is parallel to the vector $\overrightarrow{v}$, with P the point (0,0,0) and $\overrightarrow{v}$ the vector **i** + 2**j** + 3**k** = $\langle 1, 2, 3\rangle$.
Solution: The equation is $\langle x, y, z\rangle = P + t\overrightarrow{v}$ with t ranging over all real numbers. Substituting for P and $\overrightarrow{v}$ yields $\langle x, y, z\rangle = \langle 0, 0, 0\rangle + t\langle 1, 2, 3\rangle$, so $\langle x, y, z\rangle = \langle t, 2t, 3t\rangle$, or equivalently x= t, y= $2t$ and $z = 3t$.

Section 11.4: Question 2: Write parametric equations of the straight line that passes through the point P and is parallel to the vector $\overrightarrow{v}$, with P equal to (3,-4,5) and $\overrightarrow{v}$= -2**i** + 7**j** + 3**k** = $\langle -2, 7, 3\rangle$.

Solution: Again the equation of the line is $\langle x, y, z\rangle = P + t\overrightarrow{v}$. Substituting for P and $\overrightarrow{v}$ yields $\langle x, y, z\rangle = \langle 3, -4, 5\rangle + t\langle -2, 7, 3\rangle$. Thus $\langle x, y, z\rangle = \langle 4 + 2t, -4 + 7t, 5 + 37\rangle$, or expanding x= 4+2t, y= -4+7t and z= 5+3t.

Section 11.4: Question 3: Write parametric equations of the straight line that passes through the point P and is parallel to the vector $\overrightarrow{v}$, with P equal to (4,13,-3) and $\overrightarrow{v}$= 2**i** - 3**k** $= \langle 2, 0, -3\rangle$.
Solution: Again the equation of the line is $\langle x, y, z\rangle = P + t\overrightarrow{v}$. Substituting for P and $\overrightarrow{v}$ yields $\langle x, y, z\rangle = \langle 4, 13, -3\rangle + t\langle 2, 0, -3\rangle$. Thus $\langle x, y, z\rangle = \langle 3 - 2t, 13, -3 - 3t\rangle$, or x= 3-2t, y= 13 and z= -3-3t.

Section 11.4: Question 22: Write an equation of the plane with normal vector $\overrightarrow{n} = \langle -2, 7, 3\rangle$ that passes through the point $P = (3, -4, 5)$.
Solution: The equation of the plane is $(\langle x, y, z\rangle - \overrightarrow{P}) \cdot \overrightarrow{n} = 0$, or $\langle x, y, z\rangle \cdot \overrightarrow{n} = \overrightarrow{P} \cdot \overrightarrow{n}$. Substituting gives $\langle x, y, z\rangle \cdot \langle -2, 7, 3\rangle = \langle 3, -4, 5\rangle \cdot \langle -2, 7, 3\rangle$. Thus $-2x + 7y + 3z = 3(-2) + (-4)7 + 5(3) = -19$, so $-2x + 7y + 3z = -19$.

5. HW #5

5.1. **Problems: HW #5.**

Section 11.8: Question 1: Find the rectangular coordinates of the point with the given cylindrical coordinates. $(1, \frac{\pi}{2}, 2\ )$.

Section 11.8: Question 26: Describe the graph of the given equation: $\rho = 5$.

Page 908: #2:: Find the largest possible domain for $f(x, y) = \sqrt{x^2 + 2y^2}$.

Page 908: #4:: Find the largest possible domain for $f(x, y) = 1/(x - y)$.

Page 908: #5:: Find the largest possible domain for $f(x, y) = (y - x^2)^{1/3}$.

Page 908: # 27:: Describe the graph of $f(x, y) = \sqrt{4 - x^2 - y^2}$.

Page 908: #32:: Sketch level sets of $f(x, y) = x^2 - y^2$.

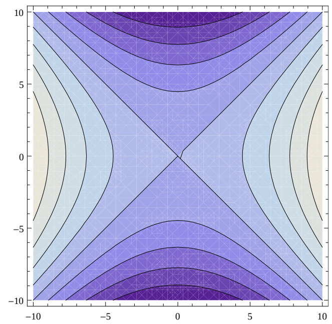

FIGURE 1. Contour Plot for Problem #32 on page 908.

5.2. **Solutions: HW #5.**

Section 11.8: Question 1: Find the rectangular coordinates of the point with the given cylindrical coordinates. $(1, \frac{\pi}{2}, 2\,)$
Solution: In cylindrical coordinates, x= r cos θ, y= r sin θ and z= z. We know that r= 1 and $\theta = \frac{\pi}{2}$ because cylindrical coordinates are written as (r, θ, z). Thus $x = 1\cos\frac{\pi}{2} = 0$, $y = 1\sin\frac{\pi}{2} = 1$, and $z = 2$, and the rectangular coordinates are (0,1,2).

Section 11.8: Question 26: Describe the graph of the given equation: $\rho = 5$.
Solution: The graph of the equation of the form ρ= c (c being a constant) can be described as sphere of radius c centered at the origin, thus the graph is a sphere with a radius 5 centered at the origin. In spherical coordinates we have $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$. Using the Pythagorean Theorem twice, we see that $x^2 + y^2 + z^2 = \rho^2$ in spherical coordinates. Thus if $\rho = 5$, which is the same as $\rho^2 = 25$ (since $\rho \geq 0$), in Cartesian coordinates this becomes $x^2 + y^2 + z^2 = 25$, which is the equation for the surface of a sphere of radius 5.

Page 908: #2:: Find the largest possible domain for $f(x, y) = \sqrt{x^2 + 2y^2}$.
Solution: The function is defined for all values of x and y (thus the domain is all of $\mathbb{R}^2$). The reason is that the only danger with the square-root function are negative numbers, and $x^2 + 2y^2$ is always non-negative.

Page 908: #4:: Find the largest possible domain for $f(x, y) = 1/(x - y)$.
Solution: The only danger with the reciprocal function is when the denominator is zero. Thus, so long as $x \neq y$ the function is defined. We can write this in set notation as the domain is $\{(x, y) \in \mathbb{R}^2 : x \neq y\}$.

Page 908: #5:: Find the largest possible domain for $f(x, y) = (y - x^2)^{1/3}$.
Solution: Note the cube-root of a negative number is a negative number, the cube-root of zero is zero, and the cube-root of a positive number is a positive number. In other words, the cube-root is defined for all real numbers. Thus the domain of this function is all of $\mathbb{R}^2$.

Page 908: # 27:: Describe the graph of $f(x, y) = \sqrt{4 - x^2 - y^2}$.
Solution: Note that the height only depends on $x^2 + y^2$; in other words, any two pairs (x_1, y_1) and (x_2, y_2) that are the same distance from the origin $(0, 0)$ give the same value to our function. There is thus enormous angular symmetry, and we see that there will be lots of circles in our plot. If we look at level sets, we want to solve $\sqrt{4 - x^2 - y^2} = c$ or $4 - x^2 - y^2 = c^2$ or $x^2 + y^2 = 4 - c^2$. Remembering that $\sqrt{\cdots}$ means take the positive square-root, we see that the admissible values of c are $0 \leq c \leq 2$. For each of these we get a circle of radius $\sqrt{4^2 - c^2}$ as the level set. The smallest is when $c = 2$, which is over the origin; the largest circle is when $c = 0$ and then we get a circle of radius 2 in the xy-plane. Another way of looking at this problem is to write $z = f(x, y)$. If we do this we get $z^2 = 4 - x^2 - y^2$ or $x^2 + y^2 + z^2 = 4$. Remembering that $z \geq 0$ (due to the square-root), we see this is just the upper hemisphere of a sphere of radius 2.

Page 908: #32:: Sketch level sets of $f(x, y) = x^2 - y^2$.
Solution: If we have $x^2 - y^2 = c$, note c can be anything. We get a series of hyperbolas *unless* $c = 0$, in which case we get two lines ($x^2 - y^2 = 0$ means $x = \pm y$). See Figure 1. The Mathematica code is:

```
ContourPlot[x^2-y^2, {x,-10,10}, {y,-10,10}]
```

(you can run Mathematica code online: go to http://www.wolframalpha.com/).

6. HW #6

6.1. **Problems: HW #6.**

Page 917: #1:: Find $\lim_{(x,y)\to(0,0)}(7 - x^2 + 5xy)$.

Page 917: #8:: Find $\lim_{(x,y)\to(2,-1)} \ln\left(\frac{1+x+2y}{3y^2-x}\right)$.

Page 917: #10:: Find $\lim_{(x,y)\to(0,0)} \frac{\cos(x^2+y^2)}{1-x^2-y^2}$.

Page 918: #24:: Find the limit or show that it does not exist: $\lim_{(x,y,z)\to(1,-1,1)} \frac{yz+xz+xy}{1+xyz}$.

Page 918: #38:: Evaluate the limit $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ by making the polar coordinates substitution.

Page 919: #54:: Discuss the continuity of the function $f(x,y)$ that is $\frac{\sin xy}{xy}$ if $xy \neq 0$ and 1 if $xy = 0$.

Page 928: #1:: Compute the first-order partial derivatives of $f(x,y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

Page 928: #4:: Compute the first-order partial derivatives of $f(x,y) = e^2e^{xy}$.

Page 928: #5:: Compute the first-order partial derivatives of $f(x,y) = \frac{x+y}{x-y}$.

6.2. **Solutions: HW #6.**

Page 917: #1:: Find $\lim_{(x,y)\to(0,0)}(7-x^2+5xy)$.
Solution: The limit laws tell us that the limit of a sum is the sum of the limits, and similarly for the difference or a product (so long as all limits are finite, and for the quotient the denominator is non-zero). We thus have

$$\lim_{(x,y)\to(0,0)}(7-x^2+5xy) \;=\; \lim_{(x,y)\to(0,0)} 7 - \lim_{(x,y)\to(0,0)} x^2 + \lim_{(x,y)\to(0,0)} 5 \lim_{(x,y)\to(0,0)} x \lim_{(x,y)\to(0,0)} y$$
$$7-0^2+5\cdot 0\cdot 0 \;=\; 7.$$

Page 917: #8:: Find $\lim_{(x,y)\to(2,-1)} \ln\left(\frac{1+x+2y}{3y^2-x}\right)$.
Solution: As $(x,y)\to(2,-1)$ the denominator goes to $3(-1)^2-2=1$ and the numerator goes to $1+2+2(-1)=1$. Thus we are taking the natural logarithm of a quantity getting closer and closer to 1. As $\ln 1 = 0$, the limit is zero. We could also attack this problem by noting $\ln(a/b)=\ln a-\ln b$ and then using the difference rule.

Page 917: #10:: Find $\lim_{(x,y)\to(0,0)} \frac{\cos(x^2+y^2)}{1-x^2-y^2}$.
Solution: As the limit of the denominator is 1, we can use the limit of a quotient is the quotient of the limits. What is the limit of the numerator? We're evaluating cosine at values closer and closer to 0. As cosine is continuous, this equals $\cos 0$ which is 1. Thus our limit is $1/1$ or 1.
Page 918: #24:: Find the limit or show that it does not exist: $\lim_{(x,y,z)\to(1,-1,1)} \frac{yz+xz+xy}{1+xyz}$.

Solution: $\lim_{(x,y,z)\to(1,-1,1)} \frac{yz+xz+xy}{1+xyz}$
The limit does not exist. As (x,y,z) approaches (1,-1,1), the numerator approaches $(-1)\cdot 1+1\cdot 1+1\cdot(-1)=-1$, while the denominator approaches $1+1\cdot(-1)\cdot 1=0$. Thus our quantity looks like $-1/0$ in the limit, which is undefined.

Page 918: #38:: Evaluate the limit $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ by making the polar coordinates substitution.
Solution: Using the textbook's advice to convert from cartesian coordinates to polar coordinates ($x=r\cos\theta$, $y=r\sin\theta$), the problem becomes significantly easier to manage. Note that $(x,y)\to(0,0)$ becomes $r\to 0$ and θ is free. The limit equals $\lim_{r\to 0} \frac{r^3\cos^3\theta-r^3\sin^3\theta}{r^2\cos^2\theta+r^2\sin^2\theta}$. By factoring out the r^2 from both the numerator and denominator, and using the identity $\cos^2\theta+\sin^2\theta=1$, the limit equals $\lim_{r\to 0}\frac{r(\cos^3\theta-\sin^3\theta)}{1}$. Here we can see the limit approaches 0, because $|\cos^3\theta=\sin^3\theta|\le 2$ and $r\to 0$.

Page 919: #54:: Discuss the continuity of the function $f(x,y)$ that is $\frac{\sin xy}{xy}$ if $xy\neq 0$ and 1 if $xy=0$.
Solution: This function is continuous. By the definition of continuity, a function f is continuous at (a,b) if it is defined at (a,b) and the limit is equal to the value there. The only troublesome points are when $a=0$, $b=0$ or both a and b equal 0. Assume first that our point is $(a,0)$ with $a\neq 0$. Then $(x,y)\to(a,0)$ means that eventually x is non-zero and close to a, and y may or may not be zero but is close to 0. We have $f(a,0)=1$. If $y=0$ then $f(x,y)=1$. If $y\neq 0$ and x is close to a and non-zero and y is close to 0, then we must show $f(x,y)$ is close to 1. We have $f(x,y)=\frac{\sin xy}{xy}$ with $xy\neq 0$ and small; however, since $\lim_{t\to 0}\frac{\sin t}{t}=1$, we see that as $(x,y)\to(a,0)$ whenever $xy\neq 0$ we have $xy\to 0$ and thus, setting $t=xy$, we'll have $\frac{\sin xy}{xy}$ arbitrarily close to 1. The analysis for points $(0,b)$ with $b\neq 0$, as well as the point $(0,0)$, is similar.

Page 928: #1:: Compute the first-order partial derivatives of $f(x,y)=x^4-x^3y+x^2y^2-xy^3+y^4$.
Solution: To find the partial derivative with respect to x, we consider y constant and apply the standard rules of differentiation, and find $\frac{\partial f}{\partial x} = 4x^3-3x^2y+2xy-y^3$. To find the partial derivative with respect to y, we consider x constant and find $\frac{\partial f}{\partial y}=-x^3+2x^2y-3xy^2+4y^3$

Page 928: #4:: Compute the first-order partial derivatives of $f(x,y)=e^2e^{xy}$.
Solution: Note that e^2 is just a constant; there is no need to use the product rule – just use the constant rule. We have $\frac{\partial f}{\partial x}=ye^2e^{xy}$ and $\frac{\partial f}{\partial y}=xe^2e^{xy}$.

Page 928: #5:: Compute the first-order partial derivatives of $f(x,y)=\frac{x+y}{x-y}$.
Solution: Applying the quotient rule of differentiation gives $\frac{\partial f}{\partial x}=\frac{1(x-y)-1(x+y)}{(x-y)^2}=\frac{-2y}{(x-y)^2}$ and $\frac{\partial f}{\partial y}=\frac{-1(x+y)+1(x-y)}{(x-y)^2}=\frac{2x}{(x-y)^2}$. Another way to do this problem is to observe the following:

$$\frac{x+y}{x-y} \;=\; \frac{x-y+2y}{x-y} \;=\; \frac{x-y}{x-y}+\frac{2y}{x-y} \;=\; 1+\frac{2y}{x-y}.$$

This is a little nicer than using the quotient rule; if we want the derivative with respect to x note we just need to use the reciprocal rule, and find $\frac{\partial f}{\partial x}=2y\cdot(-1)(x-y)^{-2}$, as before.

7. HW #7

Practice Problem #1: Multiple Derivatives: Let $z(x,y) = x^2e^{xy} + \sin(x^2+3xy) + 2$. Find z_x, z_y, z_{xy} and z_{yx}.
Solution: We have $z_x = \frac{\partial z}{\partial x}$, the partial derivative of the function $z(x,y)$ with respect to x. This means we hold y fixed, and we find

$$z_x \ = \ 2xe^{xy} + x^2e^{xy}y + \cos(x^2+3xy)\cdot(2x+3y),$$

where we used the product rule for the first piece and the quotient rule for the second. Similarly we find

$$z_y \ = \ x^2e^{xy}x + \cos(x^2+3xy)\cdot 3x \ = \ x^3e^{xy} + 3x\cos(x^2+3xy).$$

Note the notation z_{xy} means $\frac{\partial z_x}{\partial y}$; so first we take the derivative of z with respect to x and get $z_x = \frac{\partial z}{\partial x}$, and then we take the derivative of $z_x = \frac{\partial z}{\partial x}$ with respect to y and get $z_{xy} = \frac{\partial z_x}{\partial y}$. To make the notation clearer as to the order, we can use parentheses: $z_{xy} = (z_x)_y$; note we're using our variables as subscripts to indicate the variable of differentiation. We find

$$\begin{aligned} z_{xy} \ = \ \frac{\partial z_x}{\partial y} \ &= \ 2xe^{xy}x + x^2\left(e^{xy}xy + e^{xy}\right) - \sin(x^2+3xy)\cdot 3x(2x+3y) + \cos(x^2+3xy)3 \\ &= \ 3x^2e^{xy} + x^3ye^{xy} + 3\cos(x^2+3xy) - 3x(2x+3y)\sin(x^2+3xy), \\ z_{yx} \ = \ \frac{\partial z_y}{\partial x} \ &= \ 3x^2e^{xy} + x^3e^{xy}y + 3\cos(x^2+3xy) - 3x\sin(x^2+3xy)\cdot(2x+3y); \end{aligned}$$

note $z_{xy} = z_{yx}$. This last equality frequently holds; it is a theorem that it holds if the mixed partial derivatives exist and are continuous. Thus we need only compute z_{xy} to know z_{yx} in many cases, but it's good to do both as a check for calculus and algebra errors.

