Math 150: Calculus III: Multivariable Calculus

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https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 21: 4-8-2022: https://youtu.be/ygskSmmshKg

slides:

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture21.pdf

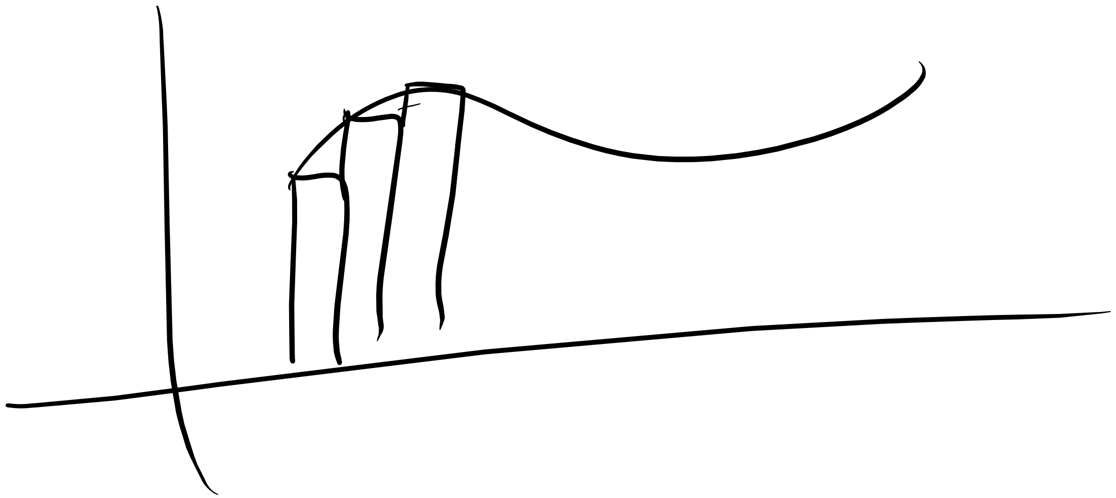
Plan for the day: Lecture 21: April 8, 2022:

Topics:

Monte Carlo Integration

Polar Integration

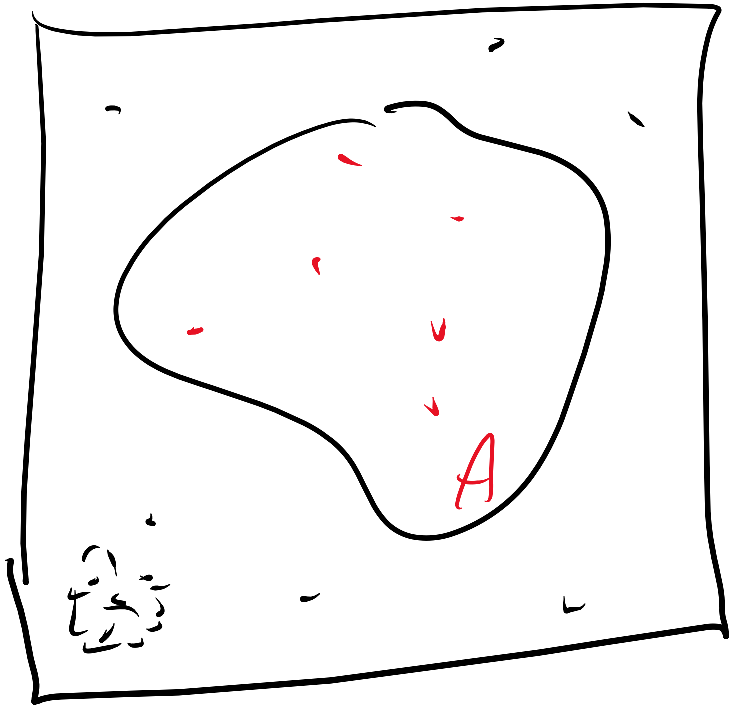
Simple Regions



Region A where easy to tell if $\vec{x} = (x_1, \dots, x_d) \in A$

A lives in a box: simplicity $[0,1]^d$

d-dim hypercube



N tosses uniformly in $[0,1]^2$

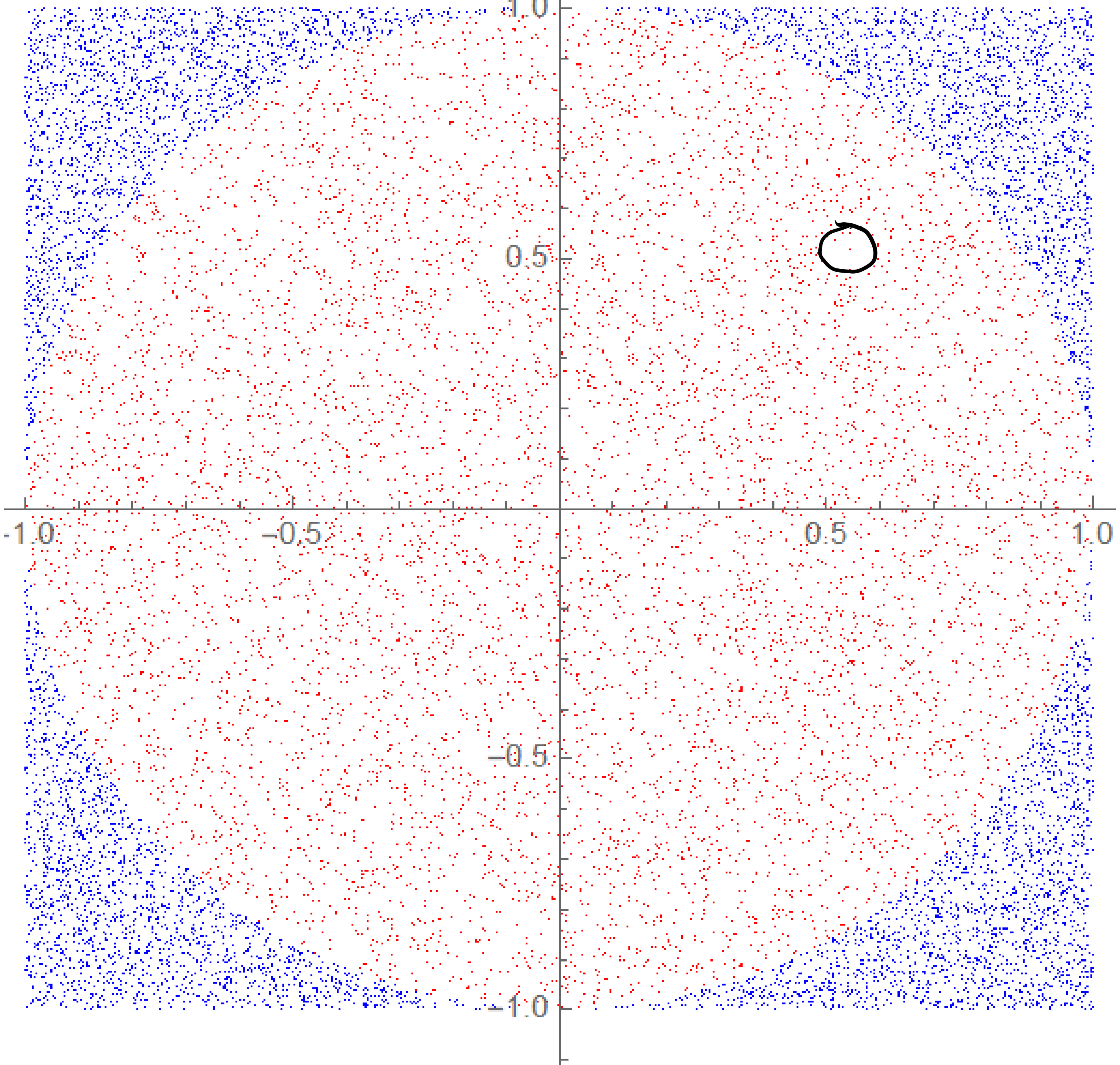
Say N_A land in A

