Math 150: Calculus III: Multivariable Calculus

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Lecture 9: 2-25-2022:

https://youtu.be/xolV2TnWvlI

http://www.youtube.com/watch?v=71_8kHSAE4w **(February 21, 2014: Limits, Defn Partial Derivatives)**

Plan for the day: Lecture 9: February 24, 2022:

Topics:

Limits

Three definitions of the Derivative (one variable)

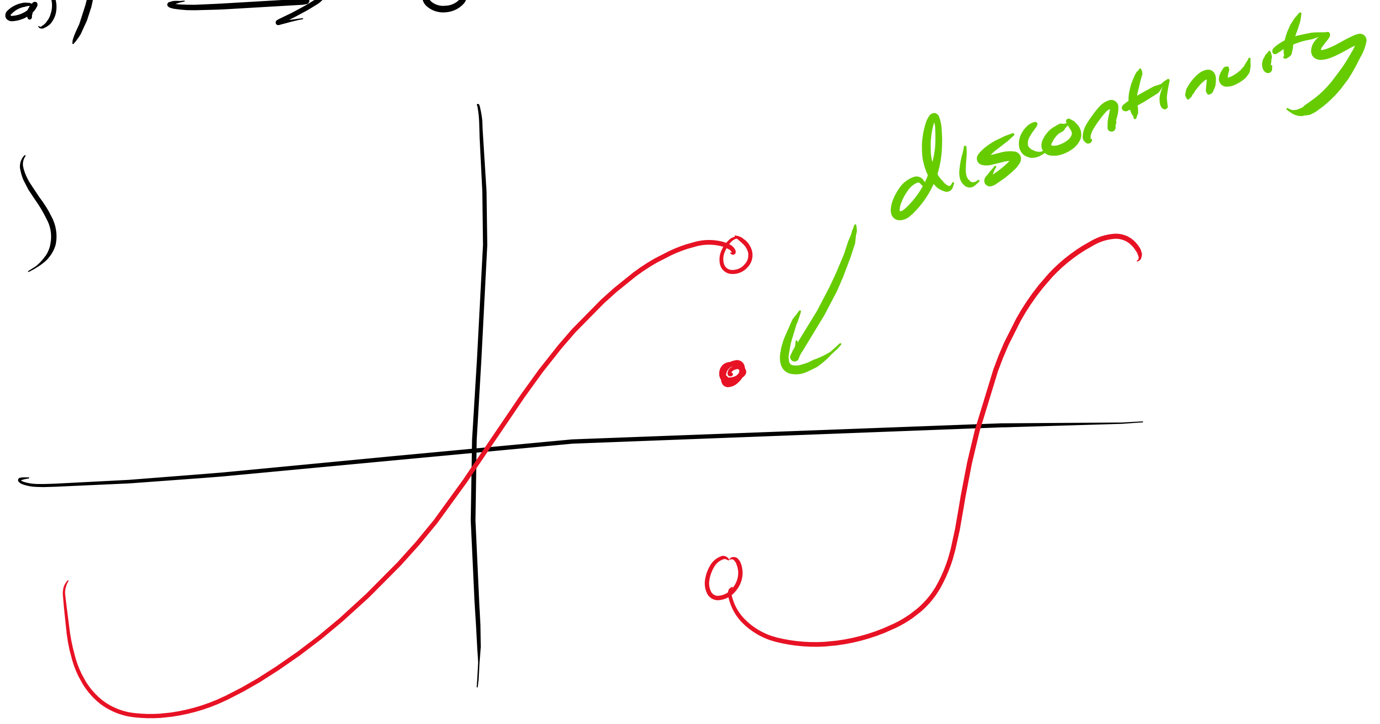
Partial Derivatives

Limits

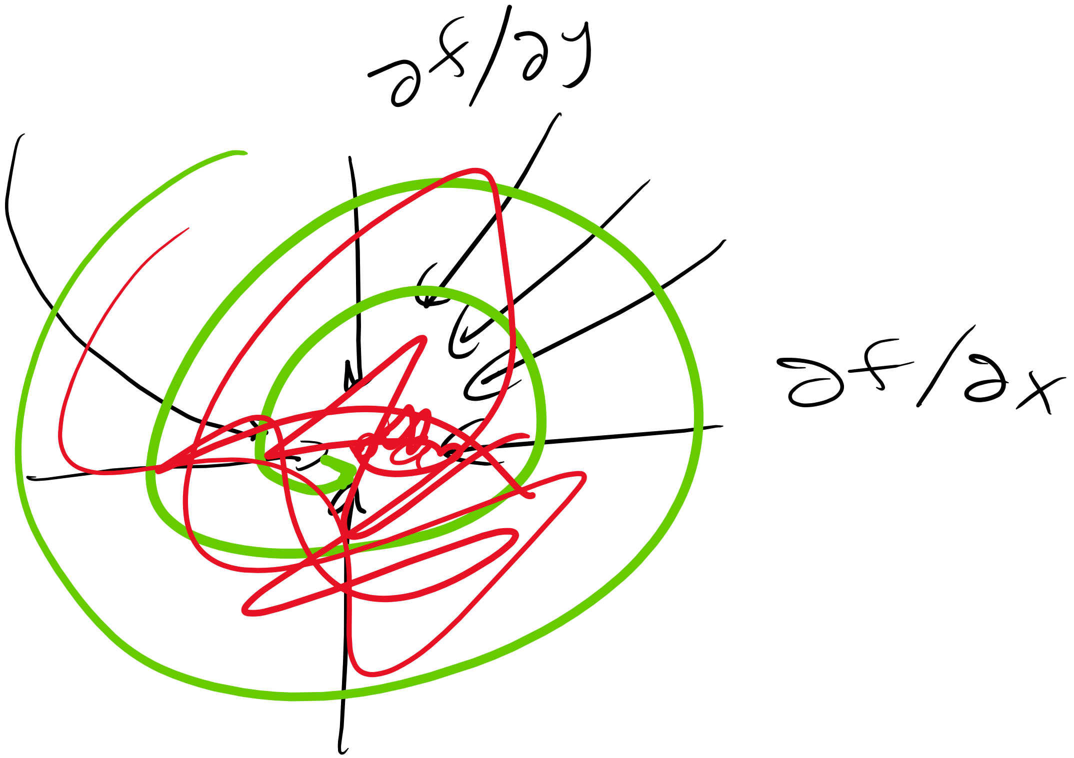
Say limit as $x \to a$ of $f(x)$ is $f(a)$ if

$|f(x) - f(a)| \to 0$ as $x \to a$

(Continuity)




a



Limit to exist along all paths!

