Math 150: Calculus III: Multivariable Calculus

Professor Steven J Miller: sjm1@williams.edu
https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 23: 4-13-2022: https://youtu.be/qUP-giFb_f8

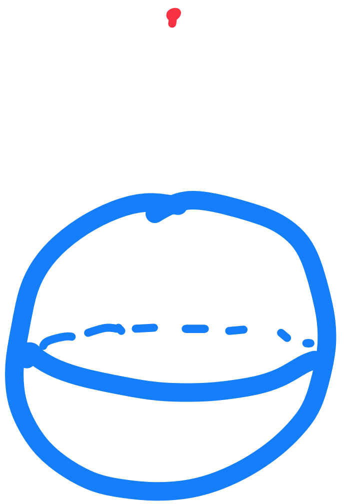
slides:

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture23.pdf

Plan for the day: Lecture 23: April 13, 2022:

Topics:

Spherical Integration

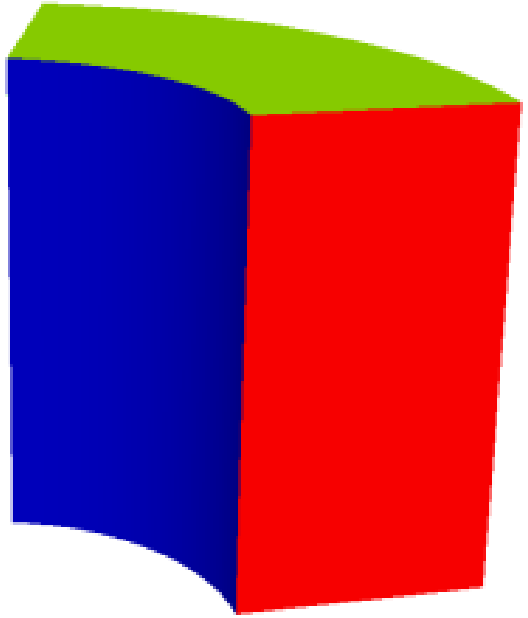


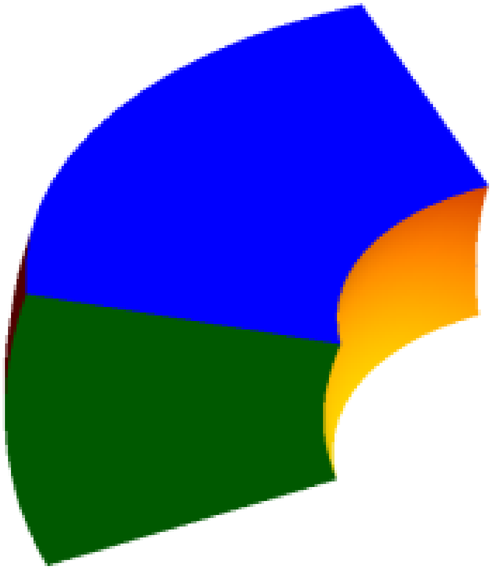
Geometric and Harmonic Series

https://legacy-www.math.harvard.edu/~knill/teaching/summer2021/handouts/lecture18.pdf

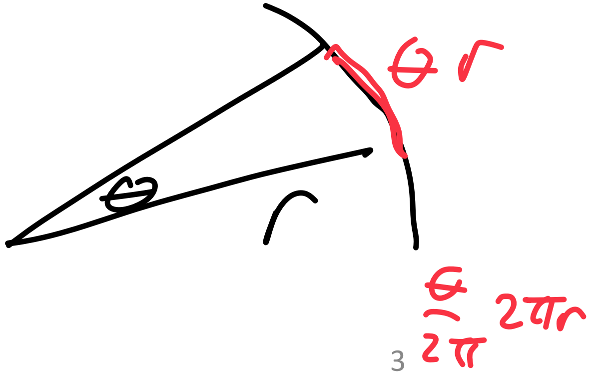
Definition: **Cylindrical coordinates** are space coordinates where polar coordinates are used in the xy-plane while the z-coordinate is not changed. The coordinate transformation $T(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$, produces the integration factor $\boxed{r}$. It is the same factor than what we are used to in polar coordinates.

$$\iiint_{T(R)} f(x,y,z)\, dx\,dy\,dz = \iiint_R g(r,\theta,z)\, \boxed{r}\, dr\,d\theta\,dz$$