Below are Mathematica commands for these derivatives. Note we write e^{xy} as $\mathrm{Exp}[x\ y]$ (it's important to put a space between x and y, as otherwise Mathematica reads it as *one* variable and not a product. Similarly sine is encoded as $\mathrm{Sin}[3x^2 + 3x\ y]$ (note the use of square brackets). We use $\mathrm{D}[\text{function}, \text{variable}]$ to denote the derivative of a function with respect to a variable; to do two derivatives we can nest the expressions.

```
D[x^2 Exp[x y] + Sin[x^2 + 3 x y] + 2, x]
D[x^2 Exp[x y] + Sin[x^2 + 3 x y] + 2, y]
D[D[x^2 Exp[x y] + Sin[x^2 + 3 x y] + 2, x], y]
D[D[x^2 Exp[x y] + Sin[x^2 + 3 x y] + 2, y], x]
```

Practice Problem #2: Tangent Planes: Find the tangent plane to $z = e^{x-y}\cos(xy^2\pi)$ at the point $(1,1,-1)$.
Solution: First, we check that this point is on the surface; it is as $-1 = e^{1-1}\cos(1\cdot 1^2\pi)$. From the book or my lecture notes for Chapter 12 (see http://web.williams.edu/Mathematics/sjmiller/public_html/150/currentnotes/Math105LecNotes_Chap12.pdf page 11 for the equation of the tangent plane), we see that if $z = f(x,y)$ (also denoted by $z(x,y)$ at times) then the tangent plane at the point (x_0,y_0,z_0) (with $z_0 = f(x_0,y_0)$) is just

$$z \ = \ f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right]\cdot(x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right]\cdot(y-y_0).$$

Here $\frac{\partial f}{\partial x}(x_0,y_0)$ means we take the partial derivative of f with respect to x and evaluate that at the point (x_0,y_0); if we wanted we could write z_0 instead of $f(x_0,y_0)$ above as $z_0 = f(x_0,y_0)$. We have $(x_0,y_0) = (1,1)$, $z_0 = -1$, and

$$\frac{\partial f}{\partial x} \ = \ e^{x-y}\cos(xy^2\pi) + e^{x-y}\left(-\sin(xy^2\pi)\cdot y^2\pi\right), \quad \text{so} \quad \frac{\partial f}{\partial x}(1,1) \ = \ 1.$$

Similarly we find

$$\frac{\partial f}{\partial y} \ = \ e^{x-y}(-1)\cos(xy^2\pi) + e^{x-y}\left(-\sin(xy^2\pi)\cdot 2xy\pi\right), \quad \text{so} \quad \frac{\partial f}{\partial y}(1,1) \ = \ -1.$$

Thus the tangent plane is

$$z \ = \ f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right]\cdot(x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right]\cdot(y-y_0) \ = \ -1 + 1(x-1) - 1(y-1),$$

which simplifies to

$$z \ = \ -1 + (x-1) - (y-1) \quad \text{or} \quad x - y - z \ = \ 1.$$

7.1. Problems: HW #7.

Page 928: #21:: Show $z_{xy} = z_{yx}$ with $z(x,y) = x^2 - 4xy + 3y^2$.

Page 928: #25:: Show $z_{xy} = z_{yx}$ with $z(x,y) = \ln(x+y)$.

Page 928: #33:: Find the tangent plane to $z = \sin\frac{\pi xy}{2}$ at the point $(3,5,-1)$.

Page 928: #36:: Find the tangent plane to $z = 3x + 4y$ at the point $(1,1,7)$.

Page 928: #63:: The ideal gas law says $pV = nRT$. Show $\frac{\partial p}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial p} = -1$. Is this surprising?

7.2. **Solutions: HW #7.**

Page 928: #21:: Show $z_{xy} = z_{yx}$ with $z(x,y) = x^2 - 4xy + 3y^2$.
Solution: We have $z_x = 2x - 4y$ and then $z_{xy} = -4$, while $z_y = -4x + 6y$ and then $z_{yx} = -4 = z_{xy}$.

Page 928: #25:: Show $z_{xy} = z_{yx}$ with $z(x,y) = \ln(x+y)$.
Solution: First, remember that $\ln(x+y) \neq \ln x + \ln y$; it is $\ln(xy)$ that equals this. We have $z_x = \frac{1}{x+y} = (x+y)^{-1}$ and thus $z_{xy} = -(x+y)^{-2}$. Similarly, $z_y = \frac{1}{x+y} = (x+y)^{-1}$ and $z_{yx} = -(x+y)^{-2} = z_{xy}$.

Page 928: #33:: Find the tangent plane to $z = \sin \frac{\pi xy}{2}$ at the point $(3, 5, -1)$.
Solution: First, we check that this point is on the surface; it is as $-1 = \sin \frac{15\pi}{2}$. Thus $x_0 = 3$, $y_0 = 5$ and $z_0 = f(x_0, y_0) = -1$, with $f(x,y) = \sin \frac{\pi xy}{2}$. We have $\frac{\partial f}{\partial x} = \frac{\pi y}{2} \cos \frac{\pi xy}{2}$, so $\frac{\partial f}{\partial x}\Big|_{(3,5)} = 0$. Similarly $\frac{\partial f}{\partial y} = \frac{\pi x}{2} \cos \frac{\pi xy}{2}$, so $\frac{\partial f}{\partial y}\Big|_{(3,5)} = 0$. The tangent plane is

$$z \;=\; f(3,5) + \frac{\partial f}{\partial x}\Big|_{(3,5)}(x-3) + \frac{\partial f}{\partial y}\Big|_{(3,5)}(y-5) \;=\; -1$$

(since the two partial derivatives vanish at the point of interest).

Page 928: #36:: Find the tangent plane to $z = 3x + 4y$ at the point $(1, 1, 7)$.
Solution: Note this is the equation of a plane, so we expect this to be the answer (this problem is thus a good check of the reasonableness of our definition of the tangent plane). First, we do observe that $7 = 3 \cdot 1 + 4 \cdot 1$. Letting $f(x,y) = 3x + 4y$, we have $x_0 = 1$, $y_0 = 1$, $z_0 = f(x_0, y_0) = 7$, $\frac{\partial f}{\partial x} = 3$ so $\frac{\partial f}{\partial x}\Big|_{(1,1)} = 3$, and $\frac{\partial f}{\partial y} = 4$ so $\frac{\partial f}{\partial y}\Big|_{(1,1)} = 4$. The tangent plane is

$$z \;=\; f(1,1)\frac{\partial f}{\partial x}\Big|_{(1,1)}(x-1) + \frac{\partial f}{\partial y}\Big|_{(1,1)}(y-1) \;=\; 7 + 3(x-1) + 4(y-1) \;=\; 3x + 4y.$$

Page 928: #63:: The ideal gas law says $pV = nRT$. Show $\frac{\partial p}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial p} = -1$.
Solution: We may write $p = nRT/V$, $V = nRT/p$ and $T = pV/nR$. Direct computation gives $\frac{\partial p}{\partial V} = -nRT/V^2$, $\frac{\partial V}{\partial T} = nR/p$ and $\frac{\partial T}{\partial p} = V/nR$. Thus

$$\frac{\partial p}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial p} \;=\; -\frac{nRT}{V^2}\,\frac{nR}{p}\,\frac{V}{nR} \;=\; -\frac{nRT}{Vp} \;=\; -1,$$

where the last equality follows from $pV = nRT$. Surprising! “Canceling” the differentials gives $\frac{\partial p}{\partial p}$, which is 1.

7.3. **Problems: HW #8.**

Page 940: #5: Find every point on the surface $f(x,y) = x^2 + y^2 - 6x + 2y + 5$ at which the tangent plane is horizontal.

Page 940: #11: Find every point on the surface $f(x,y) = (2x^2 + 3y^2)\exp(-x^2 - y^2)$ at which the tangent plane is horizontal.

Page 940: #29: Find the first octant point on the surface $12x + 4y + 3z = 169$ closest to the point $(0, 0, 0)$.

Page 941: #61a.: Suppose Alpha Inc and Beta Ltd have profit functions given by

$$P(x,y) \;=\; -2x^2 + 12x + xy - y - 10, \qquad Q(x,y) \;=\; -3y^2 + 18y + 2xy - 2x - 15,$$

where x is the price of Alpha Inc’s good and y is the price of Beta Ltd’s good. Supposing that the managers of Alpha and Beta know calculus and know that the other manager knows calculus as well, what price will the two companies set to maximize their profits?

Page 941: #61b.: Now suppose that Alpha Inc and Beta Ltd set their prices so as to maximize their combined profit. Now what will the optimal x and y be?

7.4. **Solutions: HW #8.**

Page 940: #5: Find every point on the surface $f(x,y) = x^2 + y^2 - 6x + 2y + 5$ at which the tangent plane is horizontal.
Solution: Remember that the equation for the plane tangent to the surface $z = f(x,y)$ at the point (a,b) is given by

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b),$$

with $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$. Therefore, if we want the tangent plane to be horizontal, we need to find all points (x,y) such that both partial derivatives vanish at that point, or $f_x(x,y) = f_y(x,y) = 0$. We have

$$\frac{\partial f}{\partial x} = 2x - 6, \quad \frac{\partial f}{\partial y} = 2y + 2.$$

We see $f_x(x,y) = 0$ when $x = 3$ and $f_y(x,y) = 0$ when $y = -1$ (this is the example done in class), so the only point at which the tangent plane is horizontal is $(3,-1)$.

Page 940: #11: Find every point on the surface $f(x,y) = (2x^2 + 3y^2)\exp(-x^2 - y^2)$ at which the tangent plane is horizontal.
Solution: The equation for the plane tangent to the surface $f(x,y)$ at the point (a,b) is given by

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Therefore, if we want the tangent plane to be horizontal, we need to find all points (x,y) such that $f_x(x,y) = f_y(x,y) = 0$. Taking the partial of f with respect to x, we have

$$\frac{\partial f}{\partial x} = (2x^2 + 3y^2)e^{-x^2-y^2}(-2x) + 4xe^{-x^2-y^2} = 2xe^{-x^2-y^2}(2 - 2x^2 - 3y^2).$$

Since $e^{-x^2-y^2} \neq 0$ for all pairs of real numbers (x,y), we see that $f_x(x,y) = 0$ when $x = 0$ or $2 - 2x^2 - 3y^2 = 0$. Similarly taking the partial derivative of f with respect to y gives

$$\frac{\partial f}{\partial y} = (2x^2 + 3y^2)e^{-x^2-y^2}(-2y) + 6ye^{-x^2-y^2} = 2ye^{-x^2-y^2}(3 - 2x^2 - 3y^2).$$

Therefore $f_y(x,y) = 0$ when $y = 0$ or when $3 - 2x^2 - 3y^2 = 0$.

We now need to find the points where both f_x and f_y are 0. Since we know $f_x(x,y) = 0$ whenever $x = 0$, let's first let $x = 0$. For f_y to be 0 given that $x = 0$, we need $2y(3 - 3y^2) = 0$, so $y = 0$ or $y = \pm 1$. Therefore the tangent plane is horizontal at the three points $(0,0,0)$, $(0,1,3/e)$ and $(0,-1,3/e)$ (the z-component is found by evaluating f at the x and y values).

We also know that $f_y(x,y) = 0$ whenever $y = 0$, so now let's let $y = 0$. For f_x to be 0, we need $2x(2 - 2x^2) = 0$, so $x = 0$ or $x = \pm 1$. Therefore the tangent plane is also horizontal at the points $(0,0,0)$, $(1,0,2/e)$ and $(-1,0,2e^{-1})$.

Finally, we need to make sure there aren't any other solutions we're missing. Notice that we've found every possible solution where x or y is 0. Thus any other solution we could find would have x and y not equal to 0. In this case, for f_y to be 0 we need $3 - 2x^2 - 3y^2 = 0$. Similarly, for f_x to be 0 we need $2 - 2x^2 - 3y^2 = 0$. However, it is impossible for both of these equations to be satisfied at the same time. Subtracting the two equations we find $1 = 0$, which is a clear contradiction. Therefore there are no additional solutions with both x and y not equal to 0.

Page 940: #29: Find the first octant point on the surface $12x + 4y + 3z = 169$ closest to the point $(0,0,0)$.
Solution: We want to express the distance from a point on the surface to the origin as a function of x and y. Once we've done that, we can use our optimization techniques to find the pair (x,y) which minimizes this distance. Notice that we can rewrite the equation of the plane as

$$z = \frac{169}{3} - 4x - \frac{4}{3}y.$$

Therefore any point on the plane can be written as $(x, y, 169/3 - 4x - 4y/3)$. The distance squared from this point to the origin is given by

$$h(x,y) = x^2 + y^2 + \left(\frac{169}{3} - 4x - \frac{4}{3}y\right)^2.$$

Notice that the point (x,y) which minimizes the distance from the origin to the plane also minimizes the distance squared from the origin to the plane. Therefore we can just minimize $h(x,y)$ to find our optimal point, instead of having to deal with the nasty square roots that come into play with actual distance. To minimize $h(x,y)$, we take the partial derivatives with respect to x and y, and set them equal to 0. We have

$$\frac{\partial h}{\partial x} = 2x - 8\left(\frac{169}{3} - 4x - \frac{4}{3}y\right), \quad \frac{\partial h}{\partial y} = 2y - \frac{8}{3}\left(\frac{169}{3} - 4x - \frac{4}{3}y\right).$$

So we want to solve the system of equations

$$2x - 8\left(\frac{169}{3} - 4x - \frac{4}{3}y\right) = 0$$
$$2y - \frac{8}{3}\left(\frac{169}{3} - 4x - \frac{4}{3}y\right) = 0.$$

Multiplying the second equation by 3 and we get

$$2x - 8\left(\frac{169}{3} - 4x - \frac{4}{3}y\right) = 0$$
$$6y - 8\left(\frac{169}{3} - 4x - \frac{4}{3}y\right) = 0.$$

Subtracting the second equation from the first gives $2x - 6y = 0$, so $x = 3y$. Substituting $x = 3y$ into the first equation gives

$$6y - 8\left(\frac{169}{3} - 12y - \frac{4}{3}y\right) = 0,$$

which simplifies to $y = 4$. Therefore $x = 3y = 12$, and $z = 169/3 - 4\cdot 12 - 4\cdot 4/3 = 3$, so the point on the plane $12x+4y+3z = 169$ which is closest to the origin is $(12, 4, 3)$.

Page 941: #61a.: Suppose Alpha Inc and Beta Ltd have profit functions given by

$$P(x, y) = -2x^2 + 12x + xy - y - 10$$
$$Q(x, y) = -3y^2 + 18y + 2xy - 2x - 15,$$

where x is the price of Alpha Inc's good and y is the price of Beta Ltd's good. Supposing that the managers of Alpha Inc and Beta Ltd know calculus & know that the other manager knows calc as well, what price will the two companies set to maximize their profits?
Solution: Since Alpha Inc can only control its own price, it will set its price to the point where $P_x = 0$. Similarly, Beta Ltd will set its price to the point where $Q_y = 0$. That is,

$$\frac{\partial P}{\partial x} = -4x + 12 + y = 0, \quad \frac{\partial Q}{\partial y} = -6y + 18 + 2x = 0.$$

From the first equation we find $y = 4x - 12$. Substituting this into the second equation gives $-6(4x - 12) + 18 + 2x = 0$, which simplifies to $x = 45/11$. Plugging this back into the first equation then gives $y = 48/11$.

Page 941: #61b.: Now suppose that Alpha Inc and Beta Ltd set their prices so as to maximize their combined profit. Now what will the optimal x and y be?
Solution: Now our profit function is $R(x, y) = P(x, y) + Q(x, y) = -2x^2 + 10x + -3y^2 + 17y + 3xy - 25$. To maximize this function with respect to x and y, we will take the partials with respect to x and y and set them equal to 0. This gives

$$\frac{\partial R}{\partial x} = -4x + 10 + 3y = 0, \quad \frac{\partial R}{\partial y} = -6y + 17 + 3x = 0.$$

The first equation gives $x = (3y + 10)/4$. Plugging this into the second equation yields $y = 98/15$. Substituting this value for y back into the first equation gives $x = 37/5$. Note that, with profit sharing, one company is quite willing to take one for the team!

8. Homework #9

Skim my notes on the Method of Least Squares; link on the course homepage, or go to http://www.williams.edu/Mathematics/sjmiller/public_html/150/handouts/MethodLeastSquares.pdf. Make sure you are comfortable with all the material from the exam. Try practice problems from the course homepage and the book. http://web.williams.edu/Mathematics/sjmiller/public_html/150/practiceexamindex.htm.

Homework from handout
http://www.williams.edu/Mathematics/sjmiller/public_html/150/handouts/MethodLeastSquares.pdf.

Exercise 3.3: Consider the observed data $(0, 0), (1, 1), (2, 2)$. Show that if we use (2.10) from the Least Squares handout to measure error then the line $y = 1$ yields zero error, and clearly this should not be the best fit line!

Exercise 3.9: Show that the Method of Least Squares predicts the period of orbits of planets in our system is proportional to the length of the semi-major axis to the 3/2 power.

