Math 150: Calculus III: Multivariable Calculus

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https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 17: 3-16-2022: https://youtu.be/grtkHEIdlkU

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture17.pdf

Plan for the day: Lecture 17: March 16, 2022:

Topics:

Directional Derivatives

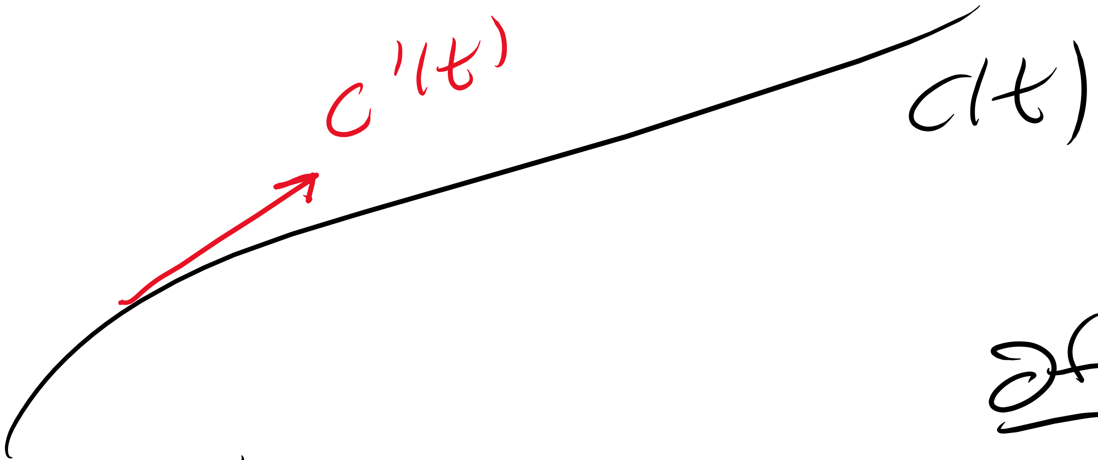
Exponential Function

Trig in a Day

(as time permits: Lagrange Multipliers introduction)

Directional Derivatives

↳ Partial derivs are examples



$$A(t) = f(c(t))$$

$$A'(t) = (\nabla f)(c(t)) \cdot c'(t)$$

$$\frac{\partial f}{\partial x_1}(c(t))\,\frac{\partial x_1}{\partial t}(t) + \frac{\partial f}{\partial x_2}(c(t))\,\frac{\partial x_2}{\partial t}(t)$$

$$+ \cdots$$

More generally:

$$(D_{\vec{v}} f)(\vec{P}) = (\nabla f)(\vec{P}) \cdot \vec{v}$$

often $\|\vec{v}\| = 1$

$$c(t) = (x_1(t), \ldots, x_n(t))$$

so $c'(t) = (x_1'(t), \ldots, x_n'(t))$

Take $\vec{v} = \vec{e}_k$ then $\left(D_{\vec{e}_k} f\right)(\vec{p}) = (\nabla f)(\vec{p}) \cdot \vec{e}_k$

$$= \frac{\partial f}{\partial x_k}(\vec{p})$$

as $e_k = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$

k^{th} spot

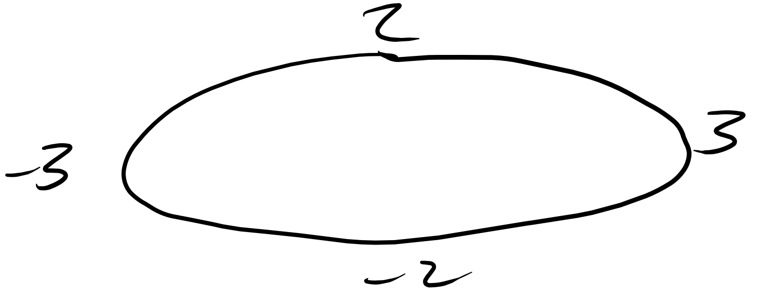
Who cares?

$$(D_{\vec{v}} f)(\vec{p}) = (\nabla f)(\vec{p}) \cdot \vec{v}$$

Candidates for max/min

↳ derivative is zero

↳ endpoints



$$c(t) = (3\cos t,\ 2\sin t) = (x(t), y(t))$$

$$\left(\frac{x(t)}{3}\right)^2 + \left(\frac{y(t)}{2}\right)^2 = 1$$

Exponential Function

Imagine interest rate of x per year, compound n times a year, how much is \$1 worth in a year?

$n=1$: $1 \longrightarrow 1+x$

$n=2$: $1 \longrightarrow \left(1+\frac{x}{2}\right) \longrightarrow \left(1+\frac{x}{2}\right) + \left(1+\frac{x}{2}\right)\frac{x}{2} = \left(1+\frac{x}{2}\right)\left(1+\frac{x}{2}\right)$

$= \left(1+\frac{x}{2}\right)^2$

n: $1 \longrightarrow \left(1+\frac{x}{n}\right)^n$

as $n\to\infty$ compounded continuously and $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^x e^y = e^{x+y}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{y^m}{m!} = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}$$

PROOF

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{y^m}{m!} = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}$$

$$\sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{x^l}{l!} \frac{y^{k-l}}{(k-l)!} \frac{k!}{k!} \qquad \frac{k!}{l!(k-l)!} = \binom{k}{l}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left[\sum_{l=0}^{k} \binom{k}{l} x^l y^{k-l} \right] \leftarrow \text{Binomial Thm}$$

$$= (x+y)^k$$

$$= \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!} = e^{x+y} \;!$$

$$e^x = 1 + x + x^2/2! + \cdots = \sum_{n=0}^{\infty} x^n/n!$$

$$(e^x)' = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n \cdot (n-1)!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$m = n-1$

