Math 150: Calculus III: Multivariable Calculus

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https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 11: 3-2-2022: https://youtu.be/J24iLtvJCFE

slides:

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture11.pdf

Plan for the day: Lecture 11: March 2, 2022:

Topics:

Optimization: Review of 1-dim

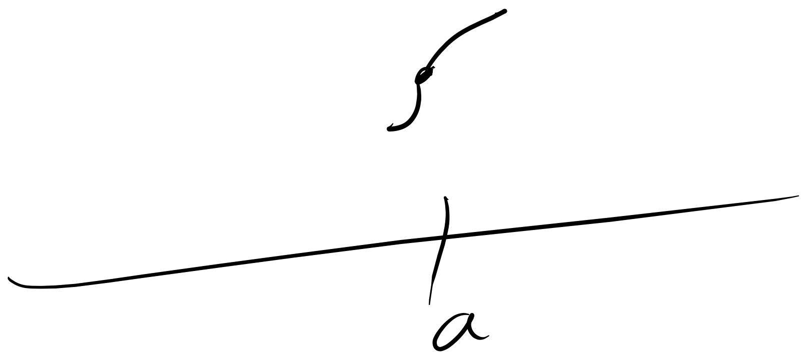
Optimization: Candidates in 2-dim

Problems: Drowning Swimmer, summands maximizing product

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Say $f'(a) > 0$: if $x > a$ and $x \searrow a$

since denom > 0, num > 0
when x is close to but greater than a.
Thus $f(x) > f(a)$ for $x > a$ by a little



If $x < a$ and $x \nearrow a$

now denom < 0 so num < 0
when x is close to but less than a.
Thus $f(x) < f(a)$ for $x < a$ by a little

Cannot have a max or min at $x = a$ if $f'(a) > 0$
Similarly no max or min if $f'(a) < 0$.

If f is differentiable and has a max or min at $x=a$,

Then $f'(a)=0$.