8.1. Solutions: HW #9.

Exercise 3.3: Consider the observed data $(0,0),(1,1),(2,2)$. Show that if we use (2.10) from the Least Squares handout to measure error then the line $y=1$ yields zero error, and clearly this should not be the best fit line!
Solution: We will use equation (2.10) to calculate the error of the line $y=1$. This gives an error function $E_2(a,b)=\sum_{n=0}^{N}(y_i-(ax_i+b))$. Evaluating the sum with the line $y=1$ (which means $a=0$ and $b=1$) gives an error of

$$E_2(0,1) \ = \ (0-1)+(1-1)+(2-1) \ = \ 0.$$

The problem with (2.10) is that the errors are signed quantities, so during the calculation the positive errors cancel out the negative errors.

Exercise 3.9: Show that the Method of Least Squares predicts the period of orbits of planets in our system is proportional to the length of the semi-major axis to the 3/2 power.
Solution: Using the numbers from the handout, namely

$$a \ = \ \frac{\sum_{n=1}^N 1\sum_{n=1}^N x_ny_n-\sum_{n=1}^N x_n\sum_{n=1}^N y_n}{\sum_{n=1}^N 1\sum_{n=1}^N x_n^2-\sum_{n=1}^N x_n\sum_{n=1}^N x_n}, \quad b \ = \ \frac{\sum_{n=1}^N x_n\sum_{n=1}^N x_ny_n-\sum_{n=1}^N x_n^2\sum_{n=1}^N y_n}{\sum_{n=1}^N x_n\sum_{n=1}^N x_n-\sum_{n=1}^N x_n^2\sum_{n=1}^N 1},$$

we find $N=8$, $\sum_{n=1}^8 1=8$, $\sum_{n=1}^8 x_n=9.409461$, $\sum_{n=1}^8 y_n=14.1140384$, $\sum_{n=1}^8 x_n^2=29.29844102$ and $\sum_{n=1}^8 x_ny_n=43.94486382$. Feeding these into the equations for a and b in the handout give best fit values of $a=1.49985642$ and $b=0.000149738$ (the reason b is so close to zero is we have chosen to measure distances in astronomical units, precisely to make the proportionality constant nice). *Note that this is not a cookbook problem; this is one of the most important calculations in the history of science, as it was one of the three guideposts that helped lead Newton to his law of universal gravitation.*

9. HW #10

9.1. Problems: HW #10.

Page 949: #18:: Use the exact value of $f(P)$ and the differential df to approximate the value $f(Q)$, where $f(x,y)=\sqrt{x^2-y^2}$, with points $P(13,5)$ and $Q(13.2,4.9)$.

Page 949: #23:: Use the exact value of $f(P)$ and the differential df to approximate the value $f(Q)$, where $f(x,y,z)=e^{-xyz}$ with the points $P=(1,0,-2)$ and $Q=(1.02,0.03,-2.02)$.

Problem #3: Briefly describe what Newton's Method is used for, and roughly how it works.

Extra Credit: to be handed in on a separate paper: Let $f(x)=\exp(-1/x^2)$ if $|x|>0$ and 0 if $x=0$. Prove that $f^{(n)}(0)=0$ (i.e., that all the derivatives at the origin are zero). This implies the Taylor series approximation to $f(x)$ is the function which is identically zero. As $f(x)=0$ only for $x=0$, this means the Taylor series (which converges for all x) only agrees with the function at $x=0$, a very unimpressive feat (as it is forced to agree there).

9.2. **Solutions: HW #10.**
qqq **Page 949: #18::** Use the exact value of $f(P)$ and the differential df to approximate the value $f(Q)$, where $f(x,y) = \sqrt{x^2 - y^2}$, with points $P(13,5)$ and $Q(13.2, 4.9)$.
Solution: Applying the differential $df = f_x(a,b)\Delta x + f_y(a,b)\Delta y$, we can approximate the value $f(Q)$. We have $f(x,y) = \sqrt{x^2 - y^2}$, with points $P(13,5)$ and $Q(13.2, 4.9)$. This means $\Delta x = 0.2$ and $\Delta y = -0.1$. Take the partial derivatives of f with respect to x and y. We get $\frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2-y^2}} = \frac{x}{\sqrt{x^2-y^2}}$ and $\frac{\partial f}{\partial y} = \frac{-2y}{2\sqrt{x^2-y^2}} = \frac{-y}{\sqrt{x^2-y^2}}$. We find $df = \frac{x}{\sqrt{x^2-y^2}}\Big|_{(13,5)} \Delta x + \frac{-y}{\sqrt{x^2-y^2}}\Big|_{(13,5)} \Delta y$. Evaluating the partial derivatives at the points and putting in the values of Δx and Δy gives $df = \frac{13}{\sqrt{13^2-5^2}}(0.2) - \frac{5}{\sqrt{13^2-5^2}}(-0.1)$, or equivalently $df = \frac{2.6}{12} + \frac{.5}{12} = 0.2583$. Now we just have to add the differential df to $f(P)$ to obtain an approximation of $f(Q)$, and obtain $f(P) = \sqrt{13^2 - 5^2} = 12$. Thus an approximation of $f(Q)$ is $12 + .2583 = 12.2583$. If you prefer to use the notation of the tangent plane, what we have is

$$z \ = \ f(13,5) + \frac{\partial f}{\partial x}\Big|_{(13,5)}(13.2 - 13) + \frac{\partial f}{\partial y}\Big|_{(13,5)}(4.9 - 5).$$

Page 949: #23:: Use the exact value of $f(P)$ and the differential df to approximate the value $f(Q)$, where $f(x,y,z) = e^{-xyz}$ with the points $P = (1,0,-2)$ and $Q = (1.02, 0.03, -2.02)$.
Solution: Similar to Problem 18, we use the differential $df = f_x(a,b)\Delta x + f_y(a,b)\Delta y$ to approximate $f(Q)$. We have $f(x,y,z) = e^{-xyz}$ with the points $P = (1,0,-2)$ and $Q = (1.02, 0.03, -2.02)$. Take the partial derivatives of f with respect to x, y, and z. This gives $\frac{\partial f}{\partial x} = -yze^{-xyz}$, $\frac{\partial f}{\partial y} = -xze^{-xyz}$, $\frac{\partial f}{\partial z} = -zye^{-xyz}$. We now find the differential df. Notice that any terms multiplied by y will be 0 because point P is $(1,0,2)$. This simplifies the math significantly. $df = -yze^{-xyz}\Big|_{(1,0,-2)} \Delta x - xze^{-xyz}\Big|_{(1,0,-2)} \Delta y - zye^{-xyz}\Big|_{(1,0,-2)} \Delta z$. Evaluating the partial derivatives gives $df = 0 - (1)(-2)e^{-(1)(0)(-2)}(0.03) - 0 = 2e^0(0.03) = 0.06$. We calculate $f(P)$ and add it to the differential df to obtain an approximation of $f(Q)$: $f(P) = e^{-(1)(0)(2)} = e^0 = 1$, so $f(Q) \approx 1 + 0.06 = 1.06$.

Problem #3: Briefly describe what Newton's Method is used for, and roughly how it works.
Solution: We use Newton's Method to find x such that $f(x) = 0$. We start with an initial guess, x_0, and use the tangent line to approximate our function with a line, and see where that intersects the x-axis. Calling that point x_1, we then find the new point on the curve with this as its x-coordinate, and approximate again with the tangent line. We look for the new intersection with the x-axis, and call that point x_2. We keep iterating and hopefully the sequence $\{x_0, x_1, x_2, \dots,\}$ converges to a solution to $f(x) = 0$.

Extra Credit: to be handed in on a separate paper: Let $f(x) = \exp(-1/x^2)$ if $|x| > 0$ and 0 if $x = 0$. Prove that $f^{(n)}(0) = 0$ (i.e., that all the derivatives at the origin are zero). This implies the Taylor series approximation to $f(x)$ is the function which is identically zero. As $f(x) = 0$ only for $x = 0$, this means the Taylor series (which converges for all x) only agrees with the function at $x = 0$, a very unimpressive feat (as it is forced to agree there).

First Proof (Professor Miller): The proof follows by induction. If you haven't seen induction, you can look it up online, check out my notes, or see me. Basically, induction is a way to prove statements for all n. Let's use L'Hopital's rule to find the derivative at 0. We start with the definition of the derivative, noting that $f(0) = 0$. We find

$$f'(0) \ = \ \lim_{h\to 0} \frac{f(h) - f(0)}{h} \ = \ \lim_{h \to 0} \frac{\exp(-1/h^2)}{h}.$$

We now change variables; let $k = 1/h$, so as $h \to 0$ we have $k \to \infty$. We find

$$f'(0) \ = \ \lim_{k\to\infty} \frac{\exp(-k^2)}{1/k} \ = \ \lim_{k\to\infty} \frac{k}{\exp(k^2)}.$$

Note this is of the form ∞/∞, and we can use L'Hopital's rule. We find

$$f'(0) \ = \ \lim_{k\to\infty} \frac{k}{\exp(k^2)} = \lim_{k\to\infty} \frac{1}{2k\exp(k^2)}.$$

As we no longer have ∞/∞ we stop, and see that $f'(0) = 0$.

To find the second derivative, we argue similarly. We now know that

$$f'(x) \ = \ \begin{cases}-\frac{1}{x^3}\exp(-1/x^2) & \text{if } x \neq 0\\ 0 & \text{if } x = 0.\end{cases}$$

We again use the definition of the derivative and L'Hopital's rule. In general the n^{th} derivative is of the form $p_n(1/x)\exp(-1/x^2)$ for $x \neq 0$ and 0 if $x = 0$, where p_n is polynomial with finitely many terms. We then just use L'Hopital!

Second Proof (2011 TA David Thompson): Let $f(x) = \exp(-1/x^2)$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. We want to show that all of the derivatives of $f(x)$ vanish when $x = 0$. Notice that it's not even clear whether this function is once differentiable, let only infinitely differentiable! However, it can be shown (using techniques from real analysis) that $f(x)$ is indeed infinitely differentiable. We will simply assume this to be true. Since $f(x)$ is infinitely differentiable (meaning all of its derivatives are continuous), we need only show that the limit of $f^{(n)}(x) = 0$ as $x \to 0$; by continuity, this will imply $f^{(n)}(0) = 0$. Making the change of variables $x \mapsto 1/y$, we see that this is equivalent to showing that all the derivatives of the function $g(y) = \exp(-y^2)$ approach 0 as $y \to \infty$.

Let's think about derivatives of $g(y)$. We see

$$g'(y) \ = \ -2y\exp(-y^2) \ = \ -2yg(y).$$

Remember that the exponential function decays faster than any polynomial; that is, if $p(y) = a_0 + a_1 y + \cdots + a_n y^n$ with $a_i \in \mathbb{R}$, then

$$\lim_{y\to\infty} \frac{p(y)}{\exp(y)} \ = \ 0.$$

Therefore $g'(y) \to 0$ as $y \to \infty$, since we can write $g'(y)$ as a polynomial in y divided by an exponential function. Suppose we knew that *every* derivative of $g(y)$ could be written as a polynomial in y times $g(y)$. By the same argument as above, this would imply that every derivative of $g(y)$ decays to 0 as y goes to infinity. Remember this would imply that every derivative of $f(x)$ is 0 when $x = 0$, which is what we want to show. Our new task, then, is to show that every derivative on $g(y)$ can be written as a polynomial in y times $g(y)$.

To prove this claim we are going to use mathematical induction (if you haven't seen this before, check out Professor Miller's notes online). Our claim is that for all positive integers n, the n^{th} derivative of $g(y)$, $g^n(y)$, can be written as $h_n(y)g(y)$ where $h_n(y)$ is a polynomial in y. Notice that we've already shown the base case $n = 1$. Suppose that our claim holds for some $n = k \geq 1$; we show it holds for $n = k + 1$.

If $g^k(y) = h_k(y)g(y)$, then we have

$$\begin{aligned} g^{k+1}(y) &= h_k'(y)g(y) + g'(y)h_k(y)\\ &= h_k'(y)g(y) - 2yg(y)h_k(y)\\ &= g(y)(h_k'(y) - 2yh_k(y)). \end{aligned}$$

Letting $h_{k+1}(y) = h_k'(y) - 2yh_k(y)$, we see that $g^{k+1}(y) = h_{k+1}(y)g(y)$, so we can indeed write $g^{k+1}(y)$ as a product of a polynomial in y times $g(y)$, and we've proven our claim.

Therefore $f(x)$ really is as strange as we claimed: despite having all of its derivatives equal 0 at the origin, $f(x)$ only equals 0 when $x = 0$. Thus the Taylor Series expansion of $f(x)$ about $x = 0$ only agrees with $f(x)$ at one point!

10. HW #11

10.1. **Problems: HW #11.**

Page 960: #2: Find dw/dt both by using the chain rule and by expressing w explicitly as a function of t before differentiating, with $w = \frac{1}{u^2+v^2}$, $u = \cos(2t)$, $v = \sin(2t)$.
Page 960: #5: Find $\partial w/\partial s$ and $\partial w/\partial t$ with $w = \ln(x^2 + y^2 + z^2)$, $x = s - t$, $y = s + t$, $z = 2\sqrt{st}$.
Page 960: #8: Find $\partial w/\partial s$ and $\partial w/\partial t$ with $w = yz + zx + xy$, $x = s^2 - t^2$, $y = s^2 + t^2$, $z = s^2 t^2$.
Page 960: #34: A rectangular box has a square base. Find the rate at which its volume and surface area are changing if its base is increasing at 2 cm/min and its height is decreasing at 3cm/min at the instant when each dimension is 1 meter (i.e., 100 cm).
Page 960 #41: Suppose that $w = f(u)$ and that $u = x + y$. Show that $\partial w/\partial x = \partial w/\partial y$.

10.2. **Solutions: HW #11.**

Page 960: #2: Find dw/dt both by using the chain rule and by expressing w explicitly as a function of t before differentiating, with $w = \frac{1}{u^2+v^2}$, $u = \cos(2t)$, $v = \sin(2t)$.
Solution: To use the chain rule, we need to consider w as a function of u and v, which are in turn functions of t; here $\frac{\partial w}{\partial t} = \frac{dw}{dt}$, $\frac{\partial u}{\partial t} = \frac{du}{dt}$ and $\frac{\partial v}{\partial y} = \frac{dv}{dt}$ as all are functions of just one variable. Let us write $w(t) = f(u(t), v(t))$, with $f(u, v) = 1/(u^2 + v^2)$, $u(t) = \cos(2t)$, $v(t) = \sin(2t)$. We have

$$\frac{\partial w}{\partial t} \ = \ \frac{\partial f}{\partial u}\Big|_{(u(t),v(t))}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\Big|_{(u(t),v(t))}\frac{\partial v}{\partial t} \quad \text{or} \quad \frac{dw}{dt} \ = \ \frac{\partial f}{\partial u}\Big|_{(u(t),v(t))}\frac{du}{dt} + \frac{\partial f}{\partial v}\Big|_{(u(t),v(t))}\frac{dv}{dt}.$$

We see $f_u = \partial f/\partial u = -(u^2 + v^2)^{-2} \cdot 2u$, $f_y = \partial f/\partial v = -(u^2 + v^2)^{-2} \cdot 2v$, $du/dt = -2\sin(2t)$ and $dv/dt = 2\cos(2t)$. Remembering to evaluate f_u and f_v at $(u(t), v(t)) = (\cos 2t, \sin 2t)$, we find

$$\frac{dw}{dt} \ = \ \frac{4\cos(2t)\sin(2t)}{\cos^2(2t) + \sin^2(2t)} - \frac{4\sin(2t)\cos(2t)}{\cos^2(2t) + \sin^2(2t)} \ = \ 0.$$

For the second approach, we write w as a function of t and differentiate. We see

$$w(t) \ = \ \frac{1}{\cos^2(2t) + \sin(2t)^2} \ = \ 1.$$

As $w(t)$ is constant, differentiating gives $dw/dt = 0$, as we found above.

Page 960: #5: Find $\partial w/\partial s$ and $\partial w/\partial t$ with $w = \ln(x^2 + y^2 + z^2)$, $x = s - t$, $y = s + t$, $z = 2\sqrt{st}$.
Solution: Again, to minimize the chance of error, we'll introducing a placeholding function f, and have $w(s, t) = f(x(s, t), y(s, t), z(s, t)$. As we vary s keeping t fixed, w can change for three reasons: a change in s can cause a change in x, which can cause a change in f; a change in s can cause a change in y, which can cause a change in f; or a change in s can cause a change in z, which can cause a change in f. The Chain Rule says

$$\frac{\partial w}{\partial s} \ = \ \frac{\partial f}{\partial x}\Big|_{(x(s,t),y(s,t),z(s,t))}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\Big|_{(x(s,t),y(s,t),z(s,t))}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\Big|_{(x(s,t),y(s,t),z(s,t))}\frac{\partial z}{\partial s}.$$

Therefore we have

$$\frac{\partial w}{\partial s} \ = \ \frac{2x(t,s)}{x(t,s)^2 + y(t,s)^2 + z(t,s)^2} + \frac{2y(t,s)}{x(t,s)^2 + y(t,s)^2 + z(t,s)^2} + \frac{2z(t,s)}{x(t,s)^2 + y(t,s)^2 + z(t,s)^2}\frac{2\sqrt{t}}{2\sqrt{s}}.$$

Substituting for $x(t, s)$, $y(t, s)$, and $z(t, s)$ gives $\frac{\partial w}{\partial s} \ = \ \frac{2s+2t}{(s+t)^2} \ = \ \frac{2}{s+t}$.