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In[•]:= Plot3D[(x^2 y^2)^(1/3), {x, -1, 1}, {y, -1, 1}] ContourPlot[(x^2 y^2)^(1/3), {x, -1, 1}, {y, -1, 1}]
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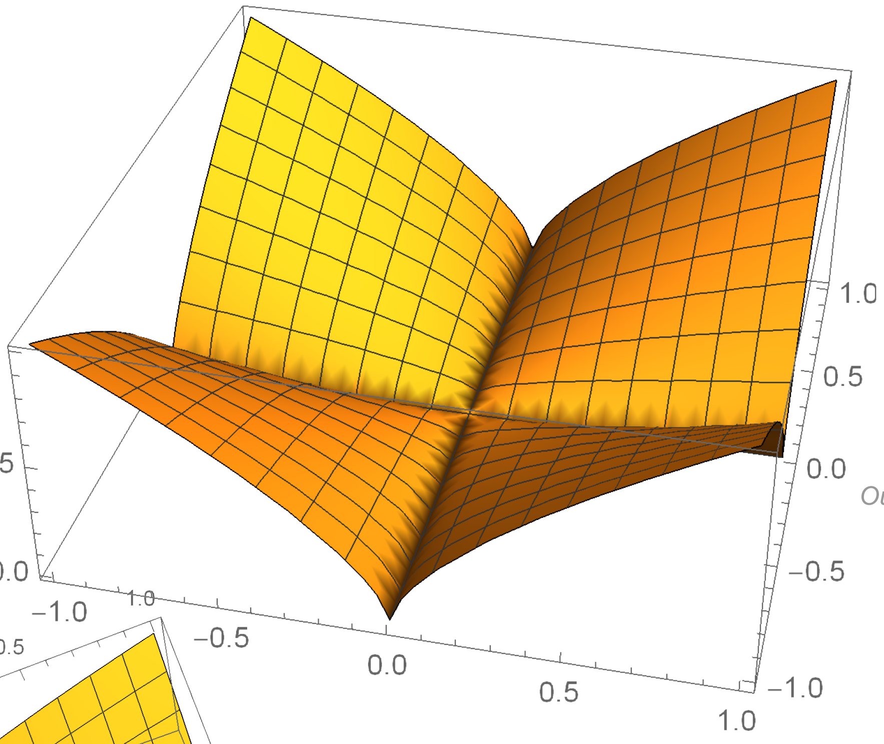
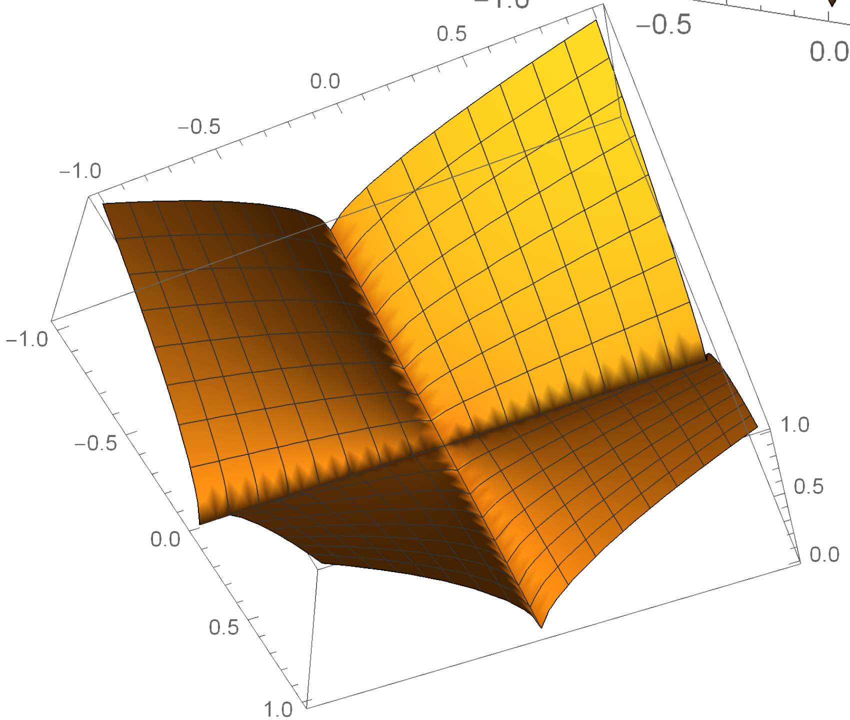
Out[•]=





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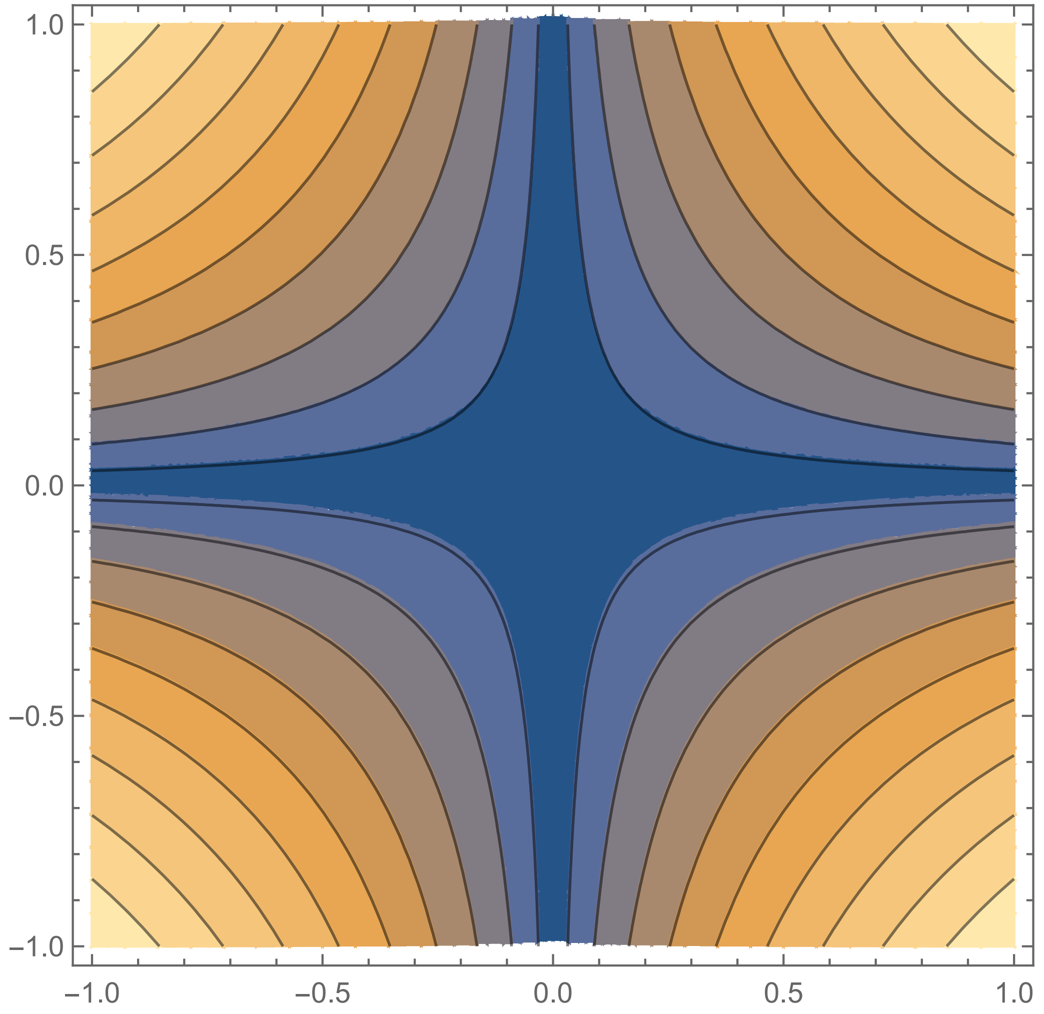