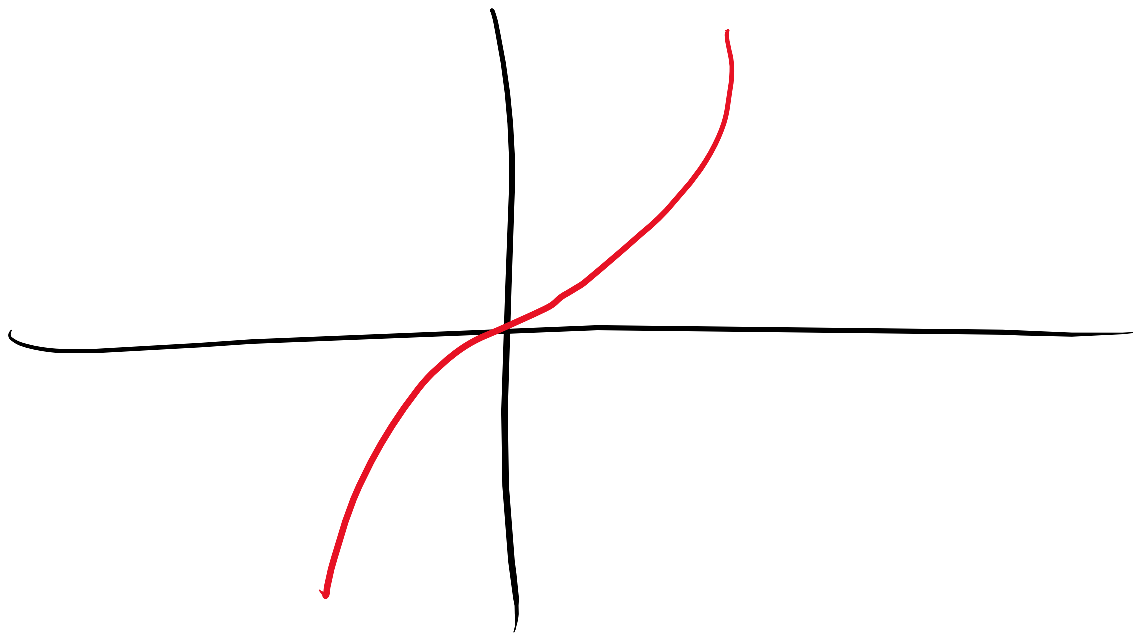
If P Then Q

$\neq$

If Q Then P

Neccessary vs sufficient conditions

$f(x) = x^3$



$f(x) = x^3$

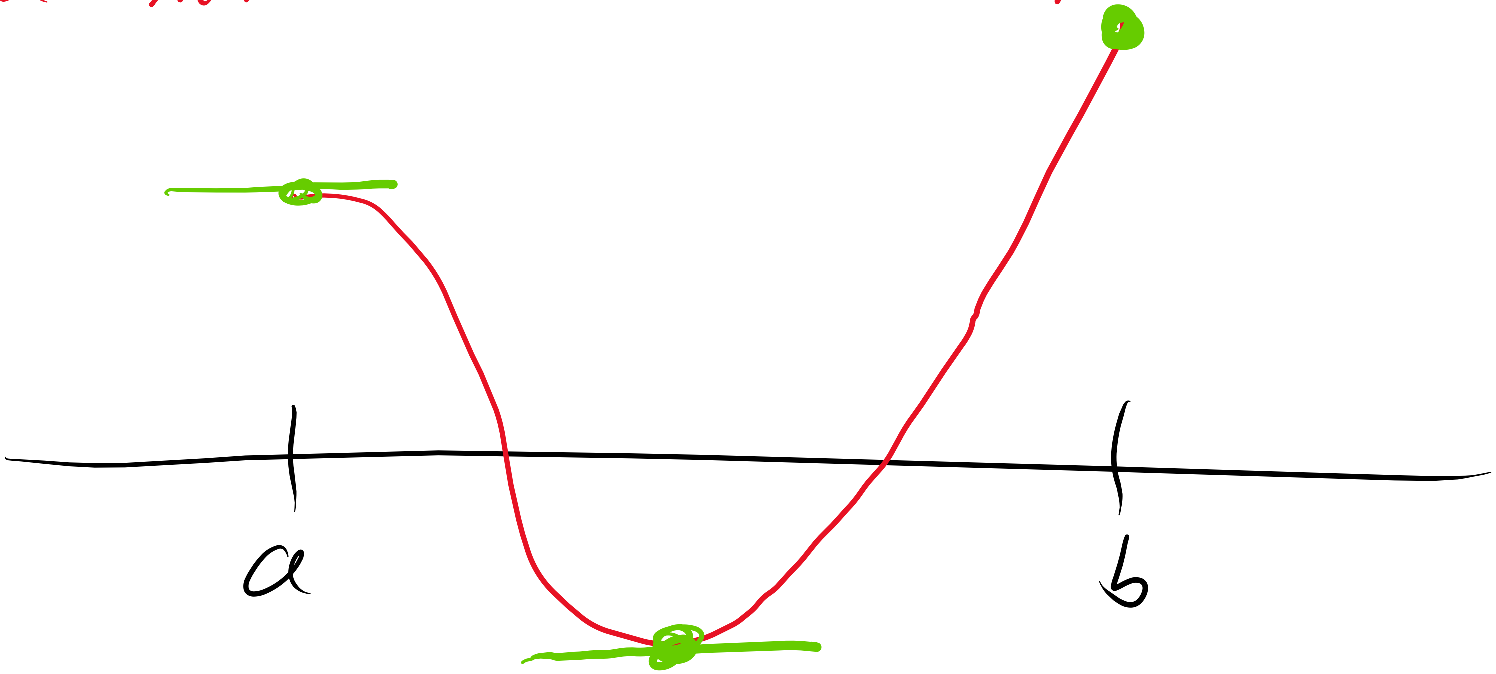
$f'(x) = 3x^2$

↳ Critical Point at $x=0$, as $f'(0)=0$

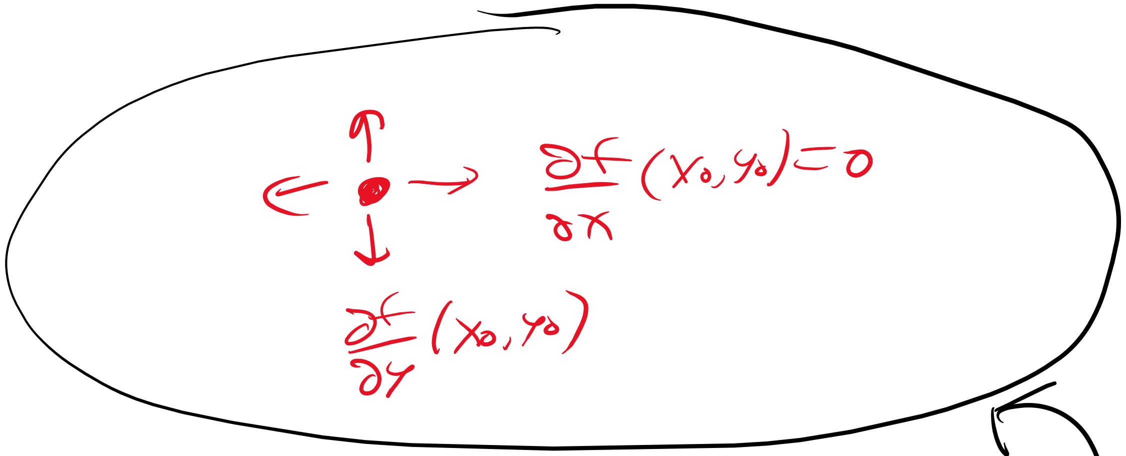
Not a max or min!

Due to $f''(x) = 6x$ is also zero at $x=0$

To find max / min: find critical points, check endpoints



at most 2 endpoints



$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\le 1$$

$f(x,y)$

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

Interior max/min are where $\nabla f=\vec{0}$

↳ Critical points, candidates for max/min

Boundary HARDER! ∞ many points to check.

Given a number S divide into pieces a_k such that all integers > 0 and prod is as large as possible.

$$\max a_1 \cdot a_2 \cdots a_n \quad \text{where } a_1 + a_2 + \cdots + a_n = S$$

and n is free

Consider $-10^{100} - 10^{100} + 2 \cdot 10^{100} + S = S$

Also if some $a_k = 0$, product $= 0$

Given a number S divide into pieces a_k such that all integers > 0 and prod is as large as possible.

$a_k = 1$ doesn't help product, so $2 \le a_k$

Some $a_k = 10$ replace it with $2+8$ Gives 16

$= 9$ " " " $2+7$ Give 14

$= 8$ " " " $2+6$ Gives 12

$= 7$ " " " $2+5$ Gives 10

$= 6$ " " " $2+4$ Gives 8

$= 5$ " " " $2+3$ Gives 6

$= 4$ " " " $2+2$ Gives 4

indifferent!

$a_k \le 4$

Given a number S divide into pieces a_k such that all integers > 0 and prod is as large as possible.

Each a_k satisfies $2 \le a_k \le 4$ better $2 \le a_k \le 3$

Only use 2's and 3's

$$\underbrace{2+2+2}_{\substack{\text{product} \\ 8}} = 6 = \underbrace{3+3}_{\substack{\text{product} \\ 9}}$$

Two 3's beats three 2's

Given a number S divide into pieces a_k such that all ~~[illegible]~~ > 0 and prod is as large as possible.