Similarly, we can write

$$\frac{\partial w}{\partial t} \ = \ \frac{\partial f}{\partial x}\Big|_{(x(s,t),y(s,t),z(s,t))}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\Big|_{(x(s,t),y(s,t),z(s,t))}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\Big|_{(x(s,t),y(s,t),z(s,t))}\frac{\partial z}{\partial t},$$

giving

$$\frac{\partial w}{\partial t} \ = \ \frac{-2x(t,s)}{x(t,s)^2 + y(t,s)^2 + z(t,s)^2} + \frac{2y(t,s)}{x(t,s)^2 + y(t,s)^2 + z(t,s)^2} + \frac{2z(t,s)}{x(t,s)^2 + y(t,s)^2 + z(t,s)^2}\frac{2\sqrt{s}}{2\sqrt{t}}.$$

Substituting for $x(t, s)$, $y(t, s)$, and $z(t, s)$ gives $\frac{\partial w}{\partial t} \ = \ \frac{2}{s+t}$.

Page 960: #8: Find $\partial w/\partial s$ and $\partial w/\partial t$ with $w = yz + zx + xy$, $x = s^2 - t^2$, $y = s^2 + t^2$, $z = s^2t^2$.
Solution: Let's write $w(s, t) = f(x(s, t), y(s, t), z(s, t))$. The Chain Rule gives

$$\frac{\partial w}{\partial s} \ = \ \frac{\partial f}{\partial x}\Big|_{(x(s,t),y(s,t),z(s,t))}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\Big|_{(x(s,t),y(s,t),z(s,t))}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\Big|_{(x(s,t),y(s,t),z(s,t))}\frac{\partial z}{\partial s}.$$

As

$$\frac{\partial f}{\partial x} \ = \ z + y, \quad \frac{\partial f}{\partial y} \ = \ z + x, \quad \frac{\partial f}{\partial z} \ = \ y + z$$

and

$$\frac{\partial x}{\partial s} \ = \ 2s, \quad \frac{\partial y}{\partial s} \ = \ 2s, \quad \frac{\partial z}{\partial s} \ = \ 2t^2s,$$

we have

$$\frac{\partial w}{\partial s} \ = \ (z(s,t) + y(s,t))(2s) + (z(s,t) + x(s,t))(2s) + (y(s,t) + x(s,t))(2t^2s).$$

Substituting for $x(s, t)$, $y(s, t)$, and $z(s, t)$ gives

$$\frac{\partial w}{\partial s} \ = \ (s^2t^2 + s^2 + t^2)(2s) + (s^2t^2 + s^2 - t^2)(2s) + (2s^2)(2t^2s) \ = \ 4s^3(1 + 2t^2).$$

Similarly, we can write

$$\frac{\partial w}{\partial t} \ = \ \frac{\partial f}{\partial x}\Big|_{(x(s,t),y(s,t),z(s,t))}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\Big|_{(x(s,t),y(s,t),z(s,t))}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\Big|_{(x(s,t),y(s,t),z(s,t))}\frac{\partial z}{\partial t}.$$

Using our values for the partials of f from above gives

$$\frac{\partial w}{\partial t} = (z(s,t)+y(s,t))(-2t) + (z(s,t)+x(s,t))(2t) + (x(s,t)+y(s,t))(2s^2t),$$

and then using

$$\frac{\partial x}{\partial t} = -2t, \quad \frac{\partial y}{\partial t} = 2t, \quad \frac{\partial z}{\partial t} = 2s^2t$$

and substituting for $x(s,t)$, $y(s,t)$ and $z(s,t)$ gives

$$\frac{\partial w}{\partial t} = (s^2t^2+s^2+t^2)(-2t) + (s^2t^2+s^2-t^2)(2t) + (2s^2)(2s^2t) = 4t(s^4-t^2),$$

Page 960: #34: A rectangular box has a square base. Find the rate at which its volume and surface area are changing if its base is increasing at 2 cm/min and its height is decreasing at 3cm/min at the instant when each dimension is 1 meter (i.e., 100 cm).

Solution: Let's call the box's length x, its width y, and its height z. Since the box has a square base, we have $x = y$. The volume of the box is given by $xyz = x^2z$. We're also going to think of x and z as functions of time t, so $V(t) = f(x(t), z(t))$ with $f(x,z) = x^2z$. The Chain Rule gives

$$\frac{\partial V}{\partial t} = \frac{\partial f}{\partial x}\Big|_{(x(t),z(t))}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial z}\Big|_{(x(t),z(t))}\frac{\partial z}{\partial t} \quad \text{or} \quad \frac{dV}{dt} = \frac{\partial f}{\partial x}\Big|_{(x(t),z(t))}\frac{dx}{dt} + \frac{\partial f}{\partial z}\Big|_{(x(t),z(t))}\frac{dz}{dt}.$$

From the statement of the problem, we know $dx/dt = 2$ cm/min and $dz/dt = -3$ cm/min. Differentiating gives $f_x = 2xz$ and $f_z = x^2$. Therefore $\frac{dV}{dt} = 4x(t)z(t) - 3x(t)^2$. When $x(t) = z(t) = 100$ cm (remember to use centimeters as the rates are in cm/min), we have $dV/dt = 10000\text{cm}^3/\text{min}$, meaning the volume is increasing at a rate of 100 cubic centimeter per minute.

To calculate the rate at which the surface area is changing, recall the surface area is $2(xy + xz + yz) = 2(x^2 + 2xz)$ (since $x = y$). Set $A(t) = g(x(t), z(t))$ with $g(x,z) = 2(x^2 + 2xz)$. The Chain rule gives

$$\frac{\partial S}{\partial t} = \frac{\partial g}{\partial x}\Big|_{(x(t),z(t))}\frac{\partial x}{\partial t} + \frac{\partial g}{\partial z}\Big|_{(x(t),z(t))}\frac{\partial z}{\partial t} \quad \text{or} \quad \frac{dS}{dt} = \frac{\partial g}{\partial x}\Big|_{(x(t),z(t))}\frac{dx}{dt} + \frac{\partial g}{\partial z}\Big|_{(x(t),z(t))}\frac{dz}{dt}.$$

Taking the derivatives and using $dx/dt = 2$ and $dz/dt = -3$ gives

$$\frac{dS}{dt} = 4(2x(t) + 2z(t)) - 12x(t) = 8z(t) - 4x(t).$$

Therefore, when $x(t) = z(t) = 100$ (remember we must use centimeters as the rates are in cm/min), the surface area is changing at a rate of 400 square centimeters per minute.

Page 960 #41: Suppose that $w = f(u)$ and that $u = x + y$. Show that $\partial w/\partial x = \partial w/\partial y$.
Solution: Let's first think about what this problem means. We have w as a function of one variable, u, which we know want to think of as a function of two variables, x and y, using the relationship $u = x + y$. Claiming that $\partial w/\partial x = \partial w/\partial y$ essentially means that we achieve the same effect by varying y a little bit as we do by varying x that same little bit. This makes sense, since if we increase x by 0.1 and leave y constant, u increases by 0.1; alternatively, if we increase y by 0.1 and leave x constant, u again increases by 0.1.

More formally, let's write $w(x,y) = f(u(x,y))$, so

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\Big|_{u(x,y)}\frac{\partial u}{\partial x} = f'(u(x,y)) \cdot 1 = f'(u(x,y)),$$
$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u}\Big|_{u(x,y)}\frac{\partial u}{\partial y} = f'(u(x,y)) \cdot 1 = f'(u(x,y)).$$

So $\partial w/\partial x$ is indeed equal to $\partial w/\partial y$.

11. HW #12

11.1. **Problems: HW #12.**

Page 971: Question 3:: Find the gradient ∇f at P where $f(x,y) = \exp(-x^2 - y^2)$ and P is (0,0).

Page 971: Question 10:: Find the gradient ∇f at P where $f(x,y,z) = (2x - 3y + 5z)^5$ and P is (-5,1,3).

Page 971: Question 11:: Find the directional derivative of $f(x,y) = x^2 + 2xy + 3y^2$ at $P(2,1)$ in the direction $\vec{v} = \langle 1, 1\rangle$. In other words, compute $(D_{\vec{u}}f)(P)$ where $\vec{u} = \vec{v}/|\vec{v}|$.

Page 971: Question 19:: Find the directional derivative of $f(x,y,z) = \exp(xyz)$ at $P(4,0,-3)$ in the direction $\vec{v} = \langle 0, 1, -1\rangle$ (which is $\mathbf{j} - \mathbf{k}$). In other words, compute $(D_{\vec{u}}f)(P)$ where $\vec{u} = \vec{v}/|\vec{v}|$.

Page 971: Question 21:: Find the maximum directional derivative of $f(x,y) = 2x^2 + 3xy + 4y^2$ at $P(1,1)$ and the direction in which it occurs.

11.2. **Solutions: HW #12.**

Page 971: Question 3:: Find the gradient ∇f at P where $f(x, y) = \exp(-x^2 - y^2)$ and P is (0,0).
Solution: $\langle f_x, f_y\rangle = \langle -2x\exp(-x^2 - y^2), -2y\exp(-x^2 - y^2)\rangle$. Plugging in $P(0, 0)$, we have the gradient at P is $\langle 0, 0\rangle$.

Page 971: Question 10:: Find the gradient ∇f at P where $f(x, y, z) = (2x - 3y + 5z)^5$ and P is (-5,1,3).
Solution: $\langle f_x, f_y, f_z\rangle = \langle 10(2x - 3y + 5z)^4, -15(2x - 3y + 5z)^4, 25(2x - 3y + 5z)^4\rangle$, and evaluating at P gives the gradient there is just $\langle 10(16), -15(16), 25(16)\rangle$ or $\langle 160, -240, 400\rangle$.

Page 971: Question 11:: Find the directional derivative of $f(x, y) = x^2 + 2xy + 3y^2$ at $P(2, 1)$ in the direction $\vec{v} = \langle 1, 1\rangle$. In other words, compute $(D_{\vec{u}}f)(P)$ where $\vec{u} = \vec{v}/|\vec{v}|$.
Solution: As $|\vec{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$, normalizing gives $\vec{u} = \vec{v}/\sqrt{2} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$. Using our formula we have $(D_{\vec{u}}f)(P) = \langle f_x, f_y\rangle\Big|_P \cdot \vec{u} = \langle 2x + 2y, 2x + 6y\rangle\Big|_P \cdot \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$. Plugging in the values for x and y, we have $(D_{\vec{u}}f)(P) = \langle 6, 10\rangle \cdot \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle = 8\sqrt{2}$.

Page 971: Question 19:: Find the directional derivative of $f(x, y, z) = \exp(xyz)$ at $P(4, 0, -3)$ in the direction $\vec{v} = \langle 0, 1, -1\rangle$ (which is $\mathbf{j} - \mathbf{k}$). In other words, compute $(D_{\vec{u}}f)(P)$ where $\vec{u} = \vec{v}/|\vec{v}|$.
Solution: As $|\vec{v}| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$, we have $\vec{u} = \langle 0, 1, -1\rangle/\sqrt{2} = \langle 0, 1/\sqrt{2}, -1/\sqrt{2}\rangle$. The gradient is

$$Df \;=\; \langle yze^{xyz}, xze^{xyz}, xye^{xyz}\rangle \;=\; \langle 0, -12e^0, 0\rangle \;=\; \langle 0, -12, 0\rangle,$$

so $D_u f(P) = \langle 0, -12, 0\rangle\langle 0, 1/\sqrt{2}, -1/\sqrt{2}\rangle = -6\sqrt{2}$.

Page 971: Question 21:: Find the maximum directional derivative of $f(x, y) = 2x^2 + 3xy + 4y^2$ at $P(1, 1)$ and the direction in which it occurs.
Solution: The maximum directional derivative is in the direction of the gradient (the minimum is in the opposite direction). The gradient of f is $Df = \langle 4x + 3y, 3x + 8y\rangle$, which at P is $\langle 7, 11\rangle$. A unit vector in this direction is $\vec{u} = \langle 7, 11\rangle/|\langle 7, 11\rangle|$. As $|\langle 7, 11\rangle| = \sqrt{7^2 + 11^2} = \sqrt{170}$, the directional derivative is largest in the direction $\vec{u} = \langle 7/\sqrt{170}, 11/\sqrt{170}\rangle$. To find the maximum value, we just need to compute $(D_{\vec{u}}f)(P) = (Df)(P) \cdot \vec{u}$, which is $\langle 7, 11\rangle \cdot 7/\sqrt{170}, 11/\sqrt{170}$. This is just $(7^2 + 11^2)/\sqrt{170} = \sqrt{170}$; it is not a coincidence that this is the magnitude of the gradient!

12. HW #13

12.1. **Problems: HW #13.**

Question 1: Use Newton's Method to find a rational number that estimates the square-root of 5 correctly to at least 4 decimal places.

Question 2: Let $w(r, s, t) = f(u(r, s, t), v(r, s, t))$ with $f(u, v) = u^2 + v^2$, $u(r, s, t) = t\cos(rs)$ and $v(r, s, t) = t\sin(rs)$. Find the partial derivatives of w with respect to r, s and t both by direct substitution (which is very nice here!) and by the chain rule.

Question 3: Write $(1/2, \sqrt{3}/2)$ in polar coordinates.

Question 4: Find the tangent plane to $z = f(x, y)$ with $f(x, y) = x^2y + \sqrt{x + y}$ at $(1, 3)$, and approximate the function at $(.9, 1.2)$.

General comments: These problems are all done the same way. Let's say we have functions of three variables, x, y, z. Find the function to maximize f, the constraint function g, and then solve $\nabla f(x, y, z) = \lambda\nabla g(x, y, z)$ and $g(x, y, z) = c$. Explicitly, solve:

$$\begin{aligned}
\frac{\partial f}{\partial x}(x, y, z) &= \lambda\frac{\partial g}{\partial x}(x, y, z)\\
\frac{\partial f}{\partial y}(x, y, z) &= \lambda\frac{\partial g}{\partial y}(x, y, z)\\
\frac{\partial f}{\partial z}(x, y, z) &= \lambda\frac{\partial g}{\partial z}(x, y, z)\\
g(x, y, z) &= c.
\end{aligned}$$

For example, if we want to maximize xy^2z^3 subject to $x + y + z = 4$, then $f(x, y, z) = xy^2z^3$ and $g(x, y, z) = x + y + z = 4$. The hardest part is the algebra to solve the system of equations. ***Remember to be on the lookout for dividing by zero. That is never allowed, and thus you need to deal with those cases separately. Specifically, if the quantity you want to divide by can be zero, you have to consider as a separate case what happens when it is zero, and as another case what happens when it is not zero.***

Page 981: Question 1: Find the maximum and minimum values, if any, of $f(x, y) = 2x + y$ subject to the constraint $x^2 + y^2 = 1$.

Page 981: Question 14: Find the maximum and minimum values, if any, of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x^4 + y^4 + z^4 = 3$.

12.2. **Solutions: HW #13.**

Question 1: Use Newton's Method to find a rational number that estimates the square-root of 5 correctly to at least 4 decimal places.
Solution: We use the function $f(x) = x^2 - 5$ and we start with $x_0 = 2$. The equation of the tangent line is $y - f(x_0) = f'(x_0)(x - x_0)$. As $f(2) = -1$ and $f'(2) = 4$, the tangent line is $y - (-1) = 4(x - 2)$. We find the x-intercept by setting $y = 0$ in the tangent line, and this gives us our next guess, x_1. Thus $1 = 4(x_1 - 2)$ or $x_1 = 9/4$. If we worked more formally, we would have found that $x_1 = 2 - \frac{f(2)}{f'(2)} = 9/4 = 2.125$. Performing this process again gives $x_2 = 2.125 - \frac{f(2.125)}{f'(2.125)} = 2.238971$, and one more time gives $x_3 = 2.238971 - \frac{f(2.238971)}{f'(2.238971)} = 2.23607$. If we instead starting with $x_0 = 3$ as our guess, the first tangent line would be $y - f(3) = f'(3)(x - 3)$. As $f(3) = 4$ and $f'(3) = 6$, the tangent line here is $y - 4 = 6(x - 3)$. The x-intercept is where $y = 0$, so x_1 is found by solving $-4 = 6(x_1 - 3)$, which gives $x_1 = 14/6 = 7/3$. The next guess is $x_2 = 47/21$, followed by $x_3 = 2207/987 \approx 2.236068896$.

Question 2: Let $w(r, s, t) = f(u(r, s, t), v(r, s, t))$ with $f(u, v) = u^2 + v^2, u(r, s, t) = t\cos(rs)$ and $v(r, s, t) = t\sin(rs)$. Find the partial derivatives of w with respect to r, s and t both by direct substitution (which is very nice here!) and by the chain rule.
Solution: We substitute (plug in) the functional expressions for u and v, then we have $w(r, s, t) = (t\cos(rs))^2 + (t\sin(rs))^2 = t^2(\cos^2(rs) + \sin^2(rs)) = t^2$. So $\frac{\partial w}{\partial t} = 2t$, $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial r} = 0$. For the chain rule, we have

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial u}\Big|_{(u(r,s,t),v(r,s,t))} \frac{\partial u}{\partial r} + \frac{\partial f}{\partial v}\Big|_{(u(r,s,t),v(r,s,t))} \frac{\partial v}{\partial r}.$$

We have $\frac{\partial f}{\partial u} = 2u$ and $\frac{\partial f}{\partial v} = 2v$, while $\frac{\partial u}{\partial r} = -ts\sin(rs)$ and $\frac{\partial v}{\partial r} = ts\cos(rs)$. Substituting (and evaluating the derivatives at the right point) gives

$$\frac{\partial w}{\partial r} = 2u\Big|_{(u(r,s,t),v(r,s,t))}(-ts\sin(rs)) + 2v\Big|_{(u(r,s,t),v(r,s,t))}(ts\cos(rs)) = -t^2 s\cos(rs)\sin(rs) + t^2 s\sin(rs)\cos(rs) = 0.$$

The other derivatives are computed similarly.

