Math 150: Calculus III: Multivariable Calculus

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https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 10: 2-28-2022:

https://youtu.be/epJbnWCOqCA

slides:
https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture10.pdf

http://www.youtube.com/watch?v=S6wiYRCiQhs **(March 3, 2014: Derivatives)**

Plan for the day: Lecture 10: February 28, 2022:

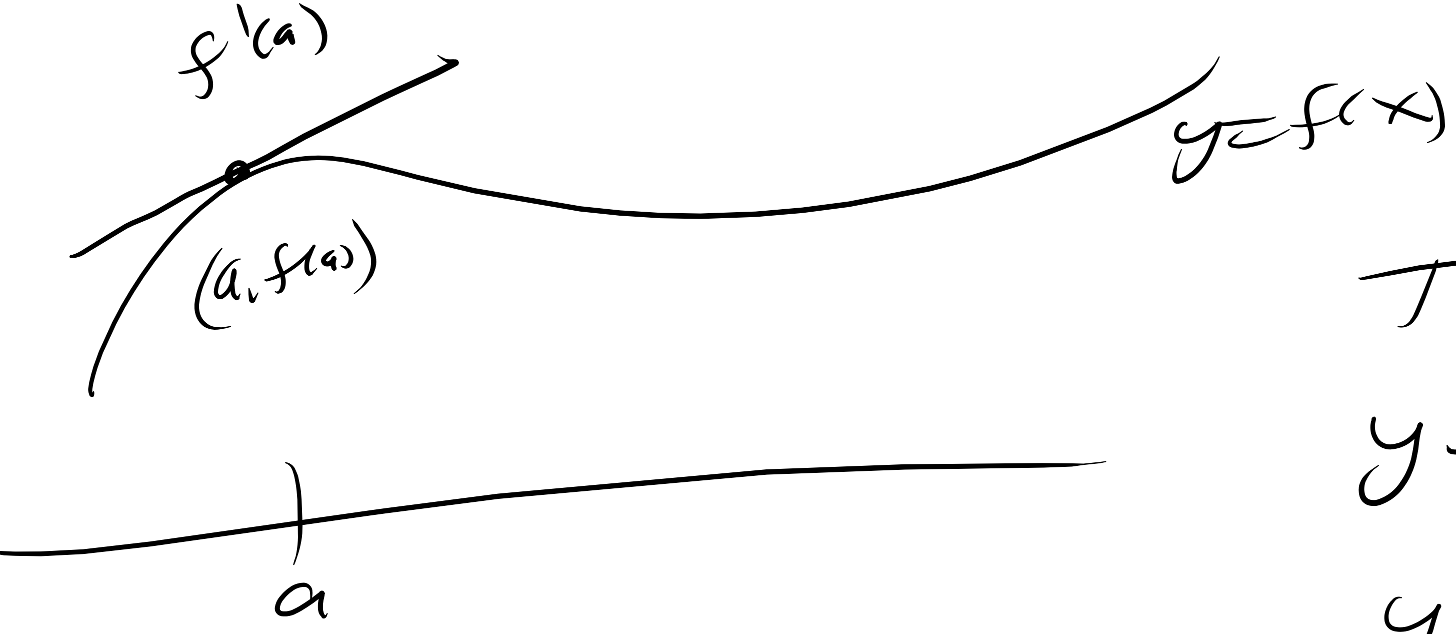
Topics:

Derivatives: Notation, Rules

Examples

Tangent Planes

Applications (heat equation, wave equation: later in the semester)



Tangent line is

$$y - f(a) = f'(a)(x-a)$$

$$y = f(a) + f'(a)(x-a)$$

$$y = f(a) + \frac{df}{dx}(a)(x-a)$$

Ex: $\sqrt{122}$ = slightly more than 11

know at a it should be easy to find $f(a)$

then approx near a by the tangent line

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ $f(121) = 11$ (taking $a = 121$)

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ $f'(a) = f'(121) = \frac{1}{2\sqrt{121}} = \frac{1}{22}$