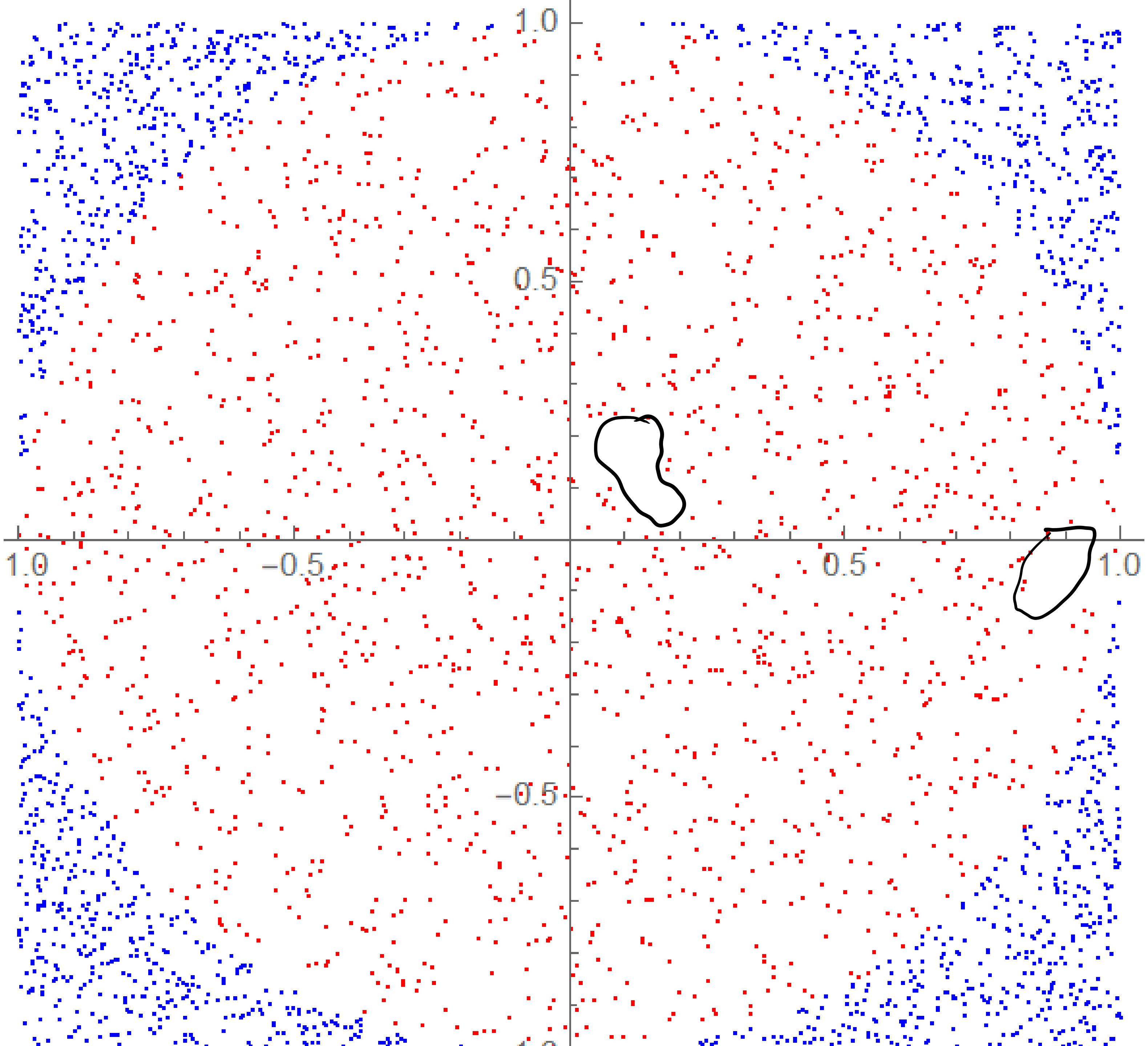
Area(A) estimate: $\frac{N_A}{N} * \mathrm{Vol}([0,1]^2)$

Prob land in A is $\frac{\mathrm{Area}(A)}{\mathrm{Area}([0,1]^2)} = \mathrm{Area}(A)$

N tosses, expect in A about $N \cdot \mathrm{Area}(A) = N_A$

Error is of size $\sqrt{N}$; Guess for Area is $\frac{N_A}{N}$ with error of size $\frac{\sqrt{N}}{N}$