Question 3: Write $(1/2, \sqrt{3}/2)$ in polar coordinates.
Solution: Polar coordinates are $x = r\cos\theta$ and $y = r\sin\theta$, or $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. We first find the radius: $r = \sqrt{x^2 + y^2} = \sqrt{1/4 + 3/4} = 1$. To find the angle, knowing $r = 1$ we see $\sin\theta = \sqrt{3}/w$ (or $\tan(\theta) = \sqrt{3}$), so $\theta = \pi/3$. Hence the expression is $(1, \pi/3)$.

Question 4: Find the tangent plane to $z = f(x, y)$ with $f(x, y) = x^2 y + \sqrt{x + y}$ at $(1, 3)$, and approximate the function at $(0.9, 1.2)$.
Solution: The equation of the tangent plane is $z = f(1, 3) + \frac{\partial f}{\partial x}(1, 3)(x - 1) + \frac{\partial f}{\partial y}(1, 3)(y - 3)$. We have $f(1, 3) = 5$, $\frac{\partial f}{\partial x} = 2xy + \frac{1}{2\sqrt{x+y)}}$, which at $(1, 3)$ equals $\frac{25}{4}$, while $\frac{\partial f}{\partial y} = x^2 + \frac{1}{2\sqrt{x+y}}$, which at $(1, 3)$ equals $5 + \frac{25}{4}(.9 - 1) + \frac{5}{4}(1.2 - 3) = \frac{17}{8}$, which is approximately 2.125. The actual value at $x = 0.9, y = 1.2$ is $z = -\frac{211}{80} = -2.6375$. The reason our approximation is off by so much is that we are at the point $(.9, 1.2)$, and 1.2 is a ways from 3.

General comments: These problems are all done the same way. Let's say we have functions of three variables, x, y, z. Find the function to maximize f, the constraint function g, and then solve $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$. Explicitly, solve:

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y, z) &= \lambda \frac{\partial g}{\partial x}(x, y, z) \\ \frac{\partial f}{\partial y}(x, y, z) &= \lambda \frac{\partial g}{\partial y}(x, y, z) \\ \frac{\partial f}{\partial z}(x, y, z) &= \lambda \frac{\partial g}{\partial z}(x, y, z) \\ g(x, y, z) &= c. \end{aligned}$$

For example, if we want to maximize xy^2z^3 subject to $x + y + z = 4$, then $f(x, y, z) = xy^2z^3$ and $g(x, y, z) = x + y + z = 4$. The hardest part is the algebra to solve the system of equations. ***Remember to be on the lookout for dividing by zero. That is never allowed, and thus you need to deal with those cases separately. Specifically, if the quantity you want to divide by can be zero, you have to consider as a separate case what happens when it is zero, and as another case what happens when it is not zero.***

Page 981: Question 1: Find the maximum and minimum values, if any, of $f(x, y) = 2x + y$ subject to the constraint $x^2 + y^2 = 1$.
Solution: We use of the method of Lagrange multipliers to solve for constrained optimization. We set up the appropriate equations by setting the gradients proportional to each other with proportionality constant λ, and remember that the constraint equation must hold as well. Thus, we are looking for (x, y, λ) such that $(\nabla f)(x, y) = \lambda(\nabla g)(x, y)$ and $g(x, y) = x^2 + y^2 = 1$. The gradient $\nabla f(x, y) = \langle 2, 1\rangle$. This is obtained by just taking the partial derivatives of the $f(x, y)$ with respect to its variables. Taking the gradient of the constraint gives $\nabla g(x, y) = \langle 2x, 2y\rangle$.

We set up the equations: $\nabla f(x, y) = \lambda \nabla g(x, y)$, so $\langle x, 1\rangle = \lambda\langle 2x, 2y\rangle$, and also $x^2 + y^2 = 1$ (it is very important not to forget this, as otherwise we have two equations in three unknowns, which is an over-determined system). We now solve the equations for each variable by setting the components of the gradients as equal. We have three equations: $2 = \lambda 2x$, $1 = \lambda 2y$ and $x^2 + y^2 = 1$.

One way to solve this is to take ratios; unfortunately, we need to be careful: what if x or y is zero? Well, if $y = 0$ then the constraint equation becomes $x^2 = 1$ so $x = \pm 1$, leading to $f(1, 0) = 2$ and $f(-1, 0) = -2$. If instead $x = 0$ then the constraint equation becomes $y^2 = 1$ so $y = \pm 1$, leading to the points $(0, 1)$ and $(0, -1)$, which evaluate under f to 1 and -1, respectively. 4 If now x does not equal zero, then dividing the second equation by the first eliminates the λ's, and we find $2/1 = x/y$, so $x = 2y$. Substituting into $x^2 + y^2 = 1$ gives $5y^2 = 1$ or $y = \pm 1/\sqrt{5}$, and thus we get the candidate points $(x, y) = (2/\sqrt{5}, 1/\sqrt{5})$ and $(-2/\sqrt{5}, -1/\sqrt{5})$. Evaluating f at the first gives $\sqrt{5}$ while evaluating f at the second gives $-\sqrt{5}$. We thus see the maximum value of f is $\sqrt{5}$, and f's minimum value is $-\sqrt{5}$. In other words, it is very important to remember the candidates that can arise from dividing by zero!

Remark: looking at the function, we see we want x to be as large (small) as possible for the maximum (minimum), and thus it is not unexpected that these occur when $y = 0$.

Page 981: Question 14: Find the maximum and minimum values, if any, of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x^4 + y^4 + z^4 = 3$.
Solution: We use Lagrange Multipliers. The gradients are $\nabla f(x, y) = \langle 2x, 2y, 2z\rangle$ and $\nabla g(x, y, z) = \langle 4x^3, 4y^3, 4z^3\rangle$. We set up the equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = x^4 + y^4 + z^4 = 3$, leading to $\langle 2x, 2y, 2z\rangle = \lambda\langle 4x^3, 4y^3, 4z^3\rangle$ and $g(x, y, z) = x^4 + y^4 + z^4 = 3$. Writing things out, we have the three equations (*plus the constraint, of course*) $2x = \lambda 4x^3$, $2y = \lambda 4y^3$, and $2z = \lambda 4z^3$.

These functions are symmetrical and simplify the algebraic work. Assume first that none of the variables equal zero. By dividing both sides of the first equation by $2x$, both sides of the second equation by $2y$, and both sides of the third equation by $2z$, we can easily see the relationship between the three variables: $1 = \lambda 2x^2$, $1 = \lambda 2y^2$ and $1 = \lambda 2z^2$. This leads the squares of the three variables being equal. We have $x = \pm y$ and $x = \pm z$, since the square of any of these equals the square of another. Thus $x^4 + y^4 + z^4 = 3$ becomes $3x^4 = 3y^4 = 3z^4 = 3$, so $x, y, z \in \{1, -1\}$. The candidate points are the eight points $(\pm 1, \pm 1, \pm 1)$, all of which evaluate to 3 under f.

What about the case when some of the variables are zero? If all three are zero, the constraint cannot be satisfied. If two are zero then the third must equal $\pm 3^{1/4}$, and this point evaluates to $3^{1/2} \approx 1.732$ under f. What if only one variable is zero, for definiteness say z. Then we may divide the first equation by $2x$ and the second by $2y$, finding $1 = 2\lambda x^2 = 2\lambda y^2$, so $x = \pm y$ and $2x^4 = 3$ (from $x^4 + y^4 + z^4 = 3$). This gives $x = \pm(3/2)^{1/4} = \pm y$, and thus the candidate points $(\pm(3/2)^{1/4}, \pm(3/2)^{1/4}, 0)$ evaluate under f to $\sqrt{3/2} + \sqrt{3/2} \approx 2.44949$. There are lots more points like this: $(0, \pm(3/2)^{1/4}, \pm(3/2)^{1/4})$ and $(\pm(3/2)^{1/4}, 0, \pm(3/2)^{1/4})$, all evaluating under f to the same. Thus the maximum is 3 where all x, y, z are equal in absolute value, while the minimum is $\sqrt{3}$ where two of the variables are zero.

13. HW #14

13.1. **Problems: HW #14.**

Page 981: Question 19: Find the point on the line $3x + 4y = 100$ that is closest to the origin. Use Lagrange multipliers to minimize the SQUARE of the distance.

Page 981: Question 35: Find the point or points of the surface $z = xy + 5$ closest to the origin.

Page 981: Question 51: Find the point on the parabola $y = (x - 1)^2$ that is closest to the origin. *Note: after some algebra you'll get that x satisfies $2(x - 1)^3 + x = 0$ (depending on how you do the algebra it may look slightly different). You may use a calculator, computer program, ... to numerically approximate the solution.*

13.2. Solutions: HW #14.

General comments: These problems are all done the same way. Let's say we have functions of three variables, x, y, z. Find the function to maximize f, the constraint function g, and then solve $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z) = c$. Explicitly, solve:

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) &= \lambda \frac{\partial g}{\partial x}(x,y,z)\\
\frac{\partial f}{\partial y}(x,y,z) &= \lambda \frac{\partial g}{\partial y}(x,y,z)\\
\frac{\partial f}{\partial z}(x,y,z) &= \lambda \frac{\partial g}{\partial z}(x,y,z)\\
g(x,y,z) &= c.
\end{aligned}$$

For example, if we want to maximize xy^2z^3 subject to $x+y+z=4$, then $f(x,y,z) = xy^2z^3$ and $g(x,y,z) = x+y+z=4$. The hardest part is the algebra to solve the system of equations. *Remember to be on the lookout for dividing by zero. That is never allowed, and thus you need to deal with those cases separately. Specifically, if the quantity you want to divide by can be zero, you have to consider as a separate case what happens when it is zero, and as another case what happens when it is not zero.*

Page 981: Question 19: Find the point on the line $3x + 4y = 100$ that is closest to the origin. Use Lagrange multipliers to minimize the SQUARE of the distance.

Solution: Because we are solving for the square of the distance, we take $f(x,y) = (\sqrt{x^2+y^2})^2 = x^2+y^2$. Since the distance is being squared, the square root is being canceled out, which significantly simplifies the algebra and the calculus. Our constraint is given to us as our point must lie on the line $3x+4y = 100$. We apply the method of Lagrange multipliers and set the gradients of both functions to be proportional to each other.

The gradient $\nabla f(x,y) = \langle 2x, 2y\rangle$, and $\nabla g(x,y) = \langle 3,4\rangle$. The equations to solve are $\nabla f(x,y) = \lambda \nabla g(x,y)$ with $3x+4y=100$, so $\langle 2x,2y\rangle = \lambda\langle 3,4\rangle$ with $3x+4y=100$. We now solve the equations for each variable by setting the components of the gradients as equal. We have the two equations $2x = 3\lambda$ and $2y = 4\lambda$, plus of course the constraint $3x+4y = 100$.

We solve each of the first two equations for λ, as that will allow us to find a nice relation between x and y. If we divide both sides of the first equation by 3, we can isolate λ. So $2x/3 = \lambda$; similarly the second equation gives $y/2 = \lambda$. Setting these equal to each other gives $2x/3 = y/2$ or $x = 3y/4$ or $y = 4x/3$. By plugging that value into the constraint function, we can find the candidate point. We have $3x + 4(\frac{4x}{3}) = 100$, so $25x = 300$ or $x = 12$, giving $y = \frac{4(12)}{3} = 16$. The optimal point is $(12, 16)$.

*Alternate geometric solution (advanced): We can also solve this geometrically, if we remember the product of the slopes of perpendicular lines is -1. As this line has slope -3/4, from $y = -3x/4 + 25$, the slope of any perpendicular line must be 4/3. A point on that line is (0,0), thus the equation of that line is $y - 0 = (4/3)(x-0)$ or $y = 4x/3$. We need the intersection of this and our original line, so we want (x,y) such that $y = 4x/3$ and $3x+4y=100$. The second equation becomes $3x + 16x/3 = 100$ or $25x/3 = 100$ and thus $x = 12$, exactly as before! **NO CALCULUS!***

Page 981: Question 35: Find the point or points of the surface $z = xy+5$ closest to the origin.
Solution: Again we'll be minimizing the square of the distance to simplify the algebra. We take $f(x,y,z) = (\sqrt{x^2+y^2+z^2})^2 = x^2+y^2+z^2$. Our constraint is $z = xy+5$ or $g(x,y,z) = xy - z = -5$. We apply the method of Lagrange multipliers and set the gradients of both functions proportional to each other (with proportionality constant λ). The gradients are $\nabla f(x,y,z) = \langle 2x,2y,2z\rangle$ and $\nabla g(x,y) = \langle y, x, -1\rangle$.

We set up the equations $\nabla f(x,y) = \lambda \nabla g(x,y)$ with $g(x,y) = -5$, so $\langle 2x, 2y, 2z\rangle = \lambda\langle y, x, -1\rangle$. We solve the equations for each variable by setting the components of the gradients as equal. We have three equations: $2x = \lambda y$, $2y = \lambda x$, $2z = \lambda(-1)$, and of course $g(x,y,z) = xy - z = -5$.

If $x = 0$ then since $2x = \lambda y$ we have either $\lambda = 0$ or $y = 0$ (or both). If $\lambda = 0$ then $z = 0$ from $2z = -\lambda$, but then the constraint cannot be satisfied. Thus $\lambda \neq 0$, so if $x = 0$ then we must have $y = 0$. The constraint equation (with $x = y = 0$) implies that $z = 5$, giving us the point $(0,0,5)$ whose distance-squared to the origin is 25. We get the same answer if instead $y = 0$.

We may thus assume now that neither x nor y is zero. In this case we may divide the first equation by the second, and find $2x/2y = \lambda y/\lambda x$, or $x/y = y/x$, or $x^2 = y^2$ which implies $x = \pm y$. If $x = y$ then the first equation, $2x = \lambda y$, becomes $2x = \lambda x$. As $x \neq 0$ we see $\lambda = 2$. The third equation then gives $2z = -\lambda = -2$ so $z = 1$. The constraint $xy - z = -5$ becomes $x^2 + 2 = -5$ or $x^2 = -7$, which has no solution.

Continuing to assume neither x nor y is zero, we see that it must be the case that $x = -y$. In this case, the first equation becomes $2x = \lambda y = -\lambda x$, so $\lambda = -2$. The third equation, $2z = -\lambda = 2$ now gives $z = 1$. The constraint $xy - z = -5$ is now $-x^2 - 1 = -5$ or $x^2 = 4$. Thus $x = \pm 2$, and $y = -x$ and $z = 1$, giving us the candidate points $(2,-2,1)$ and $(-2,2,1)$, whose distance-squared to the origin is 9, smaller than the 25 we saw above. Thus, these are the two closest points.

Page 981: Question 51: Find the point on the parabola $y = (x-1)^2$ that is closest to the origin. *Note: after some algebra you'll get that x satisfies $2(x-1)^3 + x = 0$ (depending on how you do the algebra it may look slightly different). You may use a calculator, computer program, ... to numerically approximate the solution.*
Solution: Again we'll be solving for distance squared using the method of Lagrange Multipliers. Our $f(x,y) = (\sqrt{x^2+y^2})^2 = x^2+y^2$ and our constraint is $y = (x-1)^2$ or $g(x,y) = (x-1)^2 - y = 0$. The gradients are $\nabla f(x,y,z) = \langle 2x, 2y\rangle$ and $\nabla g(x,y) = \langle 2(x-1), -1\rangle$.

We set up the equations: $\nabla f(x,y) = \lambda\nabla g(x,y)$ and $(x-1)^2 - y = 0$, so $\langle 2x, 2y\rangle = \lambda\langle 2(x-1), -1\rangle$ (and of course the constraint holds). Now solve the equation for each variable by setting the components of the gradients equal. We have two equations: $2x = \lambda 2(x-1)$ and $2y = \lambda(-1)$. We use the second equation to isolate λ, which is $\lambda = -2y$. Substitute that into the first equation to obtain a relationship between x and y: $2x = (-2y)(2(x-1))$ or $\frac{2x}{2(x-1)} = -2y$, which becomes $\frac{-x}{2(x-1)} = y$ so long as $x \neq 1$. Note that if $x = 1$ then $y = 0$, giving a distance-squared of 1.

Now solve for the optimal using the constraint function through substitution. Our constraint becomes $0 = (x-1)^2 - \frac{-x}{2(x-1)}$, or cross multiplying gives $2(x-1)^3 + x = 0$. Taking the advice of the book, we can enter that equation into a graphing calculator or Mathematica to solve for the optimal point which is $(0.410245, 0.347810)$. We could also use divide and conquer or Newton's method to find the root!

14. HW #15

14.1. **Problems: HW #15.**

Page 1004: Question 15: Evaluate $\int_0^3 \int_0^3 (xy + 7x + y) dxdy$.

Page 1004: Question 24: Evaluate $\int_0^1 \int_0^1 e^{x+y} dxdy$

Page 1004: Question 25: Evaluate $\int_0^\pi \int_0^\pi (xy + \sin x) dxdy$.

Page 1005: Question 37: Use Riemann sums to show, without calculating the value of the integral, that $0 \leq \int_0^\pi \int_0^\pi \sin \sqrt{xy} dxdy \leq \pi^2$.

Extra credit: Let $G(x) = \int_{t=0}^{x^3} g(t) dt$. Find a nice formula for $G'(x)$ in terms of the functions in this problem.