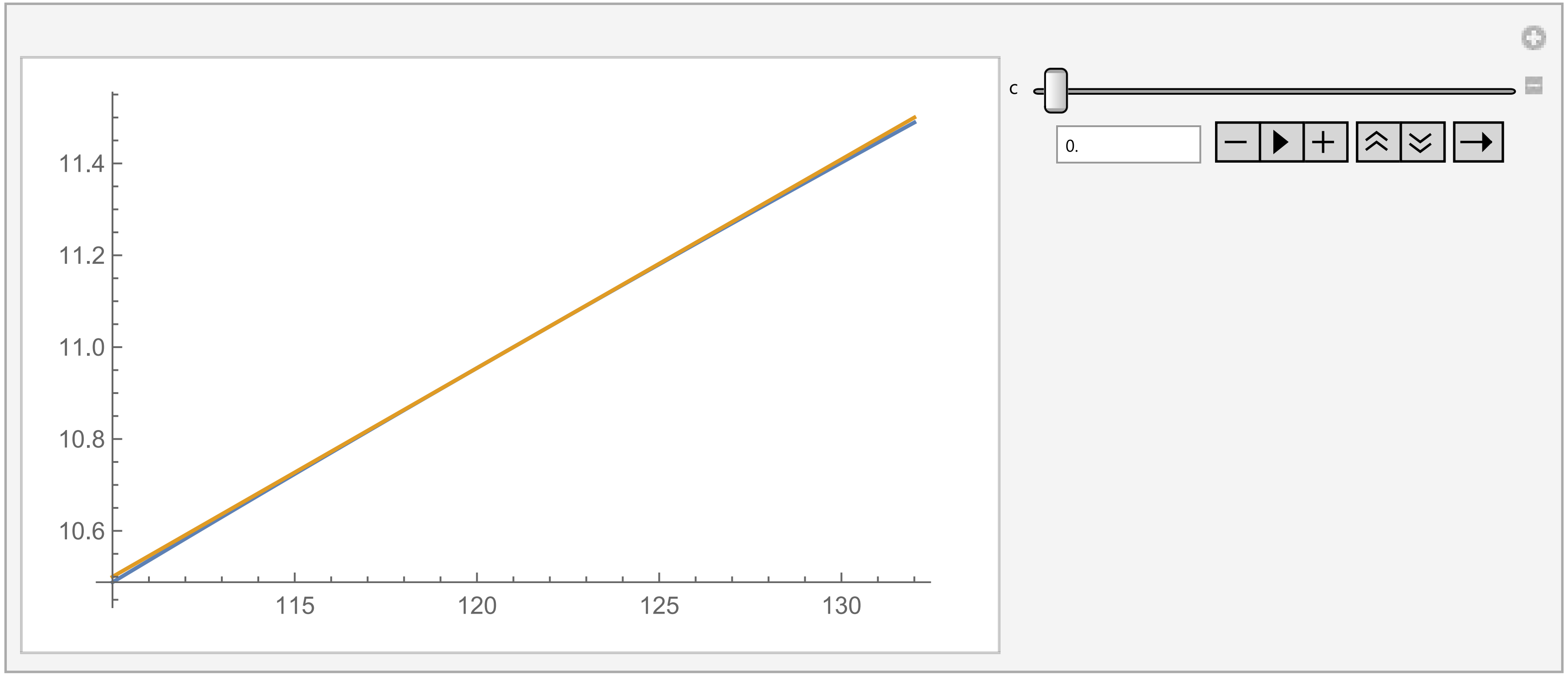
$y = f(a) + f'(a)(x-a)$ is a good approx to $f(x)$

If $x = 122$, $f(122) = \sqrt{122} \approx 11 + \frac{1}{22}(122 - 121) = 11\frac{1}{22}$