1.0
0.5
-1.0
-0.5
0.5
1.0
-0.5
-1.0


1.0
0.5
−0.5
1.0
−0.5
0.5
1.0

Area of Ellipse

a	b	Area Estimate
1	1/2	$\pi/2$

if $a = b = r$ get πr^2

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

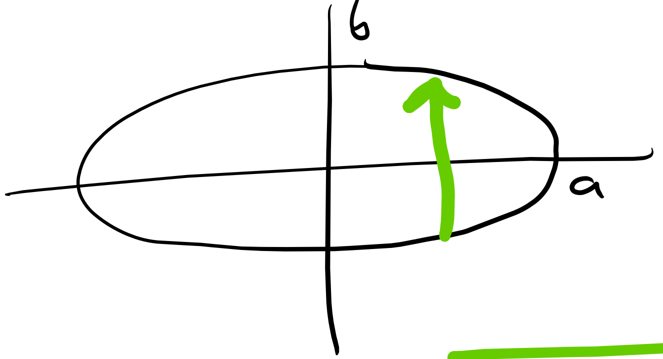
Area Conjectures

πab

$$\iint_{\text{ellipse}} 1 \, dA$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\sqrt{1-\left(\frac{x}{a}\right)^2}\; b$$



$$y = -\sqrt{1-\left(\frac{x}{a}\right)^2}\; b$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{1-\left(\frac{x}{a}\right)^2}\, b}^{\sqrt{1-\left(\frac{x}{a}\right)^2}\, b} 1 \, dy \, dx$$

$$u = \frac{x}{a} \qquad du = \frac{dx}{a} \text{ or } dx = a\,du$$

$$x: -a \to a \qquad u: -1 \text{ to } 1$$

$$= 2b \int_{x=-a}^{a} \sqrt{1-\left(\frac{x}{a}\right)^2} \, dx$$

$$= 2ba \int_{u=-1}^{1} \sqrt{1-u^2} \, du = \pi ab$$

as $\int$ is indep of a, b and is $\pi/2$ as circle
if $a=b=1$ area is π

$$\iint_{\text{ellipse}} 1\, dA = \iint_{\text{ellipse}} 1\, dx\,dy \qquad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \le 1$$

$$u = \frac{x}{a} \quad \text{and} \quad v = \frac{y}{b} \quad \text{so} \quad u^2 + v^2 \le 1$$

$$du = \frac{dx}{a} \quad \text{and} \quad dv = \frac{dy}{b} \quad \text{so} \quad dx\,dy = ab\, du\,dv$$

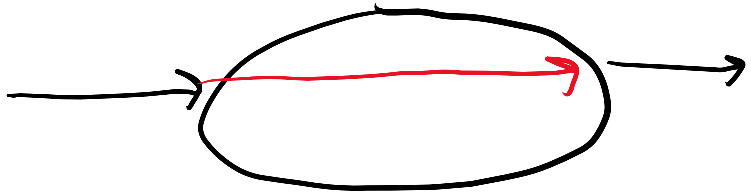
$$\iint_{\text{ellipse}} 1\, dx\,dy = \iint_{\substack{\text{unit}\\ \text{circle}}} 1\, ab\, du\,dv = ab \iint_{\substack{\text{unit}\\ \text{circle}}} du\,dv$$

$$= \pi ab$$

Vol (Ellipsoid) $\quad \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2 \leq 1$

$$= \frac{4}{3}\pi abc$$

Iterated Integrals

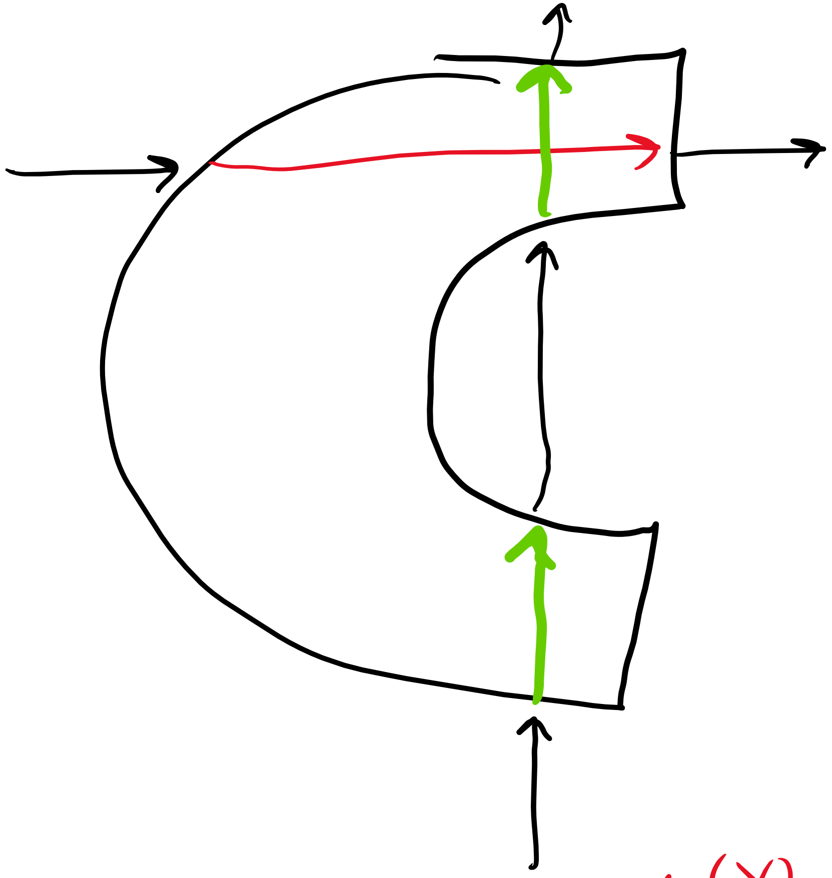


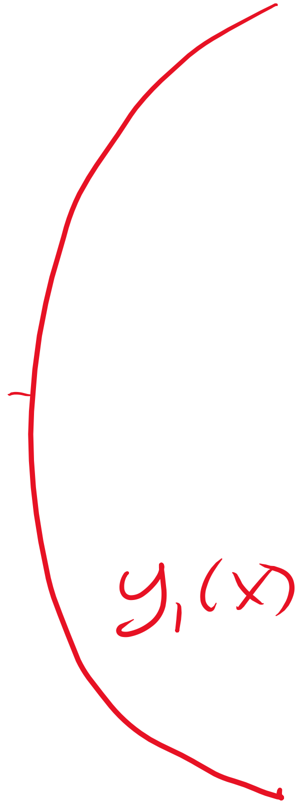
Simple Regions:

horizontally simple if walk horizontally you enter region, stay in region, leave, never return.