Know $1 \le a_k \le 4$

Study n pieces: do $n = 1, 2, 3, \ldots$

Given a number S divide into pieces a_k such that all ~~integers~~ > 0 and prod is as large as possible.

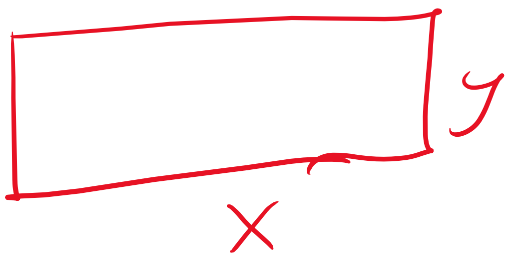
Case $n=1$

$a_1 = S$, done

Given a number S divide into pieces a_k such that all ~~integers~~ > 0 and prod is as large as possible.

$\underline{n=2}$

$S = x+y = a_1+a_2$ max $xy = a_1 \cdot a_2$



Semi-perimeter is $S = x+y$ fixed

maximize the product

$A(x) = x(S-x) = Sx - x^2$

$A'(x) = S - 2x$, $A'(x) = 0 \to x = S/2$

$A(0) = A(\frac{S}{2}) = 0$ so max $x = y = \frac{S}{2}$

Farmer Brown problem

Given a number S divide into pieces a_k such that all ~~integers~~ > 0 and prod is as large as possible.

$n=3$: $S = x+y+z$

Assume all not equal, say $x<y$

look at $\underbrace{\frac{x+y}{2} + \frac{x+y}{2}}_{\text{Farmer Brown or } x+y} + z$ $\therefore$ same sum, larger product

Using real analysis result that a max and a min exist as have a closed set $(1 \le x,y,z \le 4$ vs $1<x,y,z<4)$

Given a number S divide into pieces a_i such that all ~~integers~~ > 0 and prod is as large as possible.

n pieces, max is $a_1 = a_2 = \cdots = a_n = S/n$

largest product with n pieces is $\left(\frac{S}{n}\right)^n$ $n \in \{1, 2, 3, \ldots\}$

$\max \left(S/n\right)^n$
n pos int

Need f that reduces to this when input is an integer.

Try $f(x) = \left(S/x\right)^x$

put x for n

$$\max_x f(x) = \max_x (S/x)^x \qquad 1 \le x \le S$$

$f(1) = S \qquad f(S) = 1 \qquad$ endpoints

Critical Points of $\quad f(x) = (S/x)^x = e^{x \log(S/x)}$

$$\left(\frac{S}{x}\right)^x = e^{g(x)} \qquad \text{take logs} \qquad (\log = \ln)$$

$$\log\left[\left(\frac{S}{x}\right)^x\right] = \log\left[e^{g(x)}\right]$$

$$x \log\left(\frac{S}{x}\right) = g(x) \underbrace{\log e}_{1}$$

$$f(x) = e^{x \log(S/x)}$$

$$f'(x) = \underbrace{e^{x\log(S/x)}}\left[x \log\left(\frac{S}{x}\right)\right]'$$

$\log\left(\frac{S}{x}\right)$

$= \log S - \log x$

deriv is $0 - \frac{1}{x}$

$$= f(x)\left[1 \cdot \log\left(\frac{S}{x}\right) + x\left(-\frac{1}{x}\right)\right]$$