In[ ]:=

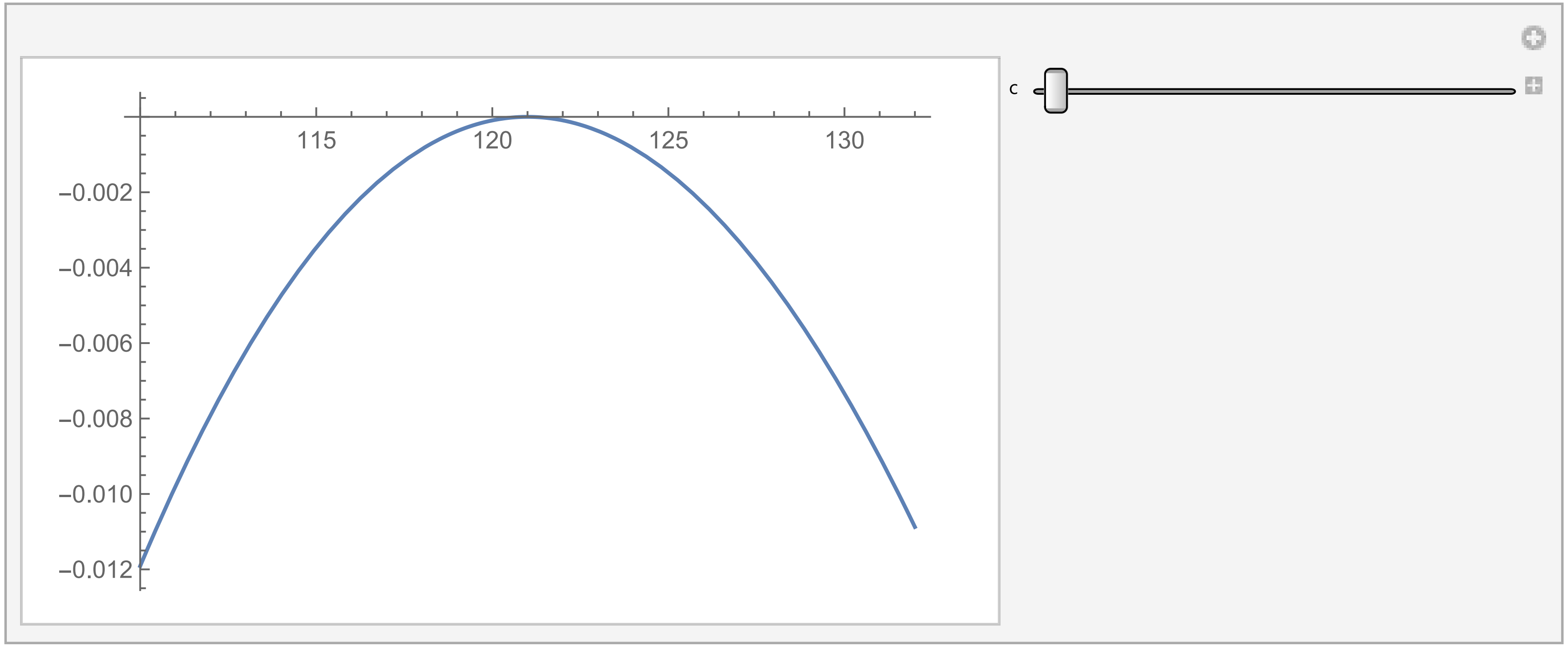
```
f[x_] := Sqrt[x]
Manipulate[Plot[{f[x], 11 + (1/22) (x - 121)}, {x, 110 - c, 132 + c}], {c, 0, 40}]
```

Out[ ]=



```
In[•]:= Manipulate[Plot[{f[x] - (11 + (1/22) (x - 121))}, {x, 110 - c, 132 + c}],
  {c, 0, 40}]
```

