Math 150: Calculus III: Multivariable Calculus

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https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 20: 4-6-2022: https://youtu.be/1OwwNrg8x3U

slides:

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture20.pdf

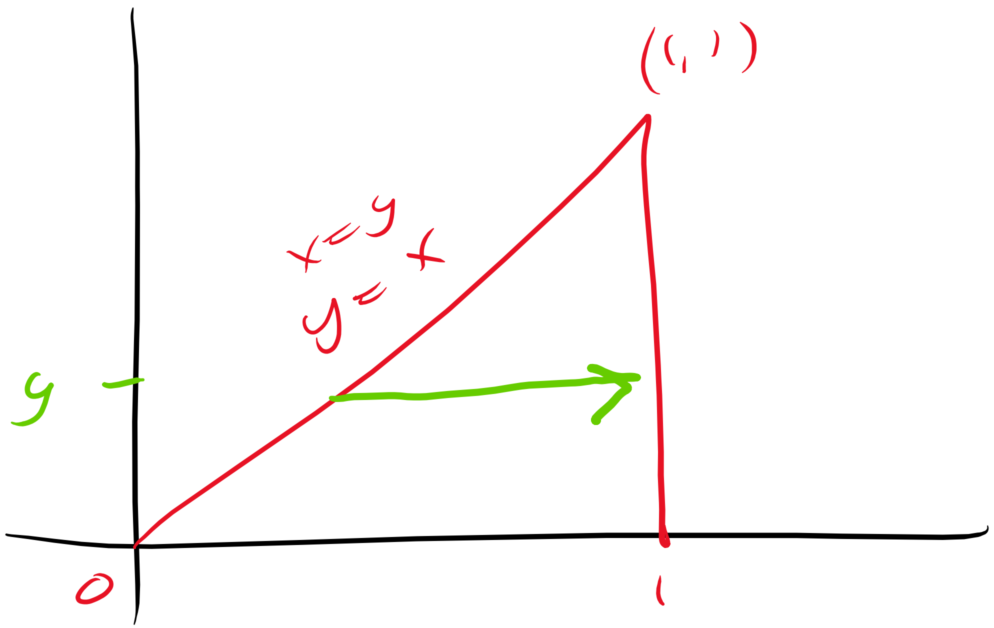
Plan for the day: Lecture 20: April 6, 2022:

Topics:

Interated Integrals: Changing Order

Polar Change of Variables: Circles and Spheres

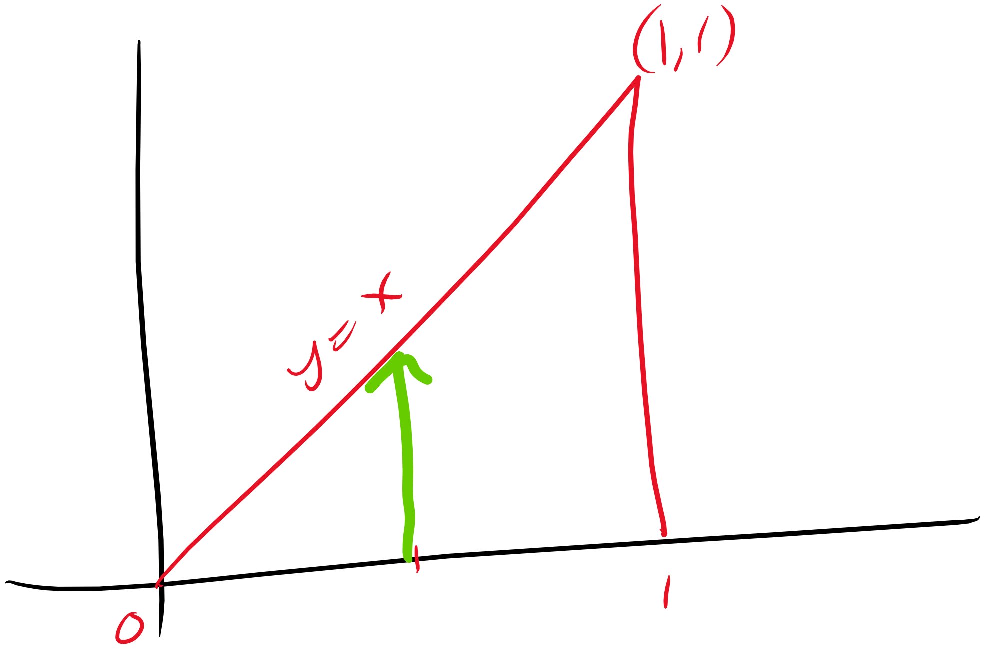
- **Sixteenth day lecture:** http://youtu.be/G9d9lcYevnM (April 9, 2014: Iterated integrals, changing order)
- **Seventeenth day lecture:** http://youtu.be/JP78q_ri-4o (April 11, 2014: Polar Change of Variables, Circles and Spheres)