14.2. Solutions: HW #15.

Page 1004: Question 15: Evaluate $\int_0^3 \int_0^3 (xy + 7x + y) dx dy$.
Solution: We start by integrating the inside first, which gives us

$$\begin{aligned}\int_0^3 \int_0^3 (xy + 7x + y) dx dy &= \int_0^3 \left[x^2 y/2 + 7x^2/2 + xy\right]_{x=0}^3 dy \\ &= \int_0^3 \left[9y/2 + 63/2 + 3y\right] dy \\ &= \int_0^3 (15y/2 + 63/2) dy \\ &= \frac{15y^2}{4}\Big|_0^3 + \frac{63y}{2}\Big|_0^3 \;=\; \frac{135}{4} + \frac{189}{2} \;=\; 128.25.\end{aligned}$$

Page 1004: Question 24: Evaluate $\int_0^1 \int_0^1 e^{x+y} dx dy$
Solution: We start by integrating the inside first. As we are integrating with respect to x, the anti-derivative of e^{x+y} with respect to x is just e^{x+y}. Note we may also write it as $e^x e^y$; written like this, we see e^y functions like a constant. Evaluating at 0 and 1 gives $e^{1+y} - e^{0+y} = e \cdot e^y - e^y = e^y(e-1)$. We now integrate this with respect to y, and find $(e-1)e^y\Big|_0^1 = (e-1)e - (e-1)1$. Alternatively, we may write this out as

$$\begin{aligned}\int_0^1 \int_0^1 e^{x+y} dx dy &= \int_0^1 \left[e^{x+y}\Big|_{x=0}^1\right] dy \\ &= \int_0^1 (e^{1+y} - e^y) dy \\ &= \int_0^1 (e-1)e^y \;=\; (e-1)e^y\Big|_0^1 \;=\; (e-1)e - (e-1)1.\end{aligned}$$

Page 1004: Question 25: Evaluate $\int_0^\pi \int_0^\pi (xy + \sin x) dx dy$.
Solution: We start by integrating the inside first, which gives $\frac{1}{2}x^2 y\Big|_{x=0}^\pi - \cos x\Big|_{x=0}^\pi$ or $\frac{\pi^2}{2}y - (-1-1) = \frac{\pi^2}{2}y + 2$. Integrating this now with respect to y gives $\frac{y^2}{4}\Big|_0^\pi + 2y\Big|_0^\pi = \pi^4/4 + 2\pi$. Alternatively, we may write this as

$$\begin{aligned}\int_0^\pi \int_0^\pi (xy + \sin x) dx dy &= \int_0^\pi \left[\frac{1}{2}x^2 y\Big|_{x=0}^\pi - \cos x\Big|_{x=0}^\pi\right] dy \\ &= \int_0^\pi \left(\frac{\pi^2}{2}y + 2\right) dy \\ &= \frac{y^2}{4}\Big|_0^\pi + 2y\Big|_0^\pi \;=\; \frac{\pi^4}{4} + 2\pi.\end{aligned}$$

Page 1005: Question 37: Use Riemann sums to show, without calculating the value of the integral, that $0 \le \int_0^\pi \int_0^\pi \sin\sqrt{xy}\, dx dy \le \pi^2$.
Solution: The idea here is to find the upper and lower bound for the integrand. We know that $\sin\sqrt{xy}$ reaches its maximum value 1 when $\sqrt{xy} = \pi/2$. We also know that since $0 \le x, y \le \pi$, $0 \le \sqrt{xy} \le \pi$. This means that $0 \le \sin\sqrt{xy} \le 1$. We use the simplest possible Riemann sum, namely just one partition (so our partition is the original rectangle). As the rectangle has area
pi^2, the lower sum is the minimum value times π^2, or 0, while the upper sum is the maximum value times the area π^2, or $1 \cdot \pi^2$. Thus $0 \le \int_0^\pi \int_0^\pi \sin\sqrt{xy}\, dx dy \le \pi^2$.

Extra credit: Let $G(x) = \int_{t=0}^{x^3} g(t) dt$. Find a nice formula for $G'(x)$ in terms of the functions in this problem.
Solution: We use the Fundamental Theorem of Calculus for this problem, which states that if F is the antiderivative of f, a.k.a. if $F' = f$, then $\int_a^b f(x) = F(b) - F(a)$.
Now in our case, we have $G(x) = F(x^3) - F(0)$. We then differentiate both sides: $G'(x) = F'(x^3) - F'(0)$. $F(0)$ is just a constant, so $F'(0) = 0$. Because F is the antiderivative of f, we have $F'(x^3) = f(x^3)$. So $G'(x) = f(x^3)$

15. HW #16

15.1. Problems: HW #16.

Page 1011: Question 4: Evaluate $\int_0^2 \int_{y/2}^1 (x+y) dx dy$; note this is $\int_{y=0}^2 \int_{x=y/2}^1 (x+y) dx dy$.

Page 1012: Question 11: Evaluate $\int_0^1 \int_0^{x^3} \exp(y/x) dy dx$; note this is $\int_{x=0}^1 \int_{y=0}^{x^3} \exp(y/x) dy dx$.

Additional Problem: Let $f(x) = x^3 - 4x^2 + \cos(2x^3) + \sin(x + 1701)$. Find a finite B such that $|f'(x)| \le B$ for all x in [2,3].

Page 1011: #13: Evaluate the iterated integral

$$\int_0^3 \int_0^y \sqrt{y^2+16}\, dx\, dy \;=\; \int_{y=0}^3 \int_{x=0}^y \sqrt{y^2+16}\, dx\, dy.$$

Page 1011: #25: Sketch the region of integration for the integral

$$\int_{-2}^2 \int_{x^2}^4 x^2 y\, dy\, dx \;=\; \int_{x=-2}^2 \int_{y=x^2}^4 x^2 y\, dy\, dx.$$

Reverse the order of integration and evaluate the integral.

Page 1011: #30: Sketch the region of integration for the integral

$$\int_0^1 \int_y^1 \exp(-x^2)\, dx\, dy \;=\; \int_{y=0}^1 \int_{x=y}^1 \exp(-x^2)\, dx\, dy.$$

Reverse the order of integration and evaluate the integral.

Additional Problem: Give an example of a region in the plane that is neither horizontally simple nor vertically simple.

15.2. **Solutions: HW #16.**

Page 1011: Question 4: Evaluate $\int_0^2 \int_{y/2}^1 (x+y)dxdy$; note this is $\int_{y=0}^2 \int_{x=y/2}^1 (x+y)dxdy$.
Solution: We start by integrating the inside first with respect to x, which gives us

$$\begin{aligned}
\int_0^2 \int_{y/2}^1 (x+y)dxdy &= \int_0^2 \left[\frac{x^2}{2} + xy\right]_{x=y/2}^1 dy \\
&= \int_0^2 \left[\left(\frac{1}{2} + y\right) - \left(\frac{y^2}{8} + \frac{y^2}{2}\right)\right] dy \\
&= \int_0^2 \left(\frac{1}{2} + y - \frac{5y^2}{8}\right) dy \\
&= \left[\frac{y}{2} + \frac{y^2}{2} - \frac{5y^3}{24}\right]_{y=0}^2 \\
&= 1 + 2 - \frac{5}{3} \;=\; \frac{4}{3}.
\end{aligned}$$

Remember that to do multiple integrals, do them one at a time, treating the variables we aren't integrating as constant. If you are not sure whether your integral is correct, you can always take the derivative and check whether it equals the original integrand.

Page 1012: Question 11: Evaluate $\int_0^1 \int_0^{x^3} \exp(y/x)dydx$; note this is $\int_{x=0}^1 \int_{y=0}^{x^3} \exp(y/x)dydx$.
Solution: We start by integrating the inside first with respect to y as we have $dydx$ and not $dxdy$. Note that the integral of $\exp(y/x)$ with respect to y is $x\exp(y/x)$, as can be verified by taking the derivative with respect to y. We thus find

$$\begin{aligned}
\int_0^1 \int_0^{x^3} \exp(y/x)dydx &= \int_0^1 \left[x\exp(y/x)\right]_{y=0}^{x^3} dx \\
&= \int_0^1 \left(xe^{x^2} - xe^0\right) dx \\
&= \int_0^1 \left(x\exp(x^2) - x\right) dx \\
&= \left[\frac{1}{2}\exp(x^2) - \frac{x^2}{2}\right]_{x=0}^1 \\
&= \left(\frac{1}{2}e^1 - \frac{1}{2}\right) - \left(\frac{1}{2}e^0 - 0\right) \;=\; \frac{e-2}{2}.
\end{aligned}$$

For help on the integral $\int_0^1 xe^{x^2} dx$, use the u-substitution technique. Let $u = x^2$, so $du = 2xdx$ and $\exp(x^2) = \exp(u)$.

Additional Problem: Let $f(x) = x^3 - 4x^2 + \cos(2x^3) + \sin(x+1701)$. Find a finite B such that $|f'(x)| \le B$ for all x in [2,3].
Solution: Differentiating f with the normal differentiation rules gives

$$f'(x) \;=\; 3x^2 - 8x - 6x^2\sin(2x^3) + \cos(x+1701).$$

Now we'll find an *upper* bound for the absolute maximum of $f'(x)$. We constantly use the absolute value of a sum/difference is less than or equal to the sum of the absolute values of the pieces. We also use the maximum of the absolute value of a product is at most the product of the maximums.

$$\begin{aligned}
|f'(x)| &= \left|3x^2 - 8x - 6x^2\sin(2x^3) + \cos(x+1701)\right| \\
&\le |3x^2| + |8x| + |6x^2| \cdot |\sin(2x^3)| + |\cos(x+1701)| \\
&= 3|x^2| + 8|x| + 6|x^2| \cdot |\sin(2x^3)| + |\cos(x+1701)| \\
&\le 3(3^2) + 8(3) + 6(3)^2 \cdot 1 + 1 \;=\; 27 + 24 + 54 + 1 \;=\; 106.
\end{aligned}$$

We may take any B greater than 106. Note the maximum of the absolute value of sine or cosine is 1, which helps in the arguments above.

Page 1011: #13: Evaluate the iterated integral

$$\int_0^3 \int_0^y \sqrt{y^2+16}\, dx\, dy \;=\; \int_{y=0}^3 \int_{x=0}^y \sqrt{y^2+16}\, dx\, dy.$$

Solution: Let's first make sure we know what region we're integrating over. We see that y ranges from 0 to 3, and that for a given value of y, x ranges from 0 to y. Therefore we're integrating over a triangle in the xy-plane with vertices at $(0,0)$, $(0,3)$ and $(3,3)$. The interior integral is easy to evaluate because $\sqrt{y^2+16}$ is constant as a function of x. Therefore

$$\int_0^y \sqrt{y^2+16}\, dx = x\sqrt{y^2+16}\Big|_0^y = y\sqrt{y^2+16}.$$

We now need to integrate $y\sqrt{y^2+16}$ from 0 to 3. It isn't immediately apparent what that integral is, although making the substitution $u = y^2$ makes things a lot clearer, since $du = 2ydy$:

$$\int_0^3 y\sqrt{y^2+16}\,dy = \frac{1}{2}\int_0^9 \sqrt{u+16}\,du = \frac{1}{3}(u+16)^{3/2}\Big|_0^9 = \frac{125}{3} - \frac{64}{3} = \frac{61}{3}.$$

Notice that this problem would have been much harder to do if we had tried to integrate with respect to y first, since we would not have had the additional y term that allowed us to make an easy substitution. In that case we would have had to find the integral of $\sqrt{y^2+16}$ with respect to y, which is (not obviously) $y\sqrt{y^2+16/2} + 8\sinh^{-1}(y/4)$. Remember that switching the order of integration can sometimes make your life a lot easier! *NOTE: We could also do* $u = y^2 + 16$.

Page 1011: #25: Sketch the region of integration for the integral

$$\int_{-2}^{2}\int_{x^2}^{4} x^2 y\,dy\,dx \;=\; \int_{x=-2}^{2}\int_{y=x^2}^{4} x^2 y\,dy\,dx.$$

Reverse the order of integration and evaluate the integral.

Solution: Notice that x ranges from -2 to 2. For a fixed value of x, y ranges from x^2 to 4. Notice that when $x = \pm 2$, $y = 4$. Therefore the boundary of the region of integration is defined by the curves $y = 4$ and $y = x^2$. To reverse the order of integration, we need to consider x as a

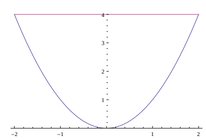

FIGURE 2. Region for Problem #25.

function of y. First notice that the minimum value of y is 0, and the maximum value of y is 4. For a fixed value of y, what values does x take? Since the bottom curve of our region of integration is given by $y = x^2$, we have $x = \pm\sqrt{y}$. Thus for a given value of y, x ranges from $-\sqrt{y}$ to $\sqrt{y}$. Our new integral is given by:

$$\int_{-2}^{2}\int_{x^2}^{4} x^2 y\,dy\,dx = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}} x^2 y\,dx\,dy.$$

We see the inner integral evaluates to

$$\int_{-\sqrt{y}}^{\sqrt{y}} x^2 y\,dx = \frac{x^3 y}{3}\Big|_{-\sqrt{y}}^{\sqrt{y}} = \frac{2y^{5/2}}{3},$$

giving our double integral as

$$\int_0^4 \frac{2y^{5/2}}{3}\,dy = \frac{2}{3}\frac{2}{7}y^{7/2}\Big|_0^4 = \frac{512}{21}.$$

Page 1011: #30: Sketch the region of integration for the integral

$$\int_0^1\int_y^1 \exp(-x^2)\,dx\,dy \;=\; \int_{y=0}^{1}\int_{x=y}^{1} \exp(-x^2)\,dx\,dy.$$

Reverse the order of integration and evaluate the integral.

Solution: Notice that y ranges from 0 to 1, and that for a given value of y, x ranges from y to 1. Therefore our region of integration is a triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$. To reverse the order of integration, notice that the minimum value of x is 0 and the maximum value of x is 1. For a given value of x, y ranges from 0 to x. Therefore our integral can be written as

$$\int_0^1\int_y^1 \exp(-x^2)\,dx\,dy = \int_0^1\int_0^x \exp(-x^2)\,dy\,dx.$$

To evaluate this integral, notice that the interior integral is

$$\int_0^x \exp(-x^2)\,dy = y\exp(-x^2)\Big|_0^x = x\exp(-x^2),$$

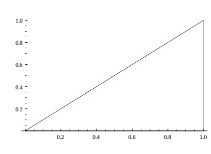

FIGURE 3. Region for Problem #30.

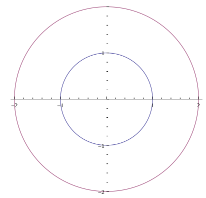

FIGURE 4. Region for Additional Problem.

so our double integral is given by

$$\int_0^1 x\exp(-x^2)\,dx.$$

Again, this might not be immediately obvious, but letting $u = x^2$, we see $du = 2xdx$ or $xdx = \frac{1}{2}du$, so our integral simplifies to

$$\int_0^1 x\exp(-x^2)\,dx = \frac{1}{2}\int_0^1 \exp(-u)\,du = -\frac{1}{2}\exp(-u)\Big|_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right).$$

As we saw in Problem 13, this integral is significantly easier to evaluate after we changed the order of integration. Without switching the order, we would have had to integrate $\exp(-x^2)$ with respect to x, which has no elementary antiderivative!

Additional Problem: Give an example of a region in the plane that is neither horizontally simple nor vertically simple.
Solution: Recall what it means for a region to be horizontally or vertically simple. A region is horizontally simply if we can express the range of x values for a given y as all x such that $g_1(y) \le x \le g_2(y)$, where g_1 and g_2 are two continuous functions with $g_1(y) \le g_2(y)$. Intuitively, a region is horizontally simple if any horizontal line intersects the region at most twice. Similarly, a region is vertically simple if any vertical line intersects the region at most twice.

One way to construct a region which is neither horizontally simple nor vertically simple is to insert a hole into a region which is horizontally and vertically simple. For example, consider the annulus in the xy plane with inner radius 1 and outer radius 2 (that is, the collection of all points between 1 and 2 units away from the origin). This region is not vertically simple, since the vertical line $x = 0$ intersects the annulus in 4 places. This region is also not horizontally simple, since the horizontal line $y = 0$ intersects the annulus in 4 places as well. Thus by taking a nice region (the circle of radius 2) and inserting a hole, we have made a region which is neither horizontally simple nor vertically simple.

16. HW #17

16.1. **Problems: HW #17.**

Page 1018: #13: Find the volume of the solid that lies below the surface $z = f(x,y) = y + e^x$ and above the region in the xy-plane bounded by the given curves: $x = 0$, $x = 1$, $y = 0$, $y = 2$.

Page 1018: #42: Find the volume of the solid bounded by the two paraboloids $z = x^2 + 2y^2$ and $z = 12 - 2x^2 - y^2$.

Page 1026: #13: Evaluate the given integral by first converting to polar coordinates:

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{1+x^2+y^2}dxdy.$$

Additional Question 1: Find $\int_{y=0}^1 \int_{x=-y}^y x^9y^8dxdy$.

Page 1026: Question 4: Evaluate $\int_{-\pi/4}^{\pi/4} \int_0^{2\cos 2\theta} rdrd\theta$.

Additional Question 2: Evaluate $\int_0^1 \int_{-y}^y \sin(xy)\cdot\exp(x^2y^2)dxdy$. *Hint: in what way is this similar to an earlier problem on this homework assignment?*

Additional Question 3: Let $f(x,y,z) = \cos(xy+z^2)$. Find $(\mathrm{Df})(x,y,z)$.

Additional Question 4: Find the maximum value of $f(x,y) = xy$ given that $g(x,y) = x^2 + 4y^2 = 1$.