Plot3D[$(x^{2}y$^2)^(1/3), {x, −1,1}, {y, −1,1}]

ContourPlot[$(x^{2}y$^2)^(1/3), {x, −1,1}, {y, −1,1}]

Definition of the deriv at $x=a$

① $$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

② $$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Ex: $f(x)=x^2$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$=\lim_{h\to 0}\frac{2xh+h^2}{h}=\lim_{h\to 0}(2x+h)$$

$$=2x$$

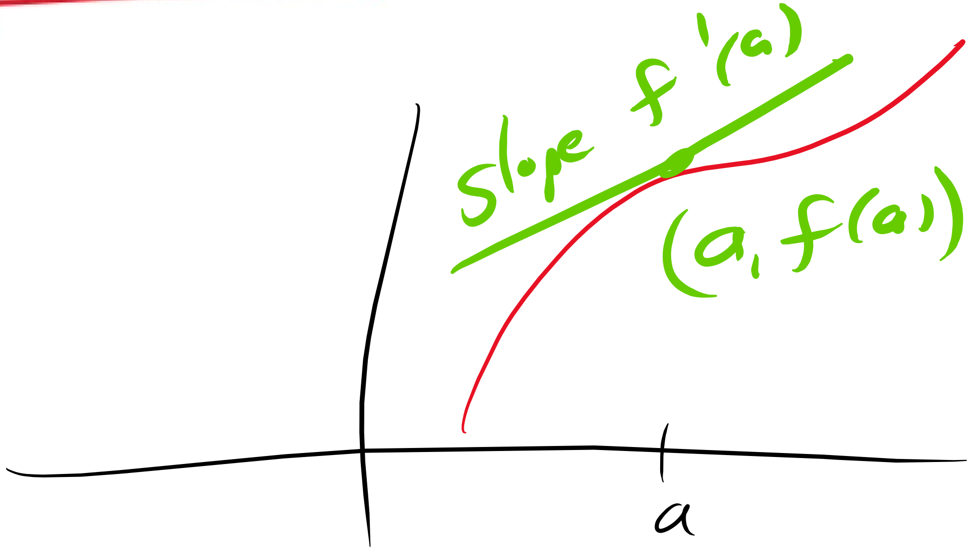
③ $$\lim_{x\to a}\frac{f(x)-f(a)-f'(a)(x-a)}{x-a}=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}-f'(a)\right]$$

$$=0$$

$$\lim_{x\to a} \frac{f(x)-f(a)-f'(a)(x-a)}{x-a}$$

$$\overbrace{f(a)}^{\text{start location}} + \overbrace{f'(a)}^{\text{speed at } a}\overbrace{(x-a)}^{\text{elapsed time}}$$

Tangent line



A fn is diff if the diff b/w f(x) and the tangent line approx at x is SO SMALL it goes to zero when we divide by x − a