$dxdy = r\,dr\,d\theta$
$= dr \cdot r\,d\theta$



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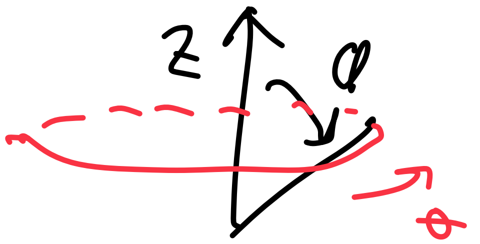
$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi$



Definition: **Spherical coordinates** use ρ, the distance to the origin as well as two **Euler angles**: $0 \le \theta < 2\pi$ the polar angle and $0 \le \phi \le \pi$, the angle between the vector and the positive z axis. The coordinate change is

$$T : (x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) .$$

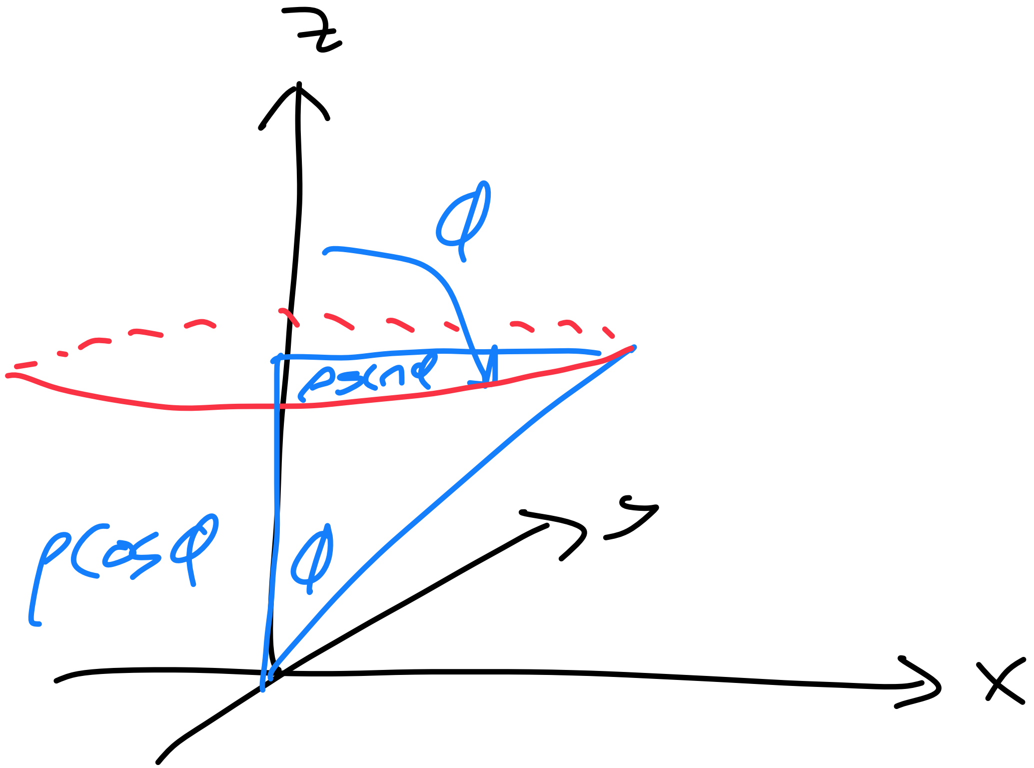
The integration factor measures the volume of a **spherical wedge** which is $d\rho \cdot \rho\sin(\phi) \cdot d\theta \cdot \rho d\phi = \rho^2 \sin(\phi) d\theta d\phi d\rho$.

$$\iiint_{T(R)} f(x,y,z)\, dxdydz = \iiint_R g(\rho,\theta,\phi) \boxed{\rho^2\sin(\phi)}\, d\rho d\theta d\phi$$

$\rho\, d\theta$

$\rho\, d\phi$

$d\rho$



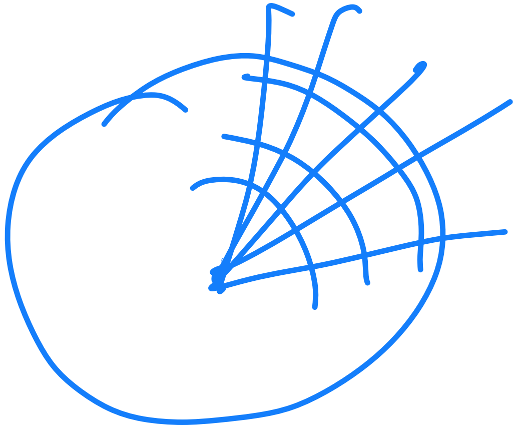
$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

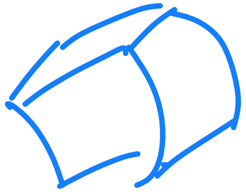
$0 \le \rho \le R$

$dx\,dy \rightarrow r\,dr\,d\theta$

$= dr \cdot r\,d\theta$



$dx\,dy\,dz \rightarrow \rho \sin\varphi \, d\rho\, d\theta\, d\varphi$



18.5. Find $\int\int\int_R z^2\, dV$ for the solid obtained by intersecting $\{1 \le x^2+y^2+z^2 \le 4\}$ with the double cone $\{z^2 \ge x^2+y^2\}$.