$$\int_{y=0}^{1}\left[\int_{x=y}^{1} x^4 e^{x^2 y}\,dx\right]dy$$

Need anti-deriv wrt x of $x^4 e^{x^2 y}$

Try $\frac{x^5}{5} e^{x^2 y} \xrightarrow{\text{deriv}} x^4 e^{x^2 y} + \frac{x^5}{5} 2xy\, e^{x^2 y}$



$$\int_{x=0}^{1}\left[\int_{y=0}^{x} x^4 e^{x^2 y}\,dy\right]dx$$

$$\int_{y=0}^{x} x^4 e^{x^2 y}\,dy = x^2\int_{y=0}^{x} e^{x^2 y}\,x^2 dy$$

$u = x^2 y$
$du = x^2 dy$
$y: 0 \to x$
$u: 0 \to x^3$

$$= x^2\int_{u=0}^{x^3} e^u\,du = x^2 e^u\Big|_0^{x^3} = \left(x^2 e^{x^3} - x^2\right)$$

$$\text{Integral} = \int_{x=0}^{1}\left(x^2 e^{x^3} - x^2\right)dx$$

$$\text{Integral} = \int_{x=0}^{1} \left(x^2 e^{x^3} - x^2\right) dx$$

u-sub for first

$u = x^3$

$du = 3x^2 dx$

$x: 0 \to 1$

$u: 0 \to 1$

$$= \int_{x=0}^{1} e^{x^3} x^2 dx - \int_{x=0}^{1} x^2 dx$$

$$= \frac{1}{3} \int_{x=0}^{1} e^{x^3} 3x^2 dx - \int_{x=0}^{1} x^2 dx$$

$$= \frac{1}{3} e^{x^3} \Big|_0^1 - \frac{x^3}{3} \Big|_0^1$$

$$= \frac{1}{3} e^1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3} e - \frac{2}{3} = \frac{1}{3}(e-2)$$

In[1]:=
```
Integrate[x^4 Exp[x^2 y], x]
```

Out[1]= $$\frac{e^{x^2 y}\, x \left(-3+2\, x^2\, y\right)}{4\, y^2}+\frac{3\, \sqrt{\pi}\, \mathrm{Erfi}\left[x\, \sqrt{y}\,\right]}{8\, y^{5/2}}$$

In[12]:=
```
Timing[Integrate[Integrate[x^4 Exp[x^2 y], {x, y, 1}],
  {y, 0, 1}]]
```

Out[12]= $$\left\{1.625,\ \frac{1}{3}\,(-2+e)\right\}$$

In[13]:=
```
Timing[Integrate[Integrate[x^4 Exp[x^2 y], {y, 0, x}],
  {x, 0, 1}]]
```

Out[13]= $$\left\{0.03125,\ \frac{1}{3}\,(-2+e)\right\}$$

```
ℛ = ImplicitRegion[0 ≤ x ≤ y ≤ 1, {x, y}];
Timing[Integrate[x^4 Exp[x^2 y],
  {x, y} ∈ ImplicitRegion[0 ≤ y ≤ x ≤ 1, {x, y}]]]
```

$$\left\{0.671875,\ \frac{1}{3}\left(-2+e\right)\right\}$$

Here is a nice way to do integrals over regions.