16.2. **Solutions: HW #17.**

Page 1018: #13: Find the volume of the solid that lies below the surface $z = f(x, y) = y + e^x$ and above the region in the xy-plane bounded by the given curves: $x = 0$, $x = 1$, $y = 0$, $y = 2$.
Solution: Note the region in the xy-plane is the rectangle $[0, 1] \times [0, 2]$, or $0 \leq x \leq 1$ and $0 \leq y \leq 2$. The height is $z = f(x, y) = y + e^x$ (which is always above the xy-axis. Thus the volume is equal to $\int_{x=0}^{1}\int_{y=0}^{2}(y + e^x)dydx$; we could have done the x-integral first since the region is both horizontally and vertically simple. The y-integral gives $\frac{1}{2}y^2 + ye^x$, which we must evaluate at 0 and 2. We thus find the volume equals

$$\begin{aligned}\int_{x=0}^{1}\int_{y=0}^{2}(y + e^x)\,dydx &= \int_{x=0}^{1}\left[\frac{y^2}{2} + ye^x\right]_{y=0}^{2} dx \\ &= \int_{x=0}^{1}(2 + 2e^x)dx \\ &= \Big[2x + 2e^x\Big]_{x=0}^{1} \\ &= 2 + 2e - 2 \;=\; 2e\end{aligned}$$

Page 1018: #42: Find the volume of the solid bounded by the two paraboloids $z = x^2 + 2y^2$ and $z = 12 - 2x^2 - y^2$.
Solution: NOTE: Should have x from -2 to 2, and then fix the rest.... We first solve for the intersection of the two paraboloids. Note the first is the bottom and the second is the top. Setting the two equal, we find $z = x^2 + 2y^2 = 12 - 2x^2 - y^2$. Doing some algebra gives $3x^2 + 3y^2 = 12$, or $x^2 + y^2 = 4$. Note this is the equation of a circle of radius 2; unlike the problems in class the height is not constant here. The distance between the top and the bottom curves at an (x, y) is $z_{\text{top}} - z_{\text{bottom}}$, which is $12 - 2x^2 - y^2 - (x^2 - y^2) = 12 - 3x^2 - 3y^2$. We have

$$\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(12 - 3x^2 - 3y^2)dydx.$$

We convert this to polar coordinates. Let $f(x, y) = 12 - 3x^3 - 3y^2$. We are integrating over the unit disk, which is easily converted to a rectangle in polar coordinates. We have $f(r\cos\theta, r\sin\theta) = 12 - 3r^2$, and thus the volume is

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{2}(12 - 3r^2)rdrd\theta.$$

We use u-substitution. Let $u = 12 - 3r^2$ so $du = -6rdr$ or $rdr = (-1/6)du$. We replace $r : 0 \to 2$ with $u : 12 \to 0$, and thus the volume is

$$\begin{aligned}\int_{\theta=0}^{2\pi}\int_{u=12}^{0}(-1/6)udud\theta &= -\frac{1}{6}\int_{\theta=0}^{2\pi}\frac{u^2}{2}\Big|_{12}^{0}d\theta \\ &= -\frac{1}{6}\frac{-144}{2}\int_{\theta=0}^{2\pi}d\theta \\ &= 12\int_{\theta=0}^{2\pi}d\theta \;=\; 12\cdot 2\pi \;=\; 24\pi.\end{aligned}$$

If you do not want to convert to polar, you can follow the hint on the book for problems 39 to 45, which says to consult the table of integrals in the back of the book for the anti-derivativfe of $(a^2 - x^2)^{3/2}$, and use that to finish solving the problem.

If you've read this far, however, you have forgotten the very sage advice of the Patron Saint of Mathematics, Henry David Thoreau, who advises us all to '*Simplify, simplify*'. Instead of trying to use u-substitution, let's just multiply things out! Then $(12 - 3r^2)r$ becomes $12r - 3r^3$, which can be integrated directly! Thus the solution to this problem is also

$$\begin{aligned}\int_{\theta=0}^{2\pi}\int_{r=0}^{2}(12 - 3r^2)rdrd\theta &= \int_{\theta=0}^{2\pi}\int_{r=0}^{2}(12r - 3r^3)drd\theta \\ &= \int_{\theta=0}^{2\pi}\left[\frac{12r^2}{2} - \frac{3r^4}{4}\right]_{r=0}^{2}d\theta \\ &= \int_{\theta=0}^{2\pi}(24 - 12)\,d\theta \\ &= 12\int_{\theta=0}^{2\pi}d\theta \\ &= 12\cdot 2\pi \;=\; 24\pi,\end{aligned}$$

not surprisingly the same answer as before.

Page 1026; #13: Evaluate the given integral by first converting to polar coordinates:

$$\int_0^1\int_0^{\sqrt{1-y^2}}\frac{1}{1 + x^2 + y^2}.dxdy$$

Solution: We notice that the region is the first-quadrant part the unit circle. Thus $0 \le r \le 1$ and $0 \le \theta \le \pi/2$. The function is $f(x,y) = 1/(1+x^2+y^2)$, so $f(r\cos\theta, r\sin\theta) = 1/(1+r^2)$. We thus have the integral equals

$$\begin{aligned}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1}\frac{1}{1+r^2}\,r dr d\theta &= \frac{1}{2}\int_{\theta=0}^{\pi/2}[\ln(1+r^2)]_{r=0}^{1} d\theta \\ &= \frac{1}{2}\int_{\theta=0}^{\pi/2}[\ln 2 - \ln 1]\,d\theta \\ &= \frac{\ln 2}{2}\int_{\theta=0}^{\pi/2} d\theta \;=\; \frac{\ln 2}{2}\cdot\frac{\pi}{2} \;=\; \frac{\pi \ln 2}{4}.\end{aligned}$$

The key step was a u-substitution. We had to integrate $\int_{r=0}^{1}\frac{rdr}{1+r^2}$. If we take $u = 1+r^2$, $du = 2rdr$ so $rdr = (1/2)du$, $r : - \to 1$ becomes $u : 1 \to 2$, and thus the r-integral becomes $\int_{u=1}^{2} du/u \;=\; \ln u\Big|_{u=1}^{2} \;=\; \ln 2$.

Additional Question #1: Find $\int_{y=0}^{1}\int_{x=-y}^{y} x^9 y^8 dxdy$.
Solution: We start by integrating with respect to x, so we have:

$$\begin{aligned}\int_{y=0}^{1}\int_{x=-y}^{y} x^9 y^8 dxdy &= \frac{1}{10}\int_{y=0}^{1}[y^8 x^{10}]_{x=-y}^{y} dy \\ &= \frac{1}{10}\int_{y=0}^{1}(y^{18} - y^{18})dy \\ &= 0.\end{aligned}$$

Note we are integrating an odd function about a symmetric interval, and thus we do get zero.

Page 1026: Question 4: Evaluate $\int_{-\pi/4}^{\pi/4}\int_{0}^{2\cos 2\theta} r dr d\theta$.
Solution: We start by integrating the inside first with respect to r, which gives us

$$\begin{aligned}\int_{-\pi/4}^{\pi/4}\int_{0}^{2\cos 2\theta} r dr d\theta &= \int_{-\pi/4}^{\pi/4}\left[\frac{r^2}{2}\right]_{0}^{2\cos 2\theta} d\theta \\ &= \int_{-\pi/4}^{\pi/4} 2\cos^2 2\theta d\theta \\ &= \int_{-\pi/2}^{\pi/2}\cos^2 u du,\end{aligned}$$

where we did u-substitution: $u = 2\theta$, $du = 2d\theta$, and $\theta : -\pi/4 \to \pi/4$ means $u : -\pi/2 \to \pi/2$. We now use a trig-identity. As

$$\cos(2u) \;=\; \cos(u+u) = \cos u \cos u - \sin u \sin u = \cos^2 u - \sin^2 u \;=\; 2\cos^2 u - 1$$

(where the last followed from $\sin^2 u = 1 - \cos^2 u$), we see that $\cos^2 u = \frac{\cos(2u)+1}{2}$. In the arguments below we'll do another substitution; we'll let $v = 2u$ so $dv = 2du$ and $u : -\pi/2 \to \pi/2$ will mean that $v : -\pi/ \to \pi$. Continuing we find

$$\begin{aligned}\int_{-\pi/4}^{\pi/4}\int_{0}^{2\cos 2\theta} r dr d\theta = \int_{-\pi/2}^{\pi/2}\cos^2 u du &= \int_{-\pi/2}^{\pi/2}\left[\frac{\cos 2u}{2} + \frac{1}{2}\right] du \\ &= \frac{1}{4}\int_{-\pi}^{\pi}\cos v dv + \frac{1}{2}\int_{-\pi/2}^{\pi/2} du \\ &= \frac{1}{4}[\sin v]_{v=-\pi}^{\pi} + \frac{1}{2}[u]_{u=-\pi/2}^{\pi/2} \\ &= \frac{1}{4}(0) + \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{\pi}{2}.\end{aligned}$$

Remember that when using u-substitution, be sure to change the bounds correctly.

There are other ways to do this problem. We could use the table of integrals in the front cover to find the anti-derivative of $\cos^2 u$; to put our expression in a form where we could do this, we would need to do a u-substitution first. Thus we have

$$\int_{-\pi/4}^{\pi/4}\int_{0}^{2\cos 2\theta} r dr d\theta \;=\; \int_{-\pi/4}^{\pi/4} 2\cos^2 2\theta d\theta \;=\; \int_{-\pi/2}^{\pi/2}\cos^2 u du \;=\; 2\int_{0}^{\pi/2}\cos^2 u du,$$

where we used the u-substitution $u = 2\theta$, $du = 2d\theta$, and as $\theta : -\pi/4 \to \pi/4$, $u : -\pi/2 \to \pi/2$. We then noted the integrand was even and the range symmetric, so we could just integrate from 0 to $\pi/2$ and double. Note that

$$\int_{0}^{\pi/2}\cos^2 u du \;=\; \frac{1}{4}\int_{0}^{2\pi}\cos^2 u du, \quad \int_{0}^{2\pi}\cos^2 u du \;=\; \int_{0}^{2\pi}\sin^2 du.$$

Thus

$$\int_{0}^{\pi/2}\cos^2 u du \;=\; \frac{1}{4}\int_{0}^{2\pi}\cos^2 u du \;=\; \frac{1}{4}\frac{1}{2}\int_{0}^{2\pi}\left[\cos^2 u + \sin^2 u\right] du \;=\; \frac{2\pi}{8},$$

as $\cos^2 u + \sin^2 u = 1$, and so the answer is $2 \cdot (2\pi/8) = \pi/2$.

Additional Question 2: Evaluate $\int_0^1 \int_{-y}^{y} \sin(xy) \cdot \exp(x^2y^2) dxdy$*. Hint: in what way is this similar to an earlier problem on this homework assignment?*

Solution: The key to this question is to realize that $\sin(xy)$ is an odd function and that $\exp(x^2y^2)$ is an even function. Recall $f(x)$ is odd if $f(-x) = -f(x)$ and even if $f(-x) = f(x)$. The product of an even function and an odd function is an odd function. Since we are first integrating over the bound $[-y, y]$ we can use the symmetry properties of integrals to simplify the calculation: the integral of an odd function over a symmetric region is zero, as the positive parts cancel with the negative parts.

$$\int_0^1 \int_{-y}^{y} \sin(xy) \cdot \exp(x^2y^2) dxdy \quad = \quad \int_0^1 0 dy \;=\; 0$$

For example, $\int_{x=-2}^{2} x dx = \frac{x^2}{2}\Big|_{x=-2}^{2} = 0$. What is very nice is that we do not need to know what the antiderivative is; the antiderivative of an odd function is an even function, and thus the difference is zero when we subtract with symmetric boundary points.

Additional Question 3: Let $f(x, y, z) = \cos(xy + z^2)$. Find $(\mathrm{D}f)(x, y, z)$.

Solution: Since we are calculating the gradient of this function, we simply need to apply the normal differentiation rules to determine the partial derivatives of $f(x, y, z)$.

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) &= -y\sin(xy+z^2) \\
\frac{\partial f}{\partial y}(x,y,z) &= -x\sin(xy+z^2) \\
\frac{\partial f}{\partial z}(x,y,z) &= -2z\sin(xy+z^2) \\
Df(x,y,z) &= \langle -y\sin(xy+z^2), -x\sin(xy+z^2), -2z\sin(xy+z^2)\rangle
\end{aligned}$$

Additional Question 4: Find the maximum value of $f(x, y) = xy$ given that $g(x, y) = x^2 + 4y^2 = 1$.

Solution: We will use the method of Lagrange multipliers to calculate the constrained maximum. Set up the appropriate equations by setting the gradients equal to each other with the constant λ. In other words, we must solve $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$. As $\nabla f = \langle y, x\rangle$ and $\nabla g = \langle 2x, 8y\rangle$, we see we must solve

$$\frac{\partial f}{\partial x}(x,y) \;=\; \lambda\frac{\partial g}{\partial x}(x,y), \quad \frac{\partial f}{\partial y}(x,y) \;=\; \lambda\frac{\partial g}{\partial y}(x,y), \quad g(x,y) \;=\; c,$$

or substituting

$$y \quad = \quad \lambda 2x, \quad x \;=\; \lambda 8y, \quad x^2+4y^2 \;=\; 1.$$

Note that if $y = 0$ then $x = 0$, but this does not satisfy the constraint. Similarly if $x = 0$ then $y = 0$. Thus neither x nor y is zero (and thus neither is λ).

If we take the ratio of the second equation over the first, we find

$$\frac{x}{y} \;=\; \frac{\lambda 8y}{\lambda 2x} \quad \text{or} \quad \frac{x}{y} \;=\; \frac{4y}{x}.$$

Cross multiplying gives $x^2 = 4y^2$. Substituting this into the constraint $x^2 + 4y^2 = 1$ gives $4y^2 + 4y^2 = 1$, so $y^2 = 1/8$ or $y = \pm 1/2\sqrt{2}$. As $x^2 = 4y^2$, we see that $x = \pm 1/\sqrt{2}$. We thus have four candidate points to check for maxima / minima: $(x, y) = (\pm 1/\sqrt{2}, \pm 1/2\sqrt{2})$. The two points where the signs are equal evaluate under f to 1/4, while the two points where the signs are opposite evaluate under f to $-1/4$; thus the maximum value is $1/4$.

17. HW #18

17.1. **Problems: HW #18.**

Page 1056: #37a.: Use spherical coordinates to evaluate the integral

$$I \;=\; \int\int\int_B \exp(-\rho^3)\, dV$$

where B is the solid ball of radius a centered at the origin.

Page 1056: #37b.: Let $a \to \infty$ in the result of part (a) to show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-(x^2+y^2+z^2)^{3/2})\, dx\, dy\, dz = \frac{4}{3}\pi.$$

17.2. **Solutions: HW #18.**

Page 1056: #37a.: Use spherical coordinates to evaluate the integral

$$I \;=\; \int\int\int_B \exp(-\rho^3)\, dV$$

where B is the solid ball of radius a centered at the origin.
Solution: We first need to figure out our limits of integration. Recall that in spherical coordinates we have the radius ρ, which will range from 0 to a, the angle θ in the xy-plane, which ranges between 0 and 2π, and the azimuthal angle ϕ, will ranges from 0 to π. Therefore our limits of integration are

$$\int_{\rho=0}^{\rho=a}\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi} \exp(-\rho^3)\, dV.$$

The volume element dV is given by $dV = \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$ (as we have a rectangular box in spherical coordinates and the bounds of integration are fixed and do not depend on each other, we may integrate in any order). Therefore our integral is given by

$$\int_{\rho=0}^{\rho=a}\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi} \exp(-\rho^3)\, \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho.$$

The inside integrates to $-\exp(-\rho^3)\rho^2\cos\phi$. Taking the difference at the endpoints we get $2\exp(-\rho^3)\,\rho^2$, and thus

$$I \;=\; 2\int_{\rho=0}^{\rho=a}\int_{\theta=0}^{\theta=2\pi} \exp(-\rho^3)\, \rho^2 d\theta\, d\rho.$$

Since the inside is constant as a function of θ, integrating with respect to θ has the same effect as multiplying by 2π, giving

$$I \;=\; 4\pi\int_{\rho=0}^{\rho=a} \exp(-\rho^3)\, \rho^2 d\rho.$$

Notice that the integral of $\exp(-\rho^3)\,\rho^2$ with respect to ρ is just $-\exp(-\rho^3)/3$, so our integral evaluates to

$$I \;=\; \frac{-4\pi}{3}\exp(-\rho^3)\Big|_0^a \;=\; \frac{4\pi}{3}(1-\exp(-a^3)).$$

Page 1056: #37b.: Let $a \to \infty$ in the result of part (a) to show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-(x^2+y^2+z^2)^{3/2})\, dx\, dy\, dz = \frac{4}{3}\pi.$$

Solution: Notice that this is the integral of $\exp(-\rho^3)$ over all of $\mathbb{R}^3$, which is exactly the integral we worked out in part (a) in the limit as $a \to \infty$. As $a \to \infty$, we see that $\exp(-a^3) \to 0$, so our integral does indeed approach $4\pi/3$.

18. HW #20

18.1. **OPTIONAL Problems: HW #–.**

THIS ASSIGNMENT IS ENTIRELY EXTRA CREDIT! IT INVOLVES YOU WATCHING THE VIDEO AND DOING THESE PROBLEMS. IT IS OPTIONAL.

Page 1071: #2: Solve for x and y in terms of u and v, and compute the Jacobian $\partial(x,y)/\partial(u,v)$ with $u = x - 2y, v = 3x + y$.