$n: 1 \longrightarrow \infty$

$m: 0 \longrightarrow \infty$

$$(e^x)' = \sum_{m=0}^{\infty} \frac{x^m}{m!} = e^x$$

$$\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \left(\frac{x}{n}\right)^{n-k}$$

$$= \binom{n}{0} + \binom{n}{1}\frac{x}{n} + \binom{n}{2}\left(\frac{x}{n}\right)^2 + \binom{n}{3}\left(\frac{x}{n}\right)^3 + \cdots + \binom{n}{n}\left(\frac{x}{n}\right)^n$$

$$= 1 + n\cdot\frac{x}{n} + \frac{n(n-1)}{2!}\frac{x^2}{n\cdot n} + \frac{n(n-1)(n-2)}{3!}\frac{x^3}{n\cdot n\cdot n} + \cdots$$

as $n \to \infty$

$$\Rightarrow 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

So equivalent definitions

$(x^3)' = 3x^2 \qquad (x^{3/2})' = \frac{3}{2}x^{\frac{1}{2}} \qquad (x^{\sqrt{2}})' = \sqrt{2}\,x^{\sqrt{2}-1}$

using $(x^r)' = r\,x^{r-1}$

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x^3: $\lim\limits_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim\limits_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \lim\limits_{h\to 0} \frac{(3x^2 + 3xh + h^2)h}{h}$

$= \lim\limits_{h\to 0} (3x^2 + 3xh + h^2)$

$= 3x^2$

Binomial Thm
for pos int n

$x^{3/2}$: $\lim\limits_{h\to 0} \dfrac{(x+h)^{3/2} - x^{3/2}}{h} \cdot \dfrac{(x+h)^{3/2} + x^{3/2}}{(x+h)^{3/2} + x^{3/2}}$

$$= \lim_{h\to 0} \underbrace{\frac{(x+h)^3 - x^3}{h}}_{\downarrow\, h\to 0} \cdot \underbrace{\frac{1}{(x+h)^{3/2} + x^{3/2}}}_{\downarrow\, h\to 0}$$

$$3x^2 \cdot \frac{1}{2x^{3/2}} = \frac{3}{2} x^{1/2}$$

$f(x) = x^{3/2}$

Set $g(x) = f(x)^2 = x^3$

$g'(x) = 2f(x)f'(x) = 3x^2$

So $f'(x) = \frac{3}{2}\frac{x^2}{f(x)}$

$f'(x) = \frac{3}{2}x^{1/2}$

$X^{\sqrt{2}}$: $\quad X^{\sqrt{2}} = e^{g(x)}$

$$\log(x^{\sqrt{2}}) = \log(e^{g(x)})$$

$$\sqrt{2}\log X = g(x)\log(e) = g(x) \Rightarrow g(x) = \sqrt{2}\log X$$

$$X^{\sqrt{2}} = e^{\sqrt{2}\log X} = \sum_{n=0}^{\infty} \frac{(\sqrt{2}\log x)^n}{n!}$$

$$(X^{\sqrt{2}})' = e^{\sqrt{2}\log X} \cdot (\sqrt{2}\log X)'$$

Chain rule: $(A(B(x)))' = A'(B(x))\,B'(x)$

$$= X^{\sqrt{2}}\ \sqrt{2}\cdot\frac{1}{X} = \sqrt{2}\,X^{\sqrt{2}-1}$$

https://en.wikipedia.org/wiki/Baker%E2%80%93Campbell%E2%80%93Hausdorff_formula

Baker–Campbell–Hausdorff formula

From Wikipedia, the free encyclopedia

In mathematics, the **Baker–Campbell–Hausdorff formula** is the solution for Z to the equation

$$e^X e^Y = e^Z$$

for possibly noncommutative X and Y in the Lie algebra of a Lie group. There are various ways of writing the formula, but all ultimately yield an expression for Z in Lie algebraic terms, that is, as a formal series (not necessarily convergent) in X and Y and iterated commutators thereof. The first few terms of this series are:

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots,$$

where "$\cdots$" indicates terms involving higher commutators of X and Y. If X and Y are sufficiently small elements of the Lie algebra $\mathfrak{g}$ of a Lie group G, the series is convergent. Meanwhile, every element g sufficiently close to the identity in G can be expressed as $g = e^X$ for a small X in $\mathfrak{g}$. Thus, we can say that *near the identity* the group multiplication in G—written as $e^X e^Y = e^Z$—can be expressed in purely Lie algebraic terms. The Baker–Campbell–Hausdorff formula can be used to give comparatively simple proofs of deep results in the Lie group–Lie algebra correspondence.

If X and Y are sufficiently small $n \times n$ matrices, then Z can be computed as the logarithm of $e^X e^Y$, where the exponentials and the logarithm can be computed as power series. The point of the Baker–Campbell–Hausdorff formula is then the highly nonobvious claim that $Z := \log\left(e^X e^Y\right)$ can be expressed as a series in repeated commutators of X and Y.

Modern expositions of the formula can be found in, among other places, the books of Rossmann[1] and Hall.[2]

GRE Practice #9: The following is Problem #14 from https://www.ets.org/s/gre/pdf/practice_book_math.pdf: Suppose g is a continuous real-valued function such that

$$3x^5 + 96 \ = \ \int_c^x g(t)dt$$

for each $x \in \mathbb{R}$, where c is a constant. What is the value of c?
(a) -96 (b) -2 (c) 4 (d) 15 (e) 32.