Solution: since the result for the double cone is twice the result for the single cone, we work with the diamond shaped region R in $\{z > 0\}$ and multiply the result at the end with 2. In spherical coordinates, the solid R is given by $1 \le \rho \le 2$ and $0 \le \phi \le \pi/4$. With $z = \rho\cos(\phi)$, we have

$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi)\sin(\phi)\; d\phi d\theta d\rho$$

$$= \left(\frac{2^5}{5} - \frac{1^5}{5}\right)2\pi\left(\frac{-\cos^3(\phi))}{3}\Big|_0^{\pi/4} = 2\pi\frac{31}{5}\left(1-2^{-3/2}\right).$$

The result for the double cone is $\boxed{4\pi(31/5)(1-1/\sqrt{2}^3)}$.

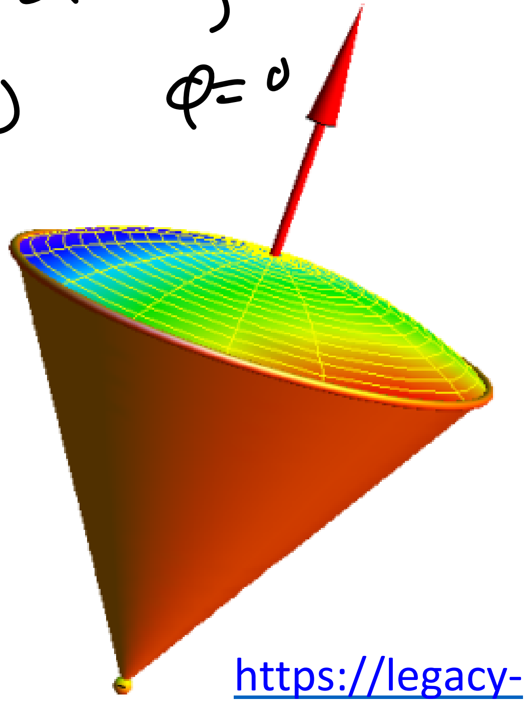
$$\int_{\rho=1}^{2} \rho^4\, d\rho \int_{\theta=0}^{2\pi} d\theta \int_{\phi=0}^{\pi/4} \cos^2(\phi)\sin\phi\, d\phi$$

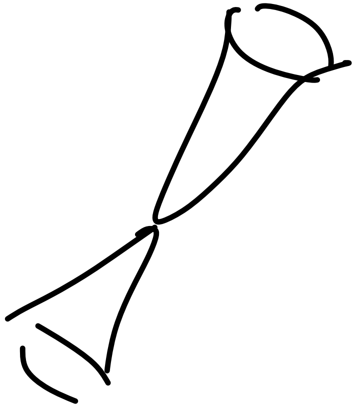
$$\int_{\rho=1}^{2}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}$$