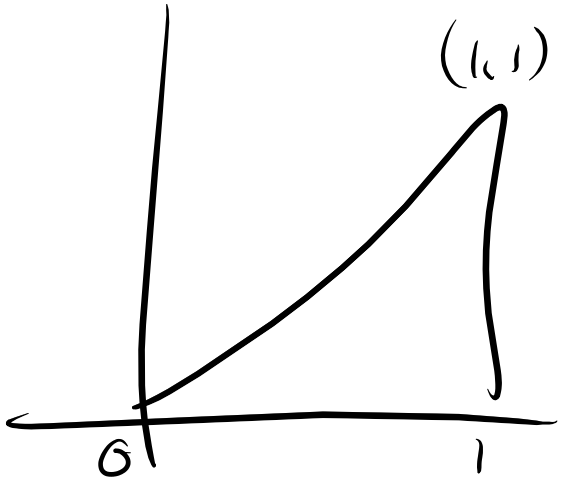
We define the region implicitly in terms of x and y.

This isn’t too bad to do.

Then we do the integration!

Note the time: faster than doing the x-integration first, slower
Than doing the y-integration first….

answer is $\frac{e-2}{3}$



$\iint x^4 e^{x^2 y}\, dA$

Integral should be positive

Should be b/w $0 \cdot \frac{1}{2}$ and $e \cdot \frac{1}{2}$

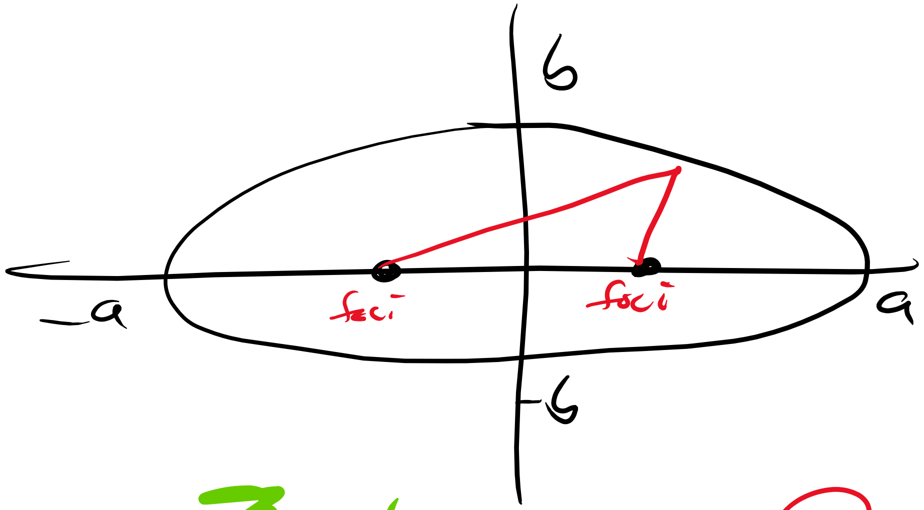
min value

max value

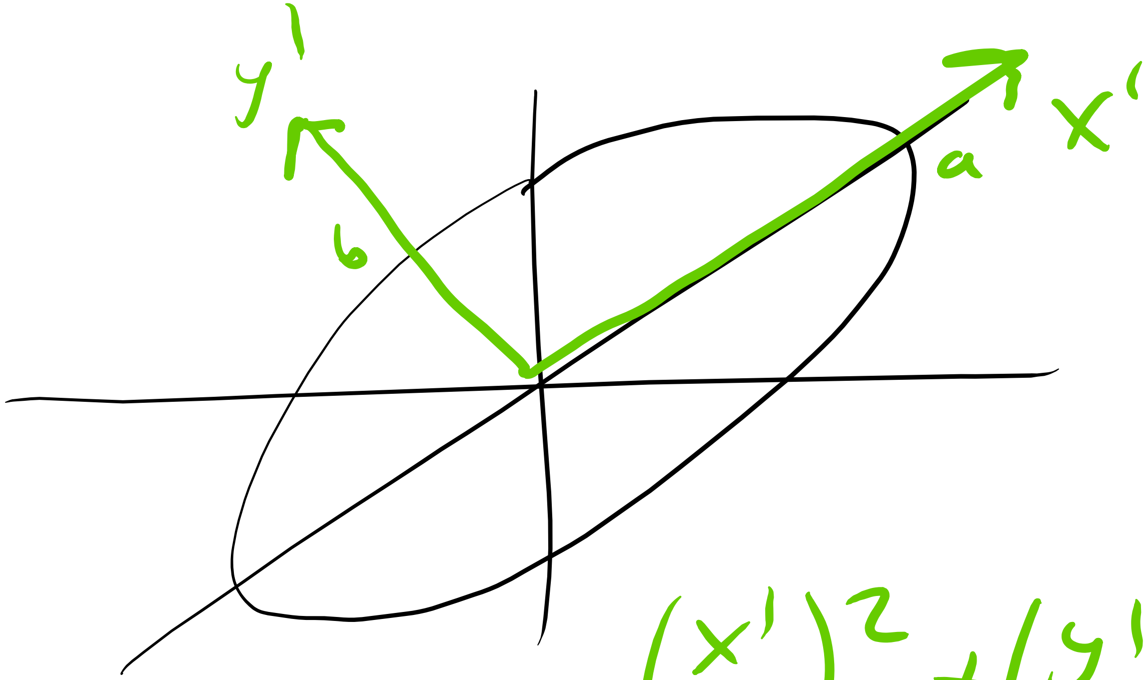
As $e^{x^2 y} \leq e$ says $\leq e \iint x^4 \, dA$