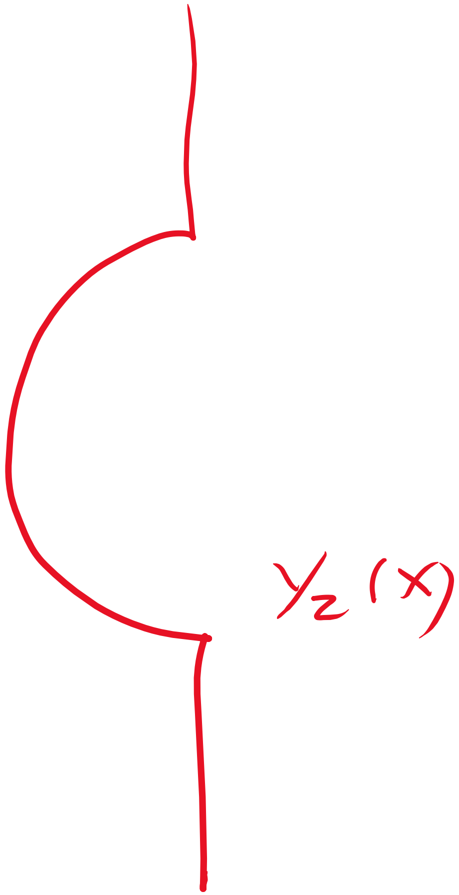
vertically simple: similar

both is horiz + vertically simple

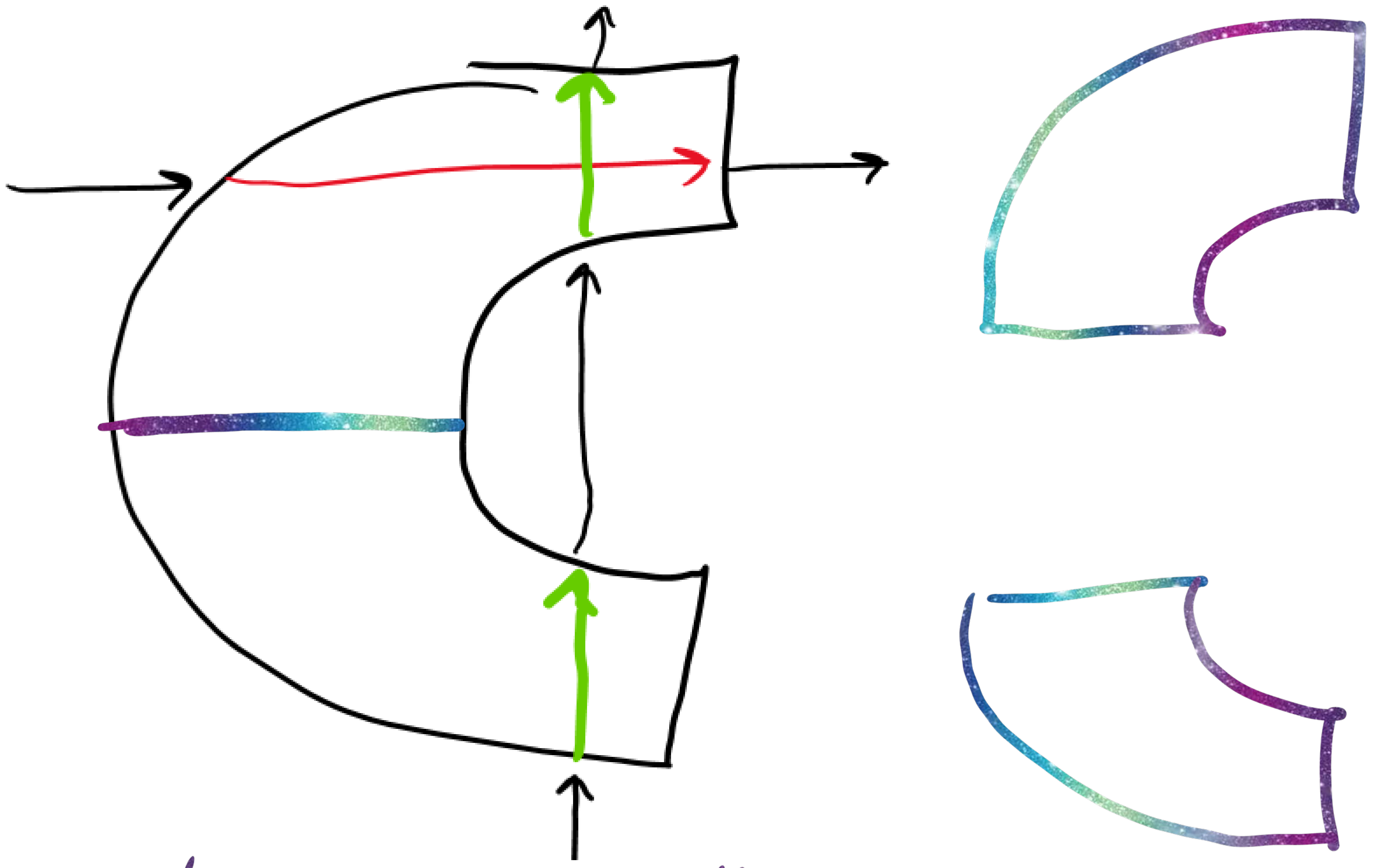




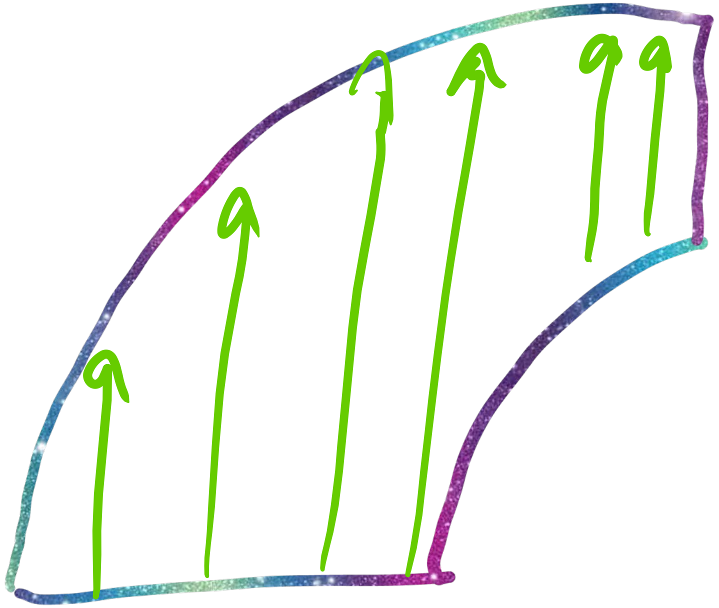
and



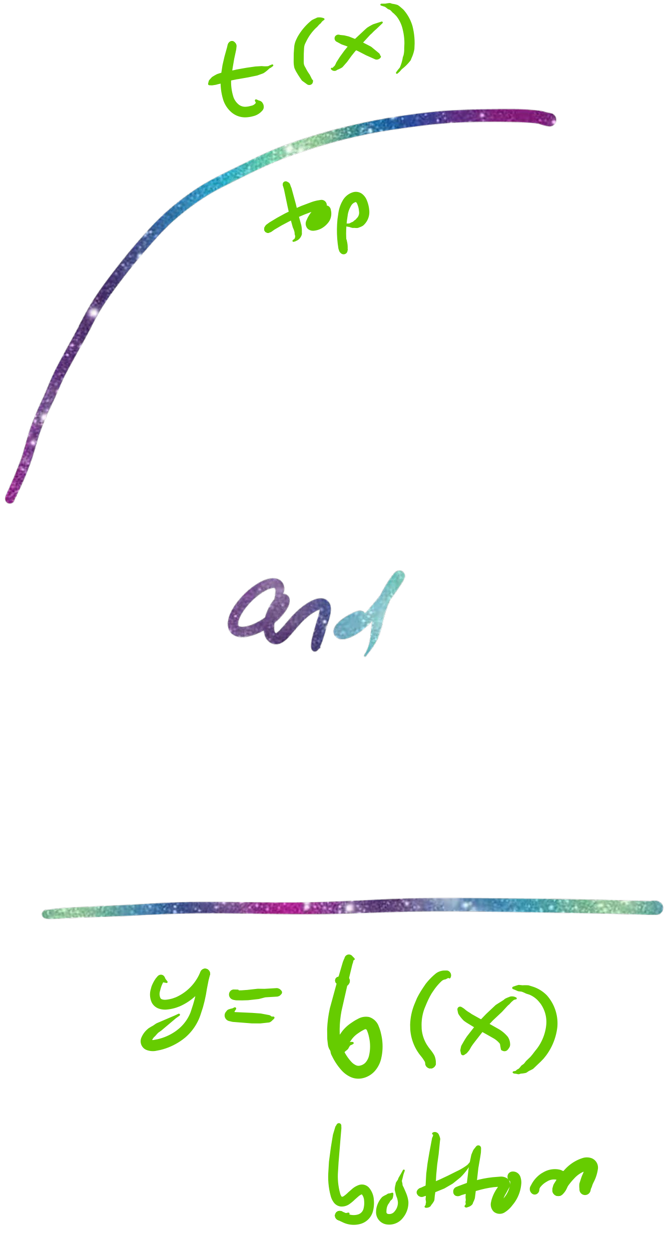
$$\int_{y=y_i}^{y_f} \left[\int_{x=y_1(x)}^{y_2(x)} f(x,y)\, dx \right] dy$$