Out[•]=



```
N[Sqrt[122],10]
N[11 + 1/22, 10]
11.04536102
11.04545455
```

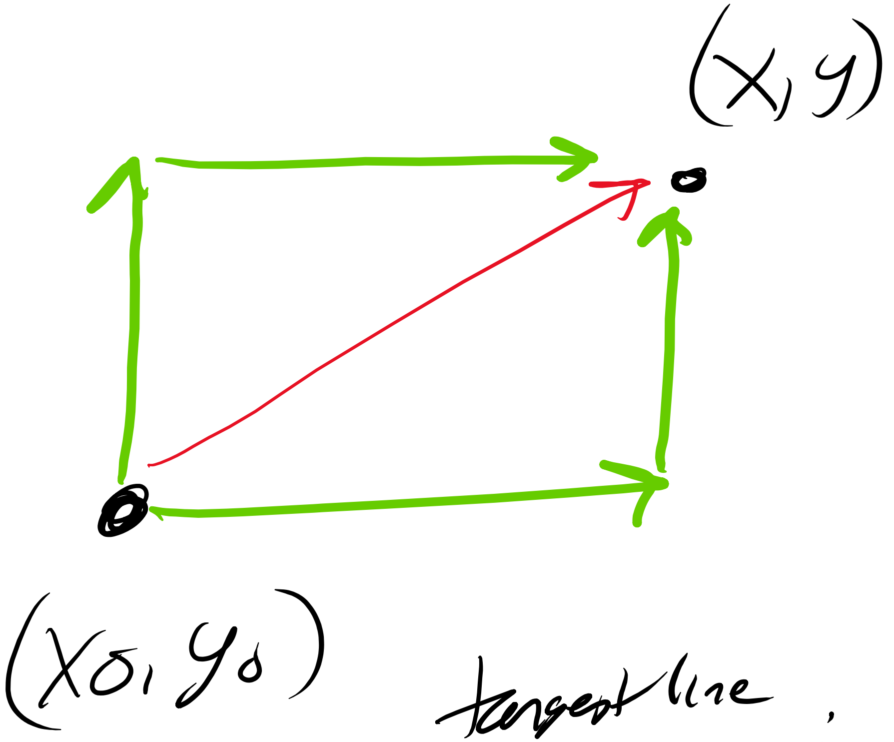
Derivative

$$Df = \left(\frac{\partial f}{\partial x_1}, \ \ldots, \ \frac{\partial f}{\partial x_n}\right)$$ if $f: \mathbb{R}^n \to \mathbb{R}$

Ex: $$Df = \left(\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y}, \ \frac{\partial f}{\partial z}\right) = \nabla f = \text{grad } f$$

gradient

Note: a **vector**



Change:

$$\Delta x = x - x_0$$

$$\Delta y = y - y_0$$

tangent line in x-dir: $f(x, y_0) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0)$

tangent line in y-dir: $f(x_0, y) = f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0)$

tangent plane: $f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$

$$f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$z = f(x,y), \qquad z_0 = f(x_0,y_0)$$

$$z = z_0 + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$Df = \nabla f = \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0)\right) = (\nabla f)(x_0,y_0)$$

$$z = \underbrace{z_0}_{\text{Start}} + \underbrace{(\nabla f)(x_0,y_0)}_{\text{Gradient}} \cdot \underbrace{(x-x_0,\ y-y_0)}_{\text{Change in location}}$$

$$z = z_0 + a(x-x_0) + b(y-y_0) \Longrightarrow ax+by+z=\delta$$

Rules for Derivatives

$$[f(x)+g(x)]' = f'(x)+g'(x)$$

$$D\left[f(\vec{x})+g(\vec{x})\right] = (Df)(\vec{x})+(Dg)(\vec{x}) \qquad \vec{x}=(x_1,\dots,x_n)$$

Similarly $D\left[f(\vec{x})-g(\vec{x})\right] = (Df)(\vec{x})-(Dg)(\vec{x})$

and $D\left[c\,f(\vec{x})\right] = c(Df)(\vec{x})$

Product: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$$D\left[f(\vec{x})g(\vec{x})\right] = \underbrace{(Df)(\vec{x})}_{\text{vectors}}\underbrace{g(\vec{x})}_{\text{number}} + \underbrace{f(x)}_{\text{number}}\underbrace{Dg(\vec{x})}_{\text{vectors}}$$

$f, g : \mathbb{R}^n \to \mathbb{R}$ scalar

$Df, Dg : \mathbb{R}^n \to \mathbb{R}^n$ vector

do component by component:

$$\frac{\partial}{\partial x_k}\left(f(\vec{x})\,g(\vec{x})\right) = \frac{\partial f}{\partial x_k}\,g(\vec{x}) + f(\vec{x})\,\frac{\partial g}{\partial x_k}$$

Quotient Rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$$D\left[\frac{f(\vec{x})}{g(\vec{x})}\right] = \frac{(Df)(\vec{x})\, g(\vec{x}) - f(\vec{x})(Dg)(\vec{x})}{g(\vec{x})^2}$$

number

Proof: Same as product, do $\frac{\partial}{\partial x_k}\left(f(\vec{x})\, g(\vec{x})\right)$

$f(\vec{x}) = x^2 y^3 z^4$ and $g(\vec{x}) = xyz$

$$\boxed{\text{Thoreau! Simplify}^2}$$

$$D\left[\frac{f(\vec{x})}{g(\vec{x})}\right] = D\left[x y^2 z^3\right]$$

$$= \left(y^2 z^3,\ 2xyz^3,\ 3xy^2z^2\right)$$

Show same by algebra

$$D\left[f(\vec{x})/g(\vec{x})\right] = \frac{(Df)(\vec{x})\,g(\vec{x}) - f(\vec{x})\,(Dg)(\vec{x})}{g(\vec{x})^2}$$

$$= \frac{\left(2xy^3z^4,\ 3x^2y^2z^4,\ 4x^2y^3z^3\right)xyz - x^2y^3z^4\left(yz,\ xz,\ xy\right)}{x^2y^2z^2}$$

$$= \left(\frac{2x^2y^4z^5 - x^2y^4z^5}{x^2y^2z^2},\ \ldots,\ \ldots\right) = \left(y^2z^3,\ \ldots,\ \ldots\right)$$