Math 150: Calculus III: Multivariable Calculus

Professor Steven J Miller: sjm1@williams.edu
https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 18: 3-18-2022: https://youtu.be/MzVRbIEf1to

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture18.pdf

lecture: http://youtu.be/pgwC2vOwRuE **(March 17, 2014: Lagrange Multipliers)**

Plan for the day: Lecture 18: March 18, 2022:

Topics:

Lagrange Multipliers

Fund Thm of Calc

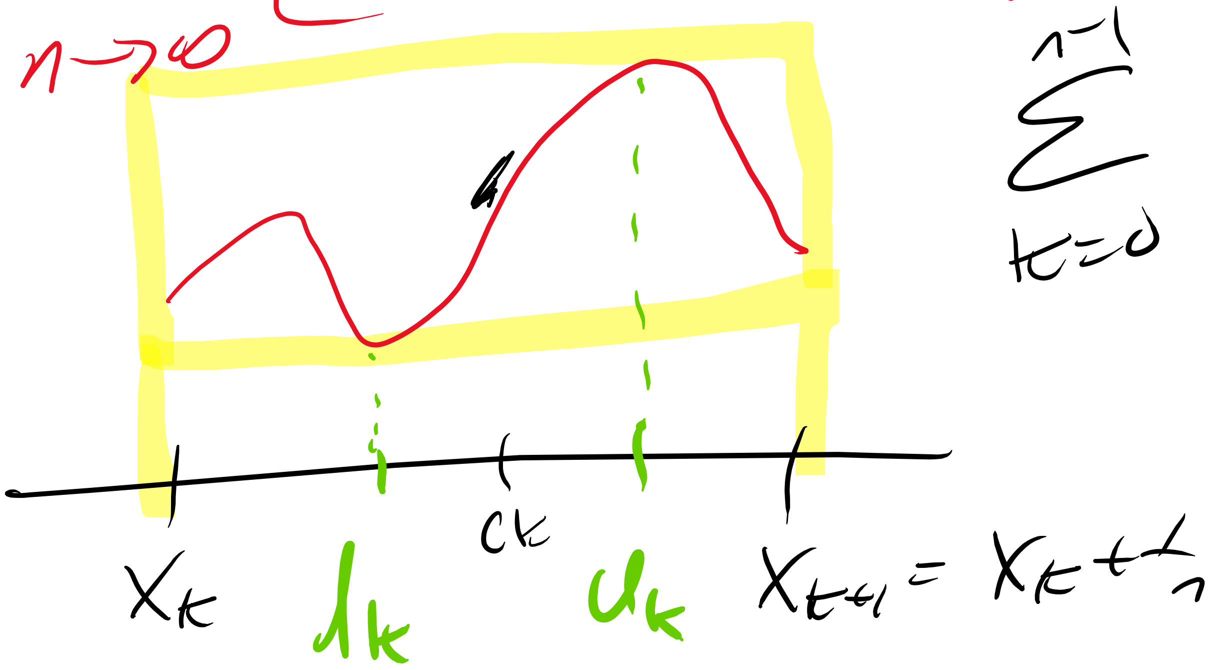
f nice function: f' is continuous and bounded

$F' = f$, $\int_a^b f(x)\,dx$ is area under $y = f(x)$ from a to b.

Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Needed f' is cont and bounded to show

$$\lim_{n\to\infty}\left[U(n)-L(n)\right]=0$$



$$U(n)-L(n)=\sum_{k=0}^{n-1}\left[f(u_k)\tfrac{1}{n}-f(l_k)\tfrac{1}{n}\right]$$

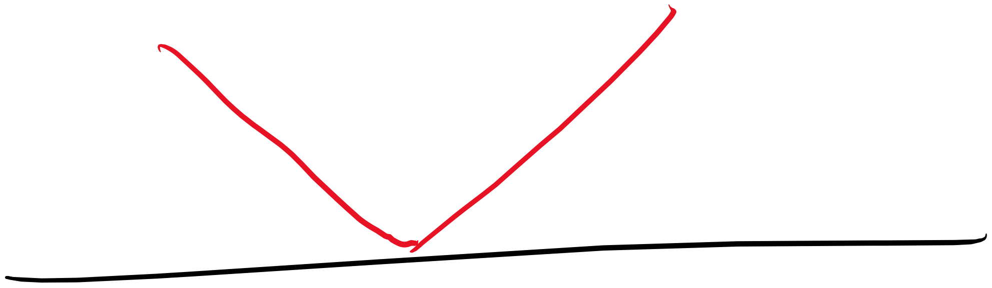
$$=\sum_{k=0}^{n-1} f'(c_k)(u_k-l_k)\tfrac{1}{n}$$