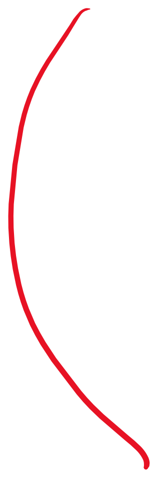


it is the sum of two vertically simple pieces

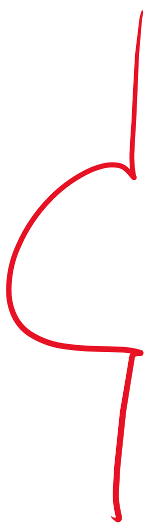


$$\int_{x=x_i}^{x_f}\left[\int_{y=b(x)}^{t(x)} f(x,y)\,dy\right]dx$$





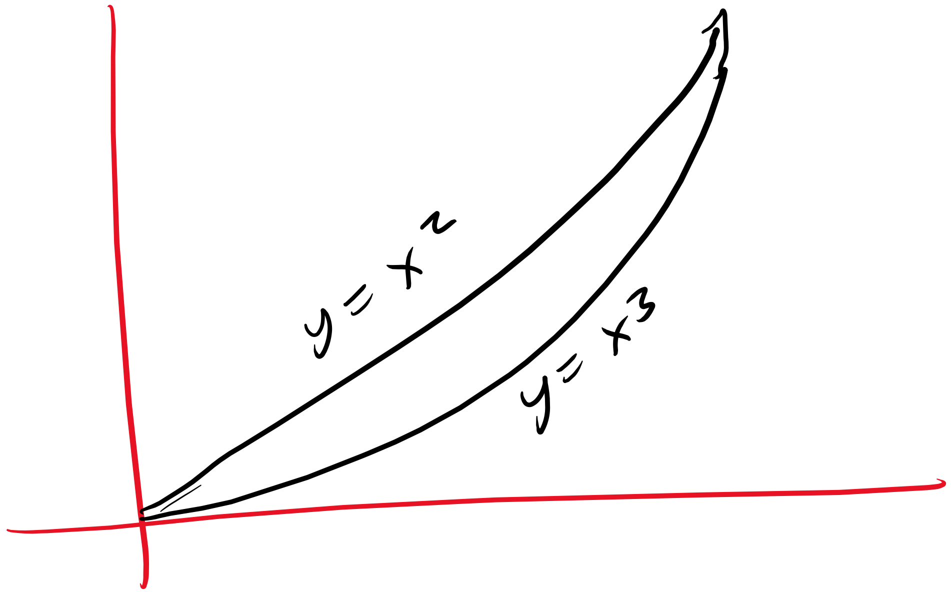
$x = L(y)$

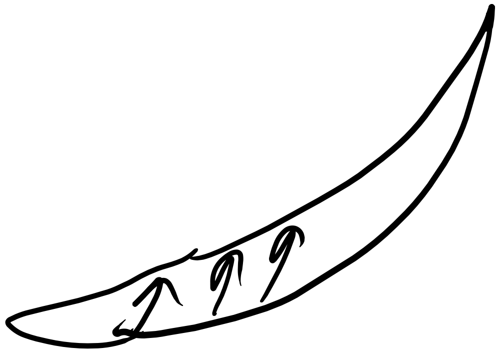


$x = R(y)$

$$\int_{y=y_i}^{y_f}\left[\int_{x=L(y)}^{R(y)} f(x,y)\,dx\right]dy$$

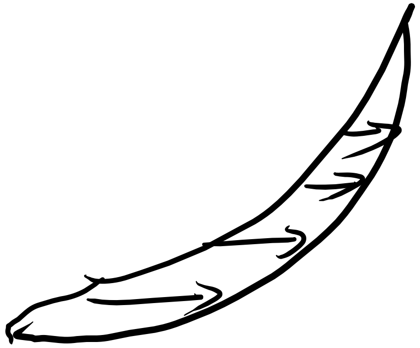
$y = x^2$ and $y = x^3$ from 0 to 1





$$\int_{x=0}^{1} \int_{y=x^3}^{x^2} f(x,y)\,dy\,dx$$

$b(x) = x^3 \quad t(x) = x^2$



$$\int_{y=0}^{1} \int_{x=\sqrt{y}}^{y^{1/3}} f(x,y)\,dx\,dy$$

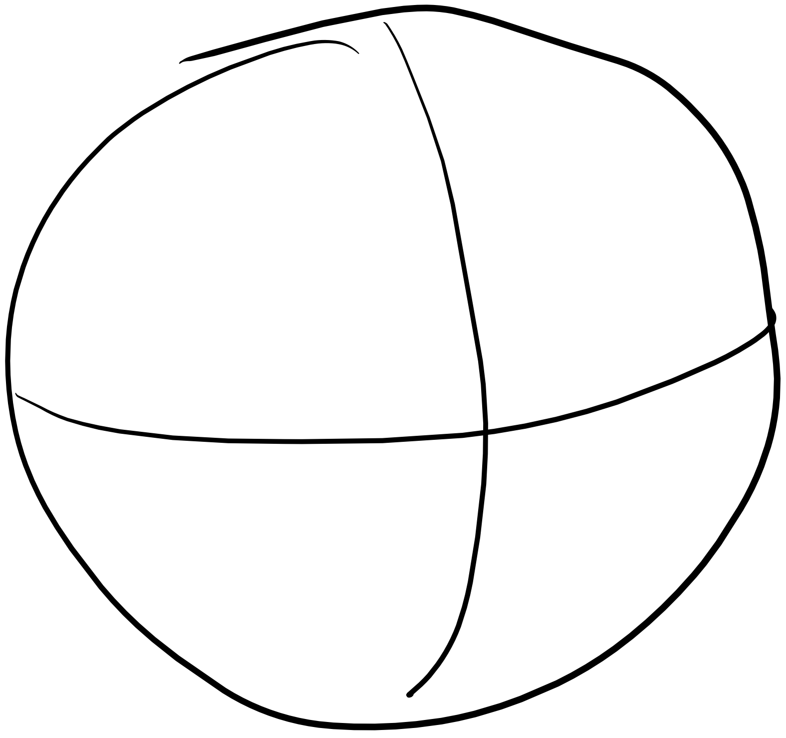
Left is from $y = x^2 \Rightarrow x = \sqrt{y}$

$L(y) = \sqrt{y}$

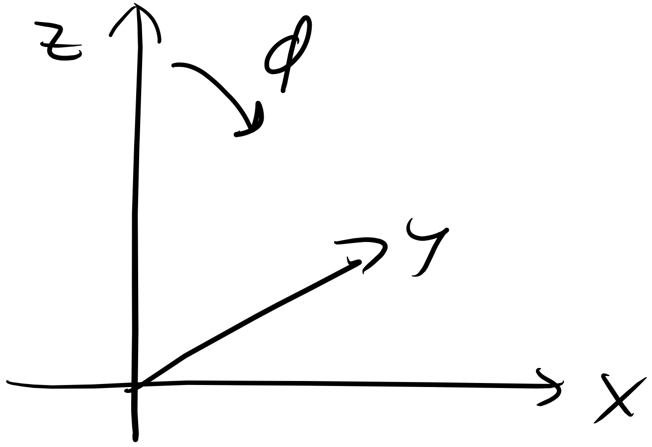
Right is from $y = x^3 \Rightarrow x = y^{1/3}$

$R(y) = y^{1/3}$

Spherical Coordinates



radius ρ

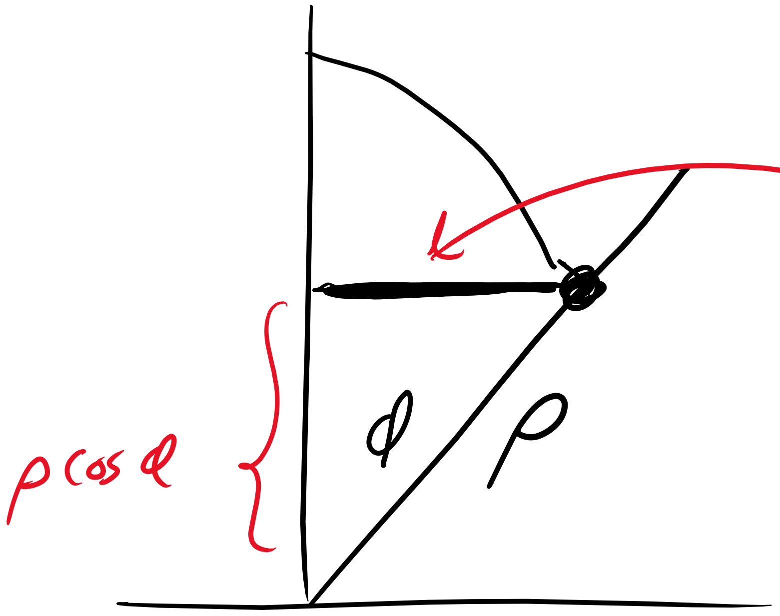


ϕ comes down from z-axis

$0 \le \phi \le \pi$

θ spins in plane // to xy-plane

$0 \le \theta \le 2\pi$



$z = \rho\cos\phi$

$\rho\sin\phi = r$

polar with $r = \rho\sin\phi$ and θ

$x = r\cos\theta = \rho\cos\theta\sin\phi$

$y = r\sin\theta = \rho\sin\theta\sin\phi$

Later: $\underbrace{dx\,dy\,dz}_{\text{meters}^3} \longleftrightarrow \underbrace{??}_{\rho^2 \text{ also } \sin\phi} \underbrace{d\rho\,d\theta\,d\phi}_{\text{meters}}$