Changing Variables

Ellipse

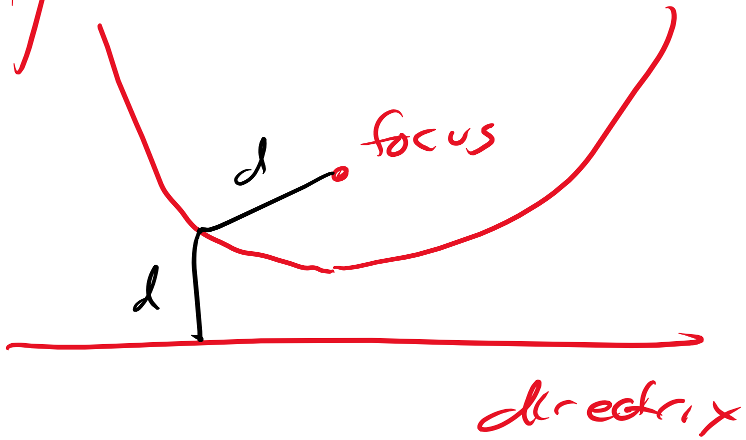


$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1$$

Parabola



Convert from x',y' to x,y Linear Alg

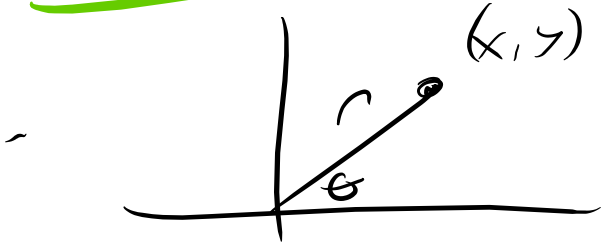
Cartesian to Polar

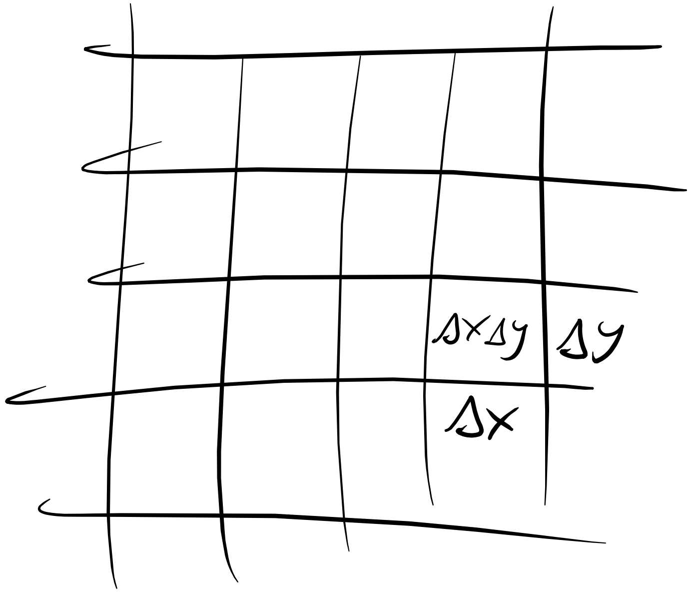
$x = r\cos\theta$

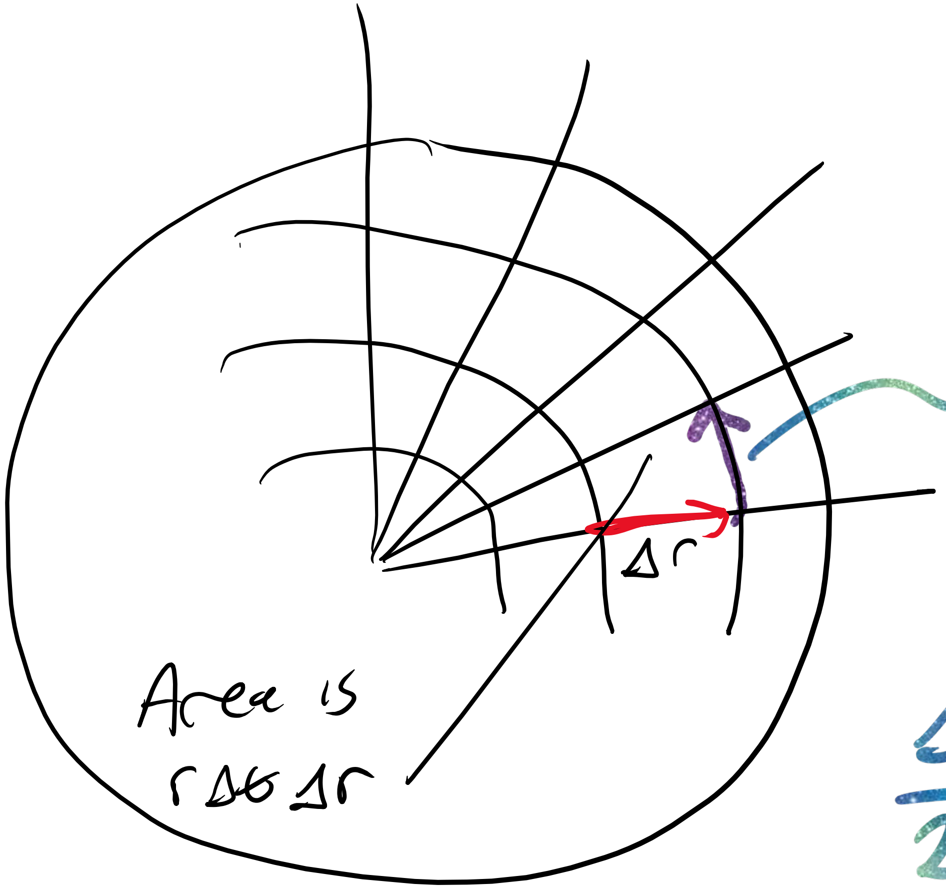
$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