$f(x, y)$ at (a, b)

(a,b) •   • (x,y)

Tangent Plane

$$f(a,b) + \underbrace{\frac{\partial f}{\partial x}}_{\text{partial deriv of } f \text{ with respect to } x\text{, keeping } y \text{ fixed}}(a,b) \cdot (x-a) + \frac{\partial f}{\partial y}(a,b) \cdot (y-b)$$

Say f is diff at (a,b) if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - f(a,b) - \frac{\partial f}{\partial x}(a,b)(x-a) - \frac{\partial f}{\partial y}(a,b)(y-b)}{\underbrace{\|(x,y) - (a,b)\|}_{\text{length: } \sqrt{(x-a)^2 + (y-b)^2}}} = 0$$

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} = f_x(a,b)$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h} = f_y(a,b)$$

f_{xy} is this $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ or $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$?

Doesn't matter, same if f is nice!

$$f(x,y,z) = x^3 y^2 + \underbrace{\cos(\sin(y))}_{\text{no } x\text{-dependence}}$$

$$\frac{\partial f}{\partial x} = 3x^2 y^2$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy^2$$

$$\frac{\partial f}{\partial z} = 0$$

$$f(x,y) = (xy)^{2/3}$$

$$y = x^{1/4}$$

$x \to 0^+$
from above, thru
positive values

$$f(0,y) = 0$$

$$f(x,0) = 0$$

$$g(x) = f(x, x^{1/4}) = (x \cdot x^{1/4})^{2/3} = x^{5/6}$$

$$g'(x) = \frac{5}{6} x^{-1/6} = \frac{5}{6 x^{1/6}}$$ as $x \to 0$, blows up!

Try $y = x$: $c(x) = f(x,x) = (x \cdot x)^{2/3} = x^{4/3}$

$$c'(x) = \frac{4}{3} x^{1/3}$$ well defined!

$$f(x,y) = (xy)^{2/3}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0$$

Similarly $\frac{\partial f}{\partial y}(0,0) = 0$

Partial derivs exist but fn is not differentiable!

If the partials exist and are continuous then f is differentiable

BIG THEOREM!

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^4+y^4}$

Try: $y=0$ $\displaystyle\lim_{x\to 0} \frac{x^2}{x^4} = \lim_{x\to 0} \frac{1}{x^2} = \infty$

limit does not exist!

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

① y-axis
$x=0$

$$\lim_{y\to 0} \frac{-y^2}{y^2} = \lim_{y\to 0}(-1) = -1$$

② x-axis
$y=0$

$$\lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$$

Limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^4+17x^2y^2+y^4}{x^2+y^2}$$

x-axis (y=0) $\lim_{x\to 0} \frac{x^4}{x^2} = \lim_{x\to 0} x^2 = 0$

also true for y-axis (Symmetric in x and y)

$y=mx$ $\lim_{x\to 0} \frac{x^4+17m^2x^4+m^4x^4}{x^2+m^2x^2}$

$$= \lim_{x\to 0} \frac{x^4}{x^2} \frac{1+17m^2+m^4}{1+m^2} = 0$$

$y=mx^2 \longrightarrow$ also zero

$$\lim_{(x,y)\to(0,0)} \frac{x^4+17x^2y^2+y^4}{x^2+y^2}$$

Polar Trick for limits

$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$

$y = r\sin\theta \qquad \theta = \arctan(y/x)$

as $(x,y)\to(0,0) \implies r\to 0$, θ FREE

$$\lim_{\substack{r\to 0\\ \theta\ \text{free}}} \frac{r^4\cos^4\theta + 17r^4\cos^2\theta\sin^2\theta + r^4\sin^4\theta}{r^2}$$

$$= \lim_{\substack{r\to 0\\ \theta\ \text{free}}} r^2\left(\cos^4\theta + 17\cos^2\theta\sin^2\theta + \sin^4\theta\right) = 0$$

Revisit: $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

$$= \lim_{\substack{r\to 0 \\ \theta \text{ free}}} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}$$

$$= \lim_{\substack{r\to 0 \\ \theta \text{ free}}} \left[\cos^2\theta - \sin^2\theta\right]$$

limit depends on what θ does!

$\theta = 0 \to 1$

$\theta = \pi/2 \to -1$

$\theta = \pi/4 \to 0$