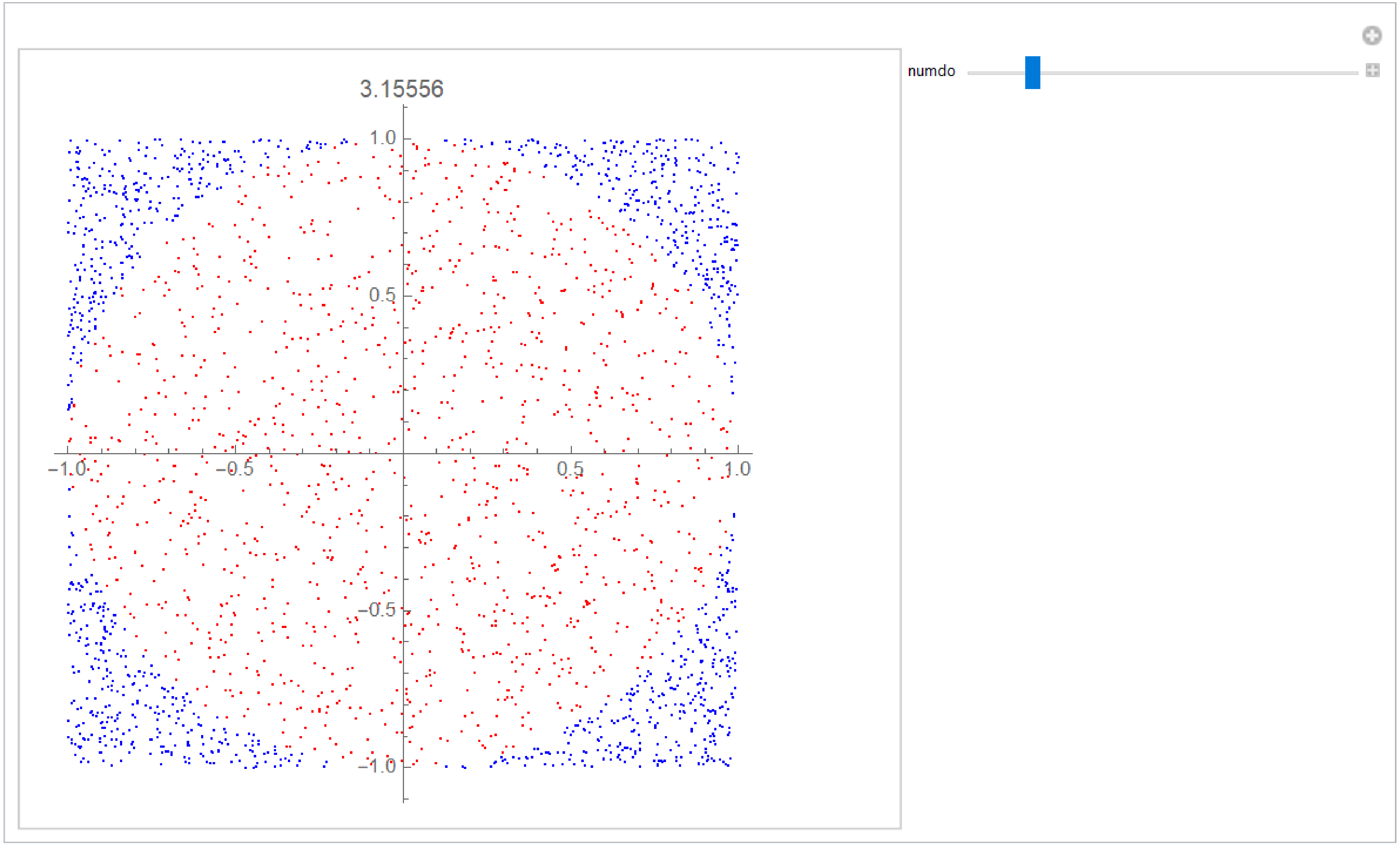
Monte Carlo Integration

Unit Circle

```
num = 40000;
Clear[inlist]; Clear[outlist];
inlist = {};
 outlist = {};
numin = 0;
For[m = 1, m ≤ num, m++,
  {
   x = 2 (Random[] - .5);
   y = 2 (Random[] - .5);
   If[x^2 + y^2 ≤ 1, loc = 1, loc = 2];
   If[loc == 1, inlist = AppendTo[inlist, {x, y}],  outlist = AppendTo[outlist, {x, y}]];
   numin = numin + 2 - loc;
   percentage[m] = 1.0 numin / m;
  }];
cut[dataset_, number_] := Module[{},
   temp = {};
   For[i = 1, i ≤ Floor[number], i++, temp = AppendTo[temp, dataset[[i]]]];
   Return[temp];
  ];
cut[inlist, 5]
```

```
Print["Plot of randomly chosen points inside and outside unit circle, whose area is ", 1.0 Pi];
Manipulate[ListPlot[{cut[inlist, numdo], cut[outlist, numdo]}, PlotStyle → {Red, Blue},
  AspectRatio → 1, PlotLabel → 4 percentage[Floor[numdo]]], {numdo, 20, .2 num}]
```

```
Plot of randomly chosen points inside and outside unit circle, whose area is 3.14159
```