MVT: $\dfrac{f(u_k)-f(l_k)}{u_k-l_k}=f'(c_k)$

$$|U(n)-L(n)|\le\sum_{k=0}^{n-1}|f'(c_k)|\cdot\frac{1}{n^2}\qquad\text{as } |u_k-l_k|\le\frac{1}{n}$$

if $|f'(x)|\le B$ sum is $\le B\cdot n\cdot\frac{1}{n^2}\to 0$

Non-differentiable function



$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

not diff at $x=0$

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \begin{cases} 1 & \text{when } h \to 0^+ \\ -1 & \text{when } h \to 0^- \end{cases}$$

for proof, just need $\left| \dfrac{f(x) - f(y)}{x - y} \right|$ is bounded

Lagrange Multipliers

Constraint: $g(\vec{x}) = 0$ $\qquad \vec{x} = (x_1, \dots, x_n)$

function $\qquad f(\vec{x})$

Candidates for max/min of f subject to $g(\vec{x}) = 0$ is

are $\vec{x}$ st

$n+1$ equations

$n+1$ variables

$(x_1, \dots, x_n, \lambda)$

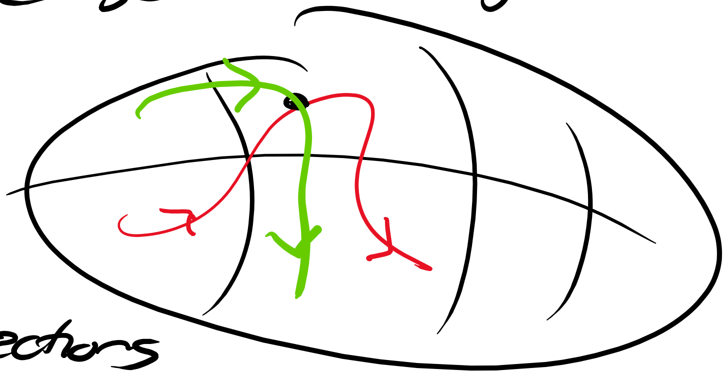
$$\begin{cases} (1) \quad g(\vec{x}) = 0 \\ (2) \quad \exists \lambda \neq 0 \text{ st } (\nabla f)(\vec{x}) = \lambda (\nabla g)(\vec{x}) \end{cases}$$

gradient is // to the normal

Looking for a point: $(x_1, x_2, \dots, x_n)$; do not care about λ

Why does it work?

$g(\vec{x}) = 0$ Curve $c(t)$ st $c(t) = (x_1(t), \dots, x_n(t))$ and lies on the surface $g(\vec{x}) = 0$



$$\left[f(c(t))\right]' = (\nabla f)(c(t)) \cdot c'(t)$$

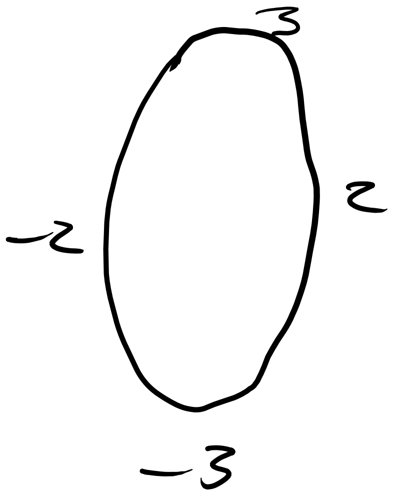
If this is 0 for all tangent directions

Then $(\nabla f)(c(t))$ is in the direction of

the normal to the surface $g(\vec{x}) = 0$,

which is $(\nabla g)(c(t))$

Make sure do not divide by zero:



$$g(x,y) = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$f(x,y) = x^4 + y^4$$

Mistake: grad(g) is (x/2, 2y/9)

$$(\nabla g)(x,y) = \left(x, \frac{2}{3}y\right)$$

$$(\nabla f)(x,y) = (4x^3, 4y^3)$$

$$4x^3 = \lambda x$$

$$4y^3 = \lambda \frac{2}{3} y$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

Some algebra error below

Correct grad(g) is (x/2, 2y/9) not (x, 2y/3)

Solve: $4x^3 = \lambda x$

$4y^3 = \lambda \frac{2}{3} y$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

Case 1: $x = 0$:

↳ Get 2nd eq always true

3rd: $\left(\frac{y}{3}\right)^2 = 1$ so $y = \pm 3$

$f(0, \pm 3) = 81$

Case 2: $y = 0$: Similarly get $f(\pm 2, 0) = 16$

Solve: $4x^3 = \lambda x$

$4y^3 = \lambda \frac{2}{3} y$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

Case 3: $x, y \neq 0$

As x, y not zero: divide (eq 1) by (eq 2)

$$\frac{4x^3}{4y^3} = \frac{\lambda x}{\lambda \frac{2}{3} y} \quad \text{or} \quad \left(\frac{x}{y}\right)^3 = \frac{3}{2}\left(\frac{x}{y}\right) \Rightarrow \left(\frac{x}{y}\right)^2 = \frac{3}{2} \Rightarrow x^2 = \frac{3}{2} y^2$$

so $x = \pm\sqrt{\frac{3}{2}}\, y$

so $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \Rightarrow \frac{3}{8} y^2 + \frac{1}{9} y^2 = 1 = \frac{35}{72} y^2 = 1$

$y = \pm\sqrt{72/35} \qquad x = \pm\sqrt{\frac{3}{2}}\sqrt{\frac{72}{35}}$ See next page for fixed math

$$f\left(\pm\sqrt{\frac{3}{2}}\sqrt{\frac{72}{35}}, \pm\sqrt{\frac{72}{35}}\right) = \left(\frac{3}{2}\right)^2\left(\frac{72}{35}\right)^2 + \left(\frac{72}{35}\right)^2 = \frac{13}{4} \cdot \left(\frac{72}{35}\right)^2$$

Some algebra error below

Correct grad(g) is (x/2, 2y/9) not (x, 2y/3)

Leads to (4x^3, 4y^3) = lambda (x/2, 2y/9) and (x/2)^2 + (y/3)^2 = 1

Now we get 4x^3/4y^3 = (x/2) / (2y/9) = (9/4) (x/y)

Thus (x/y)^2 = 9/4 so (x/y) = +- 3/2

So x^2 = (9/4) y^2

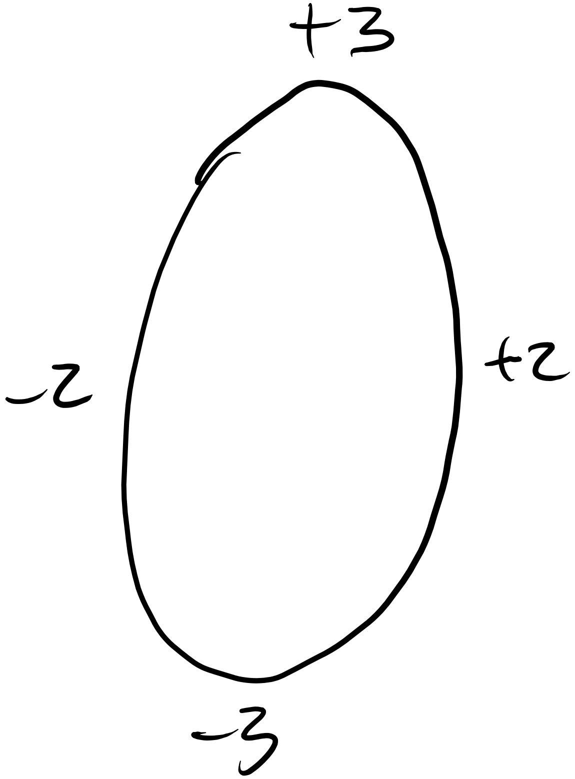
Substituting yields (9/4) y^2 / 4 + y^2/ 9 = 1

Thus (9/16) y^2 + y^2/9 = 1 or (97/144) y^2 = 1

Thus y = +- sqrt(144/97) and x = +- (3/2) sqrt(144/97)

This now works – get arccos(x/2) =approx.= .418224, as we should.

(13/4)(72/35)^2
16848/1225
(13/4)(72/35)^2.
13.7535
(Sqrt[3/2] Sqrt[72/35])^4 + 1.0 (Sqrt[72/35])^4
13.7535



Global max at $(0, \pm 3)$

Local max at $(\pm 2, 0)$

Global min at $\left(\pm\sqrt{\frac{3}{2}}\sqrt{\frac{72}{35}}, \sqrt{\frac{72}{35}}\right)$

$$c(t) = (2\cos t, 3\sin t) \quad 0 \le t \le 2\pi$$

$$f(c(t)) = (2\cos t)^4 + (3\sin t)^4$$

$$= 16\cos^4(t) + 81\sin^4(t)$$