$\theta = \arctan(y/x)$







$r\Delta\theta$

length is

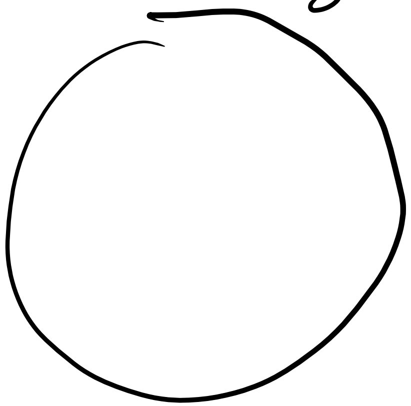
$\frac{\Delta\theta}{2\pi} * 2\pi r$

In the limit: $\Delta\theta \to d\theta$, $\Delta r \to dr$

$dx\,dy \longleftrightarrow r\,d\theta\,dr$ or $r\,dr\,d\theta$

Circle of radius R

$y = \pm\sqrt{R^2 - x^2}$



$$\int_{x=-R}^{R} \left[\int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1\,dy \right] dx$$

$$= \int_{x=-R}^{R} 2\sqrt{R^2-x^2}\,dx$$

Polar

$$\int_{\theta=0}^{2\pi}\left[\int_{r=0}^{R} 1 \quad r\,dr\right]d\theta$$

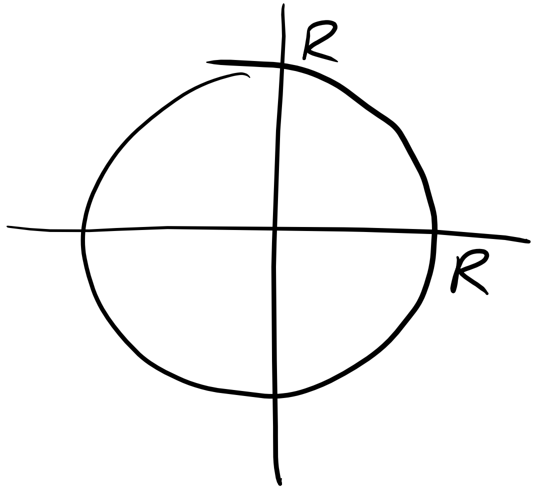
$$= \int_{\theta=0}^{2\pi} d\theta \left[\int_{r=0}^{R} r\,dr\right]$$

$$= 2\pi \left.\frac{r^2}{2}\right|_0^R = \frac{2\pi R^2}{2} = \pi R^2$$

Volume of a sphere

$x^2+y^2+z^2 \le R^2$

Fix x,y: z goes from $-\sqrt{R^2-(x^2+y^2)}$ to $\sqrt{R^2-(x^2+y^2)}$



$$\iint_{\text{circle}} 2\sqrt{R^2-(x^2+y^2)}\, dA$$

replace x with $r\cos\theta$, y with $r\sin\theta$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{R} 2\sqrt{R^2-r^2}\; r\, dr\, d\theta$$

$$= 2\int_{\theta=0}^{2\pi} d\theta \quad \int_{r=0}^{R} (R^2-r^2)^{1/2}\, r\, dr = 4\pi\int_{r=0}^{R}(R^2-r^2)^{\frac{1}{2}}\, r\, dr$$

$$\text{Vol} = 4\pi \int_{r=0}^{R} (R^2 - r^2)^{1/2}\, \underline{r}\, dr$$

$$u = R^2 - r^2$$

$$\frac{du}{dr} = -2r \quad \text{so} \quad du = -2r\,dr \quad \text{or} \quad r\,dr = -\tfrac{1}{2}\,du$$

$$r : 0 \to R \quad \text{so} \quad u : R^2 \to 0$$

$$\text{Vol} = 4\pi \int_{u=R^2}^{0} u^{1/2} \left(-\tfrac{1}{2}\right) du = 4\pi \int_{u=0}^{R^2} u^{1/2}\, \tfrac{1}{2}\, du$$

$$= 4\pi\, \frac{1}{3}\, u^{3/2} \Big|_0^{R^2} = \frac{4\pi}{3} (R^2)^{3/2} = \frac{4\pi}{3} R^3$$

Surface area is $4\pi R^2$