Page 1071: #3: Solve for x and y in terms of u and v, and compute the Jacobian $\partial(x,y)/\partial(u,v)$ with $u = xy, v = y/x$.

18.2. **Solutions: OPTIONAL Problems: HW #21.**

Page 1071: #2: Solve for x and y in terms of u and v, and compute the Jacobian $\partial(x,y)/\partial(u,v)$ with

$$u = x - 2y \qquad v = 3x + y.$$

Solution: We first notice that $u + 2v = 7x$, so $x = x(u,v) = (u+2v)/7$. Similarly, $v - 3u = 7y$, so $y = y(u,v) = (v-3u)/7$. Therefore the Jacobian $\partial(x,y)/\partial(u,v)$ is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/7 & 2/7 \\ -3/7 & 1/7 \end{vmatrix} = \frac{1}{7}$$

Page 1071: #3: Solve for x and y in terms of u and v, and computer the Jacobian $\partial(x,y)/\partial(u,v)$ with

$$u = xy \qquad v = y/x.$$

Solution: Notice that multiplying u and v together yields $uv = y^2$, so $y = y(u,v) = \pm\sqrt{uv}$. Similarly, dividing u by v gives $u/v = x^2$, so $x = x(u,v) = \pm\sqrt{u/v}$. Which of the solutions should we take? Notice that we need $xy = u$, so we must either take both positive solutions or both negative solutions. Taking both positive solutions, we find that the Jacobian is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & \frac{-\sqrt{u}}{2v\sqrt{v}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{4v} + \frac{1}{4v} = \frac{1}{2v}.$$

The exact same calculation shows that the Jacobian is again $1/(2v)$ when we take the negative solutions.

19. HW #22

19.1. **Problems: HW #22.**

Problem 1:. Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ that diverges.

Problem 2:. Give an example of a sequence of distinct terms a_n such that the sequence $\{a_n\}_{n=1}^{\infty}$ converges.

Problem 3:. Give an example of a sequence of distinct terms a_n such that $|a_n| < 2018$ and the sequence $\{a_n\}_{n=1}^{\infty}$ does not converge.

Problem 10-4 (Cain-Herod): Find the limit of the sequence $a_n = 3/n^2$, or explain why it does not converge.

Problem 10-5 (Cain-Herod): Find the limit of the sequence $a_n = \frac{3n^2+2n-7}{n^2}$.

19.2. **Solutions: HW #21.**

Problem 1:. Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ that diverges.
Solution: There are two ways for a sequence not to converge. It can either get too big (diverge to infinity), or it can bounce around forever and never settle down. For instance, the sequence given by $a_n = n$ for all $n \in \mathbb{N}$ will diverge to infinity, since given any real number $r \in \mathbb{R}$, $a_n > r$ for all $n > r$. A sequence that fails to converge because it bounces is $a_n = (-1)^n$, or more interestingly $a_n = (-1)^n + (-1)^n/n$.

Problem 2:. Give an example of a sequence of distinct terms a_n such that the sequence $\{a_n\}_{n=1}^{\infty}$ converges.
Solution: For a sequence to converge to a limit L, it must eventually get and stay arbitrarily close to L. Consider the sequence $a_n = 1/n$. We claim this converges to 0. To prove this, we need to show that given any $\epsilon > 0$, we can find an N such that $|a_n - 0| < \epsilon$ for all $n > N$. Let N be any integer exceeding $2/\epsilon$. Then for $n > N$, $a_n < \epsilon/2$, so $|a_n - 0| < \epsilon/2 < \epsilon$, so a_n does indeed converge to 0. Arguing more informally, we would say $\lim_{n\to\infty} |a_n - 0| = \lim_{n\to\infty} 1/n$, and this limit is zero, thus proving that 0 is indeed the limit of the sequence. For a more interesting example, consider the sequence $a_n = 3 + 1/n$, which converges to 3.

Problem 3:. Give an example of a sequence of distinct terms a_n such that $|a_n| < 2018$ and the sequence $\{a_n\}_{n=1}^{\infty}$ does not converge.
Solution: Here we are looking for a bounded sequence that does not converge. Since the sequence cannot diverge to infinity, it must continually bounce around. Consider the sequence

$$\{a_n\}_{n=1}^{\infty} \;=\; \left\{1,\ \frac{2}{3},\ \frac{1}{3},\ \frac{4}{5},\ \frac{1}{5},\ \frac{6}{7},\ \frac{1}{7},\ \frac{8}{9},\ \cdots,\ \right\}$$

where the odd terms are given by $a_{2k+1} = 1/(2k+1)$, and the even terms are given by $a_{2k} = 2k/(2k+1)$. Notice that this sequence is bounded since every term is less than or equal to 1, and cannot converge because the odd terms converge to 0 while the even terms converge to 1.
Problem 10-4 (Cain-Herod): Find the limit of the sequence $a_n = 3/n^2$, or explain why it does not converge.
Solution: We can use the limit of a quotient is the quotient of the limit as the limit of the denominator is not zero and we do not have ∞/∞. We see that the numerator is always 3 while the denominator increases and approaches infinity. Thus we know that $\lim_{n\to\infty} 3/n^2 = 0$.

Problem 10-5 (Cain-Herod): Find the limit of the sequence $a_n = \frac{3n^2+2n-7}{n^2}$
Solution: We cannot use the limit of a quotient is the quotient of the limits as we have ∞/∞. One approach is to use L'Hopital's rule and take derivatives of the numerator and the denominator. We have

$$\lim_{n\to\infty} \frac{3n^2+2n-7}{n^2} \;=\; \lim_{n\to\infty} \frac{6n+2}{2n} \;=\; \lim_{n\to\infty} \frac{6}{2} \;=\; \lim_{n\to\infty} 3 \;=\; 3.$$

Another approach is to pull out the highest power of n in the numerator and denominator:

$$\lim_{n\to\infty} \frac{3n^2+2n-7}{n^2} \;=\; \lim_{n\to\infty} \frac{n^2(3+2/n-7/n^2)}{n^2\cdot 1} \;=\; \lim_{n\to\infty} \frac{3+2/n-7/n^2}{1} \;=\; \lim_{n\to\infty}\left(3+\frac{2}{n}-\frac{7}{n^2}\right) \;=\; 3.$$

The analysis is easier than some of the other problems as the denominator was just n to a power.

20. HW #22

20.1. **Problems: HW #22.**

Problem 10-8 (Cain-Herod): Find the limit of the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$.

Problem 10-10 (Cain-Herod): Find a value of n that will insure that $1 + 1/2 + 1/3 + \cdots + 1/n > 10^6$. Prove your value works.

Page 10-8 (Cain-Herod): Question 14: Determine if the series $\sum_{k=0}^{\infty} \frac{1}{2e^k+k}$ converges or diverges.

Page 10-8 (Cain-Herod): Question 15: Determine if the series $\sum_{k=0}^{\infty} \frac{1}{2k+1}$ converges or diverges.

Additional question: Let $f(x) = \cos x$, and compute the first eight derivatives of $f(x)$ at $x = 0$, and determine the n^{th} derivative.

20.2. **Solutions: HW #22.**

Problem 10-8 (Cain-Herod): Find the limit of the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$.
Solution: This is the same as finding the sum of the infinite geometric sequence $1 + 1/3 + (1/3)^2 + (1/3)^3 + \cdots$ and then subtracting off 1, as we want to start the sum at $n = 1$ and not $n = 0$. We can use the formula that the sum of infinite geometric sequence with ratio r starting at $n = 0$ is $\frac{1}{1-r}$, provided of course that $|r| < 1$. For us $r = 1/3$, and thus the sum, starting from $n = 0$, is $1/(1 - \frac{1}{3}) = 1/(2/3) = 3/2$; however, we want the sum to start with the $n = 1$ term and not the $n = 0$ term, so we must subtract the $n = 0$ term, which is 1. Thus the answer is $3/2 - 1 = 1/2$.

Problem 10-10: Find a value of n that will insure that $1 + 1/2 + 1/3 + \cdots + 1/n > 10^6$. Prove your value works.
Solution: By a result stated in class, we know that for N large

$$\sum_{n=1}^{N} \frac{1}{N} \approx \ln N.$$

So we must solve $\ln N = 10^6$; the solution is $N = \exp(10^6)$, which is about $9.8 \cdot 10^{434294}$.

It is possible to solve this without using the asymptotic relation for the sum. We showed in class that if we group the terms $1/3$ and $1/4$ we get at least $1/2$, and if we group terms $1/5, 1/6, 1/7, 1/8$ we get at least $1/2$, and so on. If we go up to the term $n = 2^2$ we have at least 1/2 two times, if we go up to $n = 2^3$ we have 1/2 at least 3 times, and in general if we go up to n^k then we have 1/2 at least k times. If we want to have the sum at least 10^6, we just need to take $k = 2 \cdot 10^6$, which means $n = 2^{2 \cdot 10^6} = 4^{10^6}$, which is approximately $3.0 \cdot 10^{602059}$. Note how much larger this is than the answer we get from using the sum of the first N terms is about $\ln N$.

Page 10-8: Question 14: Determine if the series $\sum_{k=0}^{\infty} \frac{1}{2e^k + k}$ converges or diverges.
Solution: We will use the comparison test to determine if this series converges or diverges. The series $\sum_{k=0}^{\infty} \frac{1}{2e^k+k}$ is less than the series $\sum_{k=0}^{\infty} \frac{1}{2e^k}$, which is less than the convergent series $\sum_{k=0}^{\infty} \frac{1}{e^k} = \sum_{k=0}^{\infty} (1/e)^k$. This last series is a geometric series with ratio $r = 1/e$, as $|r| < 1$, the geometric series converges. Thus, by the comparison test, the original sequence converges because $|\frac{1}{2e^k+k}| \le \frac{1}{e}$.

Page 10-8: Question 15: Determine if the series $\sum_{k=0}^{\infty} \frac{1}{2k+1}$ converges or diverges.
Solution: We will use the comparison test to determine if this series converges or diverges. We want to compare this to a multiple of the harmonic series; we know the harmonic series diverges, and multiplying each term by a constant won't change if it converges or diverges. We have $4k \ge 2k+1$ for all $k \ge 1$. This implies $\frac{1}{2k+1} \ge \frac{1}{4k} = \frac{1}{4}\frac{1}{k}$. Thus our series is greater, term by term, than the harmonic series (multiplied by 1/4). As the harmonic series diverges, so too does our series.

Another proof is to note that the sum over the odd indexed terms (which are just the odd terms) in the harmonic series is at least as large as the sum over the even terms, and since the total sum diverges so too must the sum over just the odd indexed terms.

Page 10-8: Question 16: Determine if the series $\sum_{k=2}^{\infty} \frac{1}{\log k}$ converges or diverges.
Solution: We will use the comparison test to determine if this series converges or diverges. The growth of a log function is slower than a linear function: $\log k \le k$; taking the reciprocal reverses the relation, so $\frac{1}{\log k} \ge \frac{1}{k}$. Thus our series is greater, term by term, than the harmonic series. As the harmonic series diverges, so too does our series.

Additional question: Let $f(x) = \cos x$, and compute the first eight derivatives of $f(x)$ at $x = 0$, and determine the n^{th} derivative.
Solution: We will begin by computing the first eight derivatives.

$$\begin{aligned}
f'(x) &= -\sin x \\
f''(x) &= -\cos x \\
f'''(x) &= \sin x \\
f^{(iv)}(x) &= \cos x \\
f^{(v)}(x) &= -\sin x \\
f^{(vi)}(x) &= -\cos x \\
f^{(vii)}(x) &= \sin x \\
f^{(viii)}(x) &= \cos x.
\end{aligned}$$

Now compute the derivatives at $f(0)$.

$$\begin{aligned}
f'(0) &= -\sin 0 = 0, \quad f''(0) = -\cos 0 = -1 \\
f'''(0) &= \sin 0 = 0, \quad f^{(iv)}(0) = \cos 0 = 1 \\
f^{(v)}(0) &= -\sin 0 = 0, \quad f^{(vi)}(0) = -\cos 0 = -1 \\
f^{(vii)}(0) &= \sin 0 = 0, \quad f^{(viii)}(0) = \cos 0 = 1.
\end{aligned}$$

We see the pattern: 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1 and so on. Specifically, the even derivatives vanish, and if $f(x) = \cos x$ then $f^{(4k+1)}(0) = -1$ while $f^{(4k+3)}(0) = 1$.

21. HW #23

21.1. **Problems: HW #23.**

Problem Cain-Herod 10-18: Is the series $\sum_{k=0}^{n} \frac{10^k}{k!}$ convergent or divergent? Prove your answer.

Problem Cain-Herod 10-21: Is the following series convergent or divergent (and of course prove your answer)?

$$\sum_{k=1}^{n} \frac{3^k}{5^k(k^4+k+1)}$$

Problem 3: Let $a_n = \frac{1}{n \ln n}$ (one divided by n times the natural log of n). Prove that this series diverges. *Hint: what is the derivative of the natural log of x? Use u-substitution.*

Problem 4: Let $a_n = \frac{1}{n \ln^2 n}$ (one divided by n times the square of the natural log of n). Prove that this series converges. *Hint: use the same method as the previous problem.*

Problem 5: Give an example of a sequence or series that you have seen in another class, in something you've read, in something you've observed in the world, ...

21.2. **Solutions: HW #23.**

Problem 10-18: Is the series $\left(\sum_{k=0}^{n} \frac{10^k}{k!}\right)$ convergent or divergent?
Solution: We use the ratio test:

$$\rho = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{10^{k+1}}{(k+1)!}\right|\left|\frac{k!}{10^k}\right| = \lim_{k\to\infty}\left|\frac{10}{k+1}\right| = 0 < 1,$$

so the series converges as the ratio ρ is less than 1.

Problem 10-21: Is the following series convergent or divergent?

$$\sum_{k=1}^{n} \frac{3^k}{5^k(k^4+k+1)}$$

Solution: We use the Comparison Test:

$$\sum_{k=1}^{n}\left(\frac{3}{5}\right)^k \frac{1}{(k^4+k+1)} < \sum_{k=1}^{n}\frac{1}{(k^4+k+1)} < \sum_{k=1}^{n}\frac{1}{k^4},$$

which converges (it is a p-series with $p = 4$), and thus the original series also converges. Alternatively, we have $a_k \leq (3/5)^k$, and we obtain convergence by comparing with a geometric series with ratio $3/5$.

Problem 3: Let $a_n = \frac{1}{(n\ln n)}$ (one divided by n times the natural log of n). Prove that this series diverges. *Hint: what is the derivative of the natural log of x? Use u-substitution.*

Solution: We use the integral test. We start the series with $n = 2$ as $\ln 1 = 0$ and we cannot divide by zero. Set $f(x) = \frac{1}{x\ln x}$; note $f(n) = a_n$. The convergence / divergence of the series is equivalent to the convergence or divergence of the integral $\int_2^\infty \frac{1}{x\ln x}dx$. Through substitution by parts, we have $u = \ln x$, $du = \frac{dx}{x}$, and $x : 2 \to \infty$ becomes $u : \ln 2 \to \infty$. Then

$$\int_2^\infty \frac{1}{\ln x}\frac{dx}{x} = \int_{\ln 2}^\infty \frac{1}{u}du = [\ln u]_{\ln 2}^\infty.$$

As this clearly diverges, the original series diverges as well.

Problem 4: Let $a_n = \frac{1}{(n\ln^2 n)}$ (one divided by n times the square of the natural log of n). Prove that this series converges. *Hint: use the same method as the previous problem.*

Solution: We integrate $\int_2^\infty \frac{1}{x\ln^2 x}dx$, where we cannot have $n = 1$ (see previous problem). Through u-substitution, we have $u = \ln x$, $du = \frac{dx}{x}$, and $x : 2 \to \infty$ becomes $u : \ln 2 \to \infty$. Then

$$\int_2^\infty \frac{1}{\ln^2 x}\frac{dx}{x} = \int_{\ln 2}^\infty \frac{1}{u^2}du = \left[-\frac{1}{u}\right]_{\ln 2}^\infty = \frac{1}{\ln 2}.$$

As this converges, the original series converges as well.

Problem 5: Give an example of a sequence or series that you have seen in another class, in something you've read, in something you've observed in the world, ...

22. HW #24

22.1. **Problems: HW #24.**

Cain-Herod: Question 20: Does the series $\sum_{n=1}^\infty \frac{3^{2k+1}}{10^k}$ converge or diverge?
Additional Question 1: Compute the first five terms of the Taylor series expansion of $\ln(1-x)$ (the natural logarithm of x) about $x = 0$, and conjecture the answer for the full Taylor series.
Additional Question 2: Compute the first five terms of the Taylor series expansion of $\ln(1+x)$ (the natural logarithm of x) about $x = 0$, and conjecture the answer for the full Taylor series.
Additional Question 3: Give an example of a sequence or series you like.
Additional Question 4: Find the second order Taylor series expansion of $\cos(xy)$ about $(0,0)$.
Additional Question 5: Find the second order Taylor Series expansion of $\cos(\sqrt{x+y})$ about $(0,0)$.
Additional Question 6: Find the second order Taylor series expansion of $\cos(x^3y^4)$ about $(0,0)$.
Problem Extra Credit 1:: Give a product of infinitely many distinct, positive terms such that the product converges to a number c with $0 < c < \infty$.
Problem Extra Credit 2:: Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive numbers such that $\sum_{n=1}^\infty 1/a_n$ converges. Let $B_n = 1/n\sum_{k=1}^n a_k$. Prove that $\sum_{n=1}^\infty 1/B_n$ converges.

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