# Math 150: Calculus III: Multivariable Calculus

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https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 15: 3-11-2022:
https://youtu.be/HFQpLfyha5s
slides:

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture15.pdf

## Plan for the day: Lecture 15: March 11, 2022:

**Topics:**

**Chain Rule**

# One dimension:

$$A(x) = f(g(x))$$

$$
\begin{tikzcd}
x \arrow[r, "g"] \arrow[rr, bend right, "A"'] & g(x) \arrow[r, "f"] & f(g(x))
\end{tikzcd}
$$

Then $A'(x) = f'(g(x)) \cdot \underline{g'(x)}$

Build intuition: $f(x) = x^n$ $\quad g(x) = x^m$ $\quad \overbrace{f(g(x))}^{A(x)} = x^{nm}$

$f'(x) = nx^{n-1}$ $\quad g'(x) = mx^{m-1}$ $\quad A'(x) = nmx^{nm-1}$

$f'(g(x)) = n(x^m)^{n-1} = nx^{nm-m}$

vs $A'(x) = nmx^{nm-1}$

need a factor of $\underbrace{mx^{m-1}}_{g'(x)}$

Suggests $A'(x) = f'(g(x))\, g'(x)$

## Danger

$$A(x) = f(g(x))$$

$$A'(x) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

Imagine that $g(x+h) = g(x)+h$

we get $\lim_{h \to 0} \frac{f(g(x)+h) - f(g(x))}{h}$

$$= f'(g(x))$$

FALSE!

$\underline{\text{MVT}}$: $\dfrac{\phi(b)-\phi(a)}{b-a} = \phi'(c)$ for some $c \in [a,b]$

ave speed = instantaneous speed at some special point

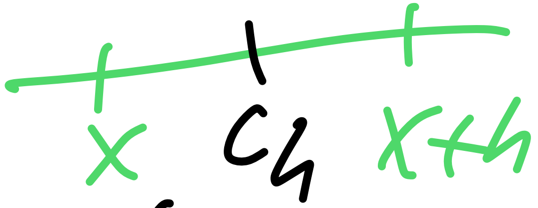
$$\frac{g(x+h)-g(x)}{x+h-x} \underset{\text{MVT}}{=} g'(c_h)$$

$a = x$

$b = x+h$

$\phi = g$

$$\hookrightarrow g(x+h) = g(x) + \underbrace{g'(c_h)}_{\text{very close to } g'(x)} \cdot h$$




as $h \to 0$, $c_h \to x$

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x)+g'(c_h)\,h) - f(g(x))}{\underbrace{g'(c_h)\,h}}$$

$g'(c_h)$ looks like $g'(x)$

as $h \to 0$, looks like $f'$ at $g(x)$

What if $g'(c_h) = 0$?

Case 1: only happens finitely often near $x$, ok if $h$ small

Case 2: infinitely often $g'(c_h) = 0$ as $h \to 0$

$$g(x) = \begin{cases} x^3 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$g'(x) = \begin{cases} 3x^2 \sin(1/x) - x\cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Danger!

# Several Variables

$$(x,y) \xrightarrow{g} (u,v,w) \xrightarrow{f} \#$$

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$g : \mathbb{R}^2 \to \mathbb{R}^3$$

$$(f \circ g) : \mathbb{R}^2 \to \mathbb{R}$$

$$f(u, v, w) = f(u(x,y), v(x,y), w(x,y))$$

↳ call this $A(x,y)$

Compute $\dfrac{\partial A}{\partial x}$ and $\dfrac{\partial A}{\partial y}$

define $g(x,y) = (u(x,y), v(x,y), w(x,y))$

Now $A(x,y) = f(g(x,y))$

Goal: input how $u, v, w$ change wrt $x, y$
input how $f$ changes wrt $u, v, w$
output how $A$ changes wrt $x, y$ in terms of above

$A(x,y) = f(u(x,y), v(x,y), w(x,y))$

$\hookrightarrow$ $\partial A/\partial x$ and $\partial A/\partial y$ should involve

$\partial f/\partial u$, $\partial f/\partial v$, $\frac{\partial f}{\partial w}$ at point $(u, v, w)$

and $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$

at the point $(x,y)$

**Thm:** Notation as before:

$$\frac{\partial A}{\partial x} = \frac{\partial f}{\partial u}(g(x,y))\frac{\partial u}{\partial x}(x,y) + \frac{\partial f}{\partial v}(g(x,y))\frac{\partial v}{\partial x}$$

$$+ \frac{\partial f}{\partial w}(g(x,y))\frac{\partial w}{\partial x}$$

Similarly for $\frac{\partial A}{\partial y}$

$$\frac{\partial A}{\partial x} = \lim_{h \to 0} \frac{f(u(x+h,y), v(x+h,y), w(x+h,y)) - \star}{h}$$

$(u(x+h,y), v(x+h,y), w(x+h,y))$

$$\star = f(u(x,y), v(x,y), w(x,y))$$

$(u(x+h,y), v(x+h,y), w(x+h,y))$

$(u(x+h,y), v(x+h,y), w(x+h,y))$

$(u(x+h,y), v(x,y), w(x,y))$

$u(x,y), v(x,y), w(x,y)$

Numerator is $\partial f/\partial w * \partial w/\partial x$

$f(u(x+h,y), v(x+h,y), w(x+h,y)) - f(u(x+h,y), v(x+h,y), w(x,y))$

$+ f(u(x+h,y), v(x+h,y), w(x,y)) - f(u(x+h,y), v(x,y), w(x,y))$

$+ f(u(x+h,y), v(x,y), w(x,y)) - f(u(x,y), v(x,y), w(x,y))$

$\frac{\partial f}{\partial v}(g(x,y)) \cdot \frac{\partial v}{\partial x}$

$\frac{\partial f}{\partial u}(g(x,y)) \cdot \frac{\partial u}{\partial x}$

$x = r\cos\theta \quad y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$

$\frac{\partial y}{\partial r} = \sin\theta$

$f(x,y) = x^3 + xy$

$A(r,\theta) = f(x(r,\theta), y(r,\theta))$

$\frac{\partial f}{\partial x} = 3x^2 + y$

$\frac{\partial f}{\partial y} = x$

$$\frac{\partial A}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$= \left[3(r\cos\theta)^2 + r\sin\theta\right]\cos\theta + \left[r\cos\theta\right]\sin\theta$$

$$= 3r^2\cos^3\theta + 2r\cos\theta\sin\theta$$

$A(r,\theta) = r^3\cos^3\theta + r^2\cos\theta\sin\theta$

$\partial A/\partial r = 3r^2\cos^3\theta + 2r\cos\theta\sin\theta$

Checks!