$dx\,dy\,dz$

$= \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

or $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$





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Taylor Series

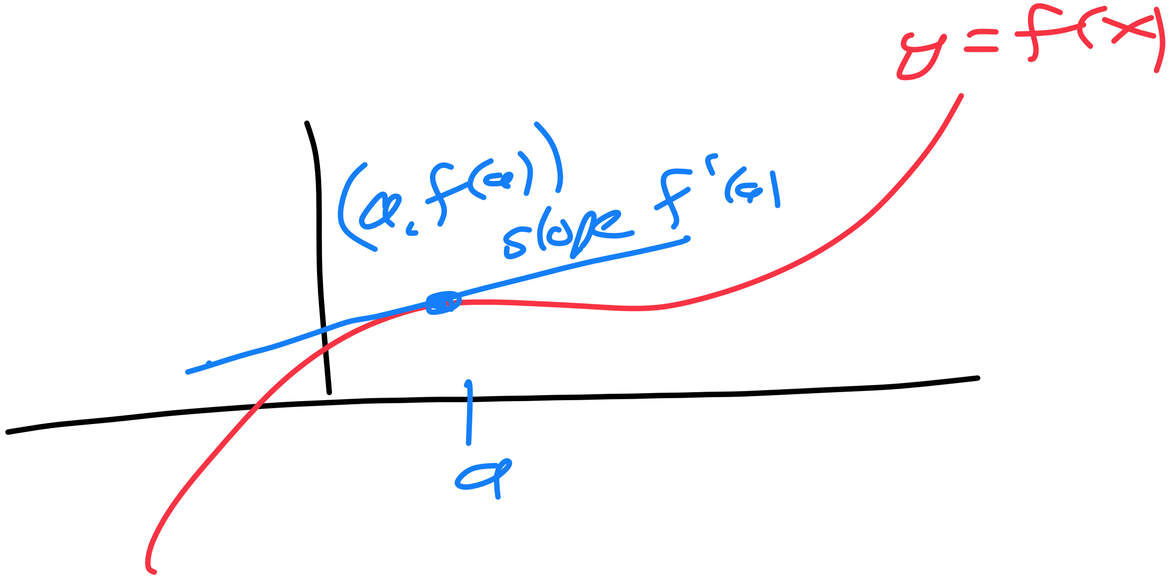
Tangent Lines

Point-Slope

$y - f(a) = f'(a)(x-a)$

$y = f(a) + f'(a)(x-a)$

- $f(a)$: start
- $f'(a)$: initial speed
- $(x-a)$: elapsed time



$$(T_N f)(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \cdots + \frac{f^{(N)}(0)}{N!}x^N$$

$$\left(\sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}x^n\right)$$

If about a use $(x-a)^n$ instead of x^n

$$(T_N f)(0) = f(0)$$

$$(T_N f)'(x) = f'(0)\cdot 1 + \frac{f''(0)\cdot 2}{2!}x + \cdots + \frac{f^{(N)}(0)\cdot N}{N!}x^{N-1}; \ (T_N f)'(0) = f'(0)$$

$$(T_N f)''(x) = \frac{f''(0)\cdot 2\cdot 1}{2!}1 + \cdots + \frac{f^{(N)}(0)\cdot N(N-1)}{N!}x^{N-2}; \ (T_N f)''(0) = f''(0)$$

First N derivs of $T_N f$ and f agree at $x=0$

From Shooting Hoops
to the Geometric Series Formula

Game of hoops: first basket wins, alternate shooting.

Bird and Magic (I’m old!) alternate shooting; first basket wins.

- Bird always gets basket with probability p.
- Magic always gets basket with probability q.

Let x be the probability Bird wins – what is x?

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1-p)(1-q)\cdot p$.
- **Bird** wins on 3rd shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.
- **Bird** wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Let $r=(1-p)(1-q)$. Then

$$\begin{aligned} x &= \text{Prob}(\textbf{Bird}\text{ wins}) \\ &= p + rp + r^2p + r^3p + \cdots \\ &= p\left(1 + r + r^2 + r^3 + \cdots\right), \end{aligned}$$

the geometric series.

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$x = \mathrm{Prob}(\textbf{Bird}\ \mathrm{wins}) = p +$$

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird}\text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$x = \text{Prob}(\textbf{Bird}\text{ wins}) = p + (1-p)(1-q)x = p + rx.$$

Thus

$$(1-r)x \;=\; p \quad\text{or}\quad x \;=\; \frac{p}{1-r}.$$

As $x = p(1 + r + r^2 + r^3 + \cdots)$, find

$$1 + r + r^2 + r^3 + \cdots \;=\; \frac{1}{1-r}.$$

Lessons from Hoop Problem

- Power of Perspective: Memoryless process.
- Can circumvent algebra with deeper understanding! (Hard)
- Depth of a problem not always what expect.
- Importance of knowing more than the minimum: connections.
- Math is fun!

Boring Proof

$$S(n) = 1 + r + r^2 + \cdots + r^{n-1} + r^n$$

$$rS(n) = \qquad r + r^2 + \cdots + r^{n-1} + r^n + r^{n+1}$$

$$(1-r)S(n) = 1 - r^{n+1}$$

So $S(n) = 1 + r + \cdots + r^n = \dfrac{1-r^{n+1}}{1-r} = \dfrac{1}{1-r} - \dfrac{r^{n+1}}{1-r}$

If $|r| < 1$ then $|r^{n+1}| \longrightarrow 0$ as $n \to \infty$

Thus $1 + r + r^2 + \cdots = \dfrac{1}{1-r}$ if $|r| < 1$