$$= f(x)\left[\log\frac{S}{x} - 1\right] \qquad 1 = \log e$$

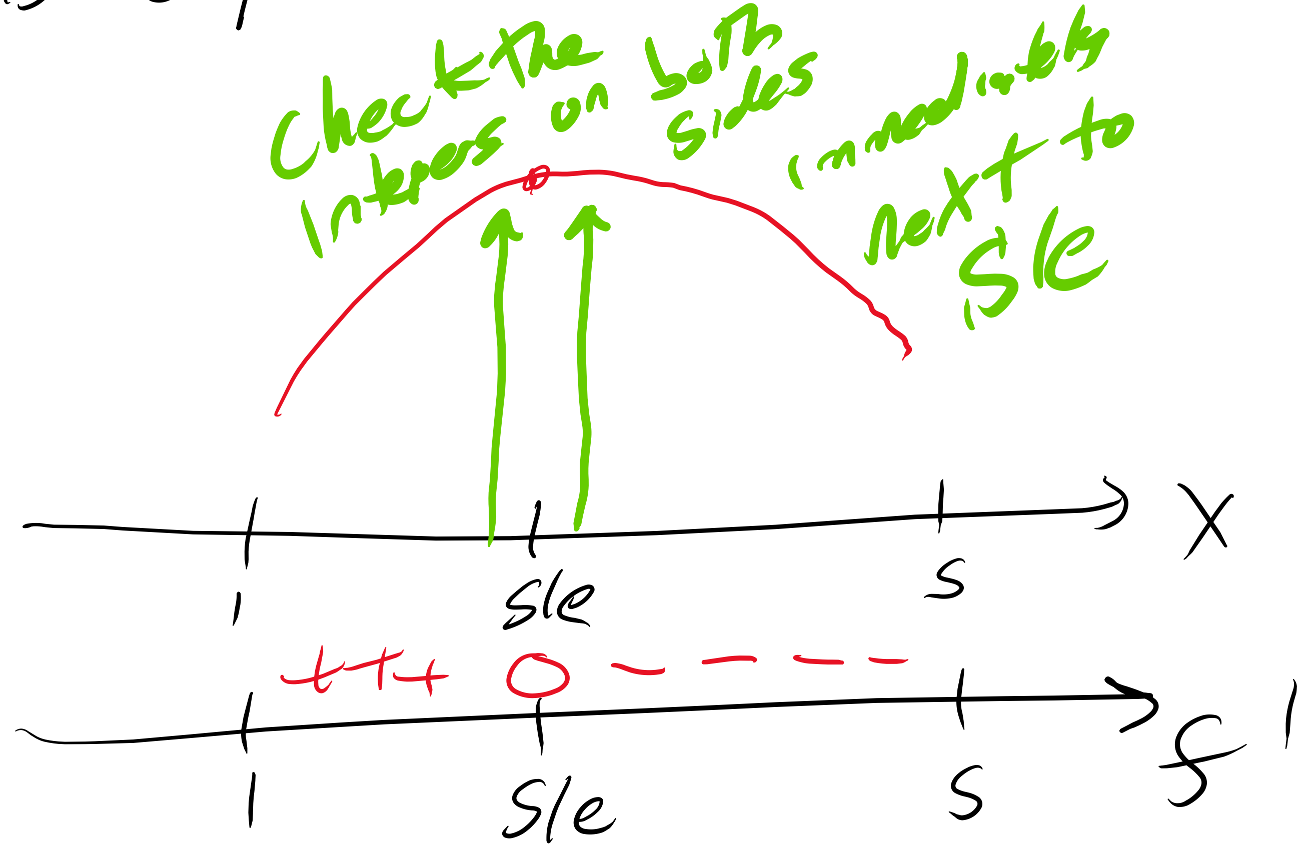
$$= f(x)\left[\log\frac{S}{x} - \log e\right]$$

$$= \underbrace{f(x)}_{\text{never zero}}\ \underbrace{\log\left(\frac{S}{xe}\right)}_{\text{need this to be zero}} \Rightarrow \frac{S}{xe} = 1 \text{ or } x = S/e$$

each piece is $\frac{S}{S/e} = e$

$f(x) = (S/x)^x$ real max is at $x = S/e$ gives $e^{S/e}$

Kills endpoints, This is the max



$$f(x) = e^{x \log S/x}$$

$$f'(x) = f(x) \cdot \log \frac{S}{xe}$$

if $x > e$, $f'(x) < 0$

if $x < e$, $f'(x) > 0$