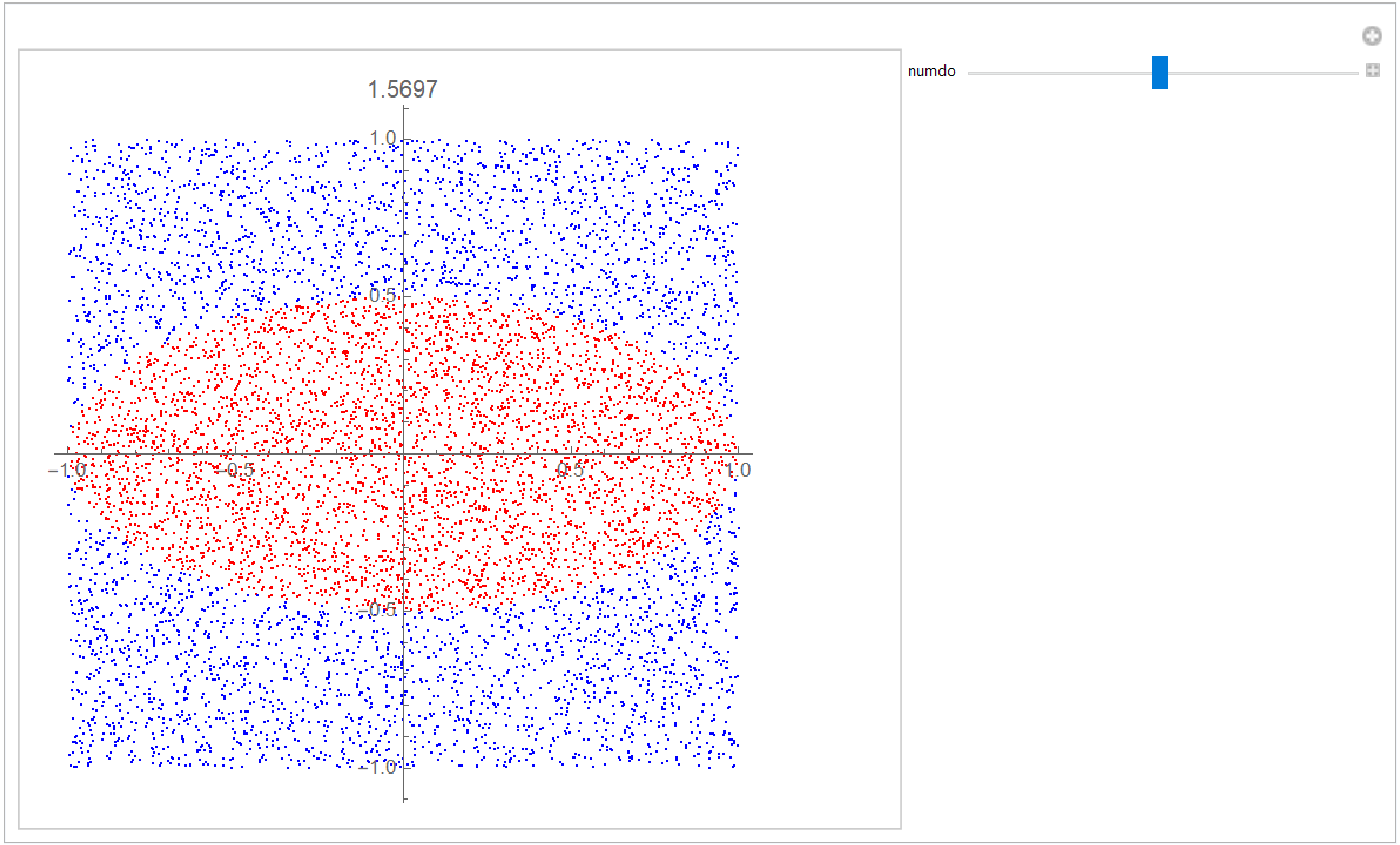
Ellipse x^2 + 4 y^2 = 1

```
num2 = 40000;
Clear[inlist2]; Clear[outlist2];
inlist2 = {};
 outlist2 = {};
numin = 0;
For[m = 1, m ≤ num2, m++,
  {
   x = 2 (Random[] - .5);
   y = 2 (Random[] - .5);
   If[x^2 + 4 y^2 ≤ 1, loc = 1, loc = 2];
   If[loc == 1, inlist2 = AppendTo[inlist2, {x, y}], outlist2 = AppendTo[outlist2, {x, y}]];
   numin = numin + 2 - loc;
   percentage2[m] = 1.0 numin / m;
  }];
cut[dataset_, number_] := Module[{},
   temp = {};
   For[i = 1, i ≤ Floor[number], i++, temp = AppendTo[temp, dataset[[i]]]];
   Return[temp];
  ];
cut[inlist2, 5]

{{0.44065, 0.220627}, {0.285419, 0.253779},
 {-0.279452, 0.285603}, {0.424153, 0.162875}, {-0.526706, -0.363985}}
```

```
Print["Plot of randomly chosen points inside and outside ellipse x^2 + 4 y^2 = 1, whose area is ",
  1.0 Pi / 2];
Manipulate[ListPlot[{cut[inlist2, numdo], cut[outlist2, numdo]}, PlotStyle → {Red, Blue},
  AspectRatio → 1, PlotLabel → 4 percentage2[Floor[numdo]]], {numdo, 20, .2 num2}]
```

Plot of randomly chosen points inside and outside ellipse x^2 + 4 y^2 = 1, whose area is 1.5708



```
Clear[inlistn]; Clear[outlistn];
inlistn = {};
outlistn = {};
numin = 0;
For[m = 1, m ≤ numn, m++,
  {
   x = randpoints[[m, 1]];
   y = randpoints[[m, 2]];
   If[16 x^2 - 1 ≥ 4 y^2, loc = 1, loc = 2];
   If[loc == 1, inlistn = AppendTo[inlistn, {x, y}], outlistn = AppendTo[outlistn, {x, y}]];
   numin = numin + 2 - loc;
   percentagen[m] = 1.0 numin / m;
  }];
cut[dataset_, number_] := Module[{},
   temp = {};
   For[i = 1, i ≤ Floor[number], i++, temp = AppendTo[temp, dataset[[i]]]];
   Return[temp];
  ];
cut[inlistn, 5]
```

```
Print["Plot of randomly chosen points inside and outside."];
Manipulate[ListPlot[{cut[inlistn, numdo], cut[outlistn, numdo]}, PlotStyle → {Red, Blue},
  AspectRatio → 1, PlotLabel → 4 percentagen[Floor[numdo]]], {numdo, 20, .2 numn}]
```

Plot of randomly chosen points inside and outside.