Math 150: Calculus III: Multivariable Calculus

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https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 24: 4-15-2022: https://youtu.be/NxpZlV3M2Zs

slides:

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture24.pdf

Plan for the day: Lecture 24: April 15, 2022:

Topics:

Sequences and Series (definition)

Necessary vs Sufficient Conditions for Convergence for Series

p-Series (especially Harmonic), applications to primes

Comparison Test

Applications of Taylor Series (if time permits)

Ratio / Root / Integral Tests (if time permits)

Sequence: collection of terms

Series: is the sum

Harmonic: diverges

converges

Geometric: Converges

$\{a_n\} = \{1/n\}$ or $\{1/n^2\}$ or $\{(1/2)^n\}$

$\downarrow$ $\downarrow$ $\downarrow$

$1, \frac{1}{2}, \frac{1}{3}, \ldots$ $\quad 1, \frac{1}{4}, \frac{1}{9}, \ldots$ $\quad 1, \frac{1}{2}, \frac{1}{4}, \ldots$

Sum? $\sum_{n=0}^{N} a_n$ and does the limit exist as $N \to \infty$?

Necessary Cond for a Sum to Converge:

Need $a_n \to 0$ as $n \to \infty$ for $\sum_{n=0}^{\infty} a_n$ to Converge

NOT SUFFICIENT

$$1, \underbrace{\frac{1}{2}, \frac{1}{2}}_{1}, \underbrace{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}}_{1}, \underbrace{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}}_{1}, \underbrace{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}}_{1}$$

Harmonic Series Diverges

$$1 + \underbrace{\frac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 2 \cdot \frac{1}{4} = \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 4 \cdot \frac{1}{8} = \frac{1}{2}} + \cdots + \frac{1}{15} + \cdots$$

$$\sum_{n=1}^{N} \frac{1}{n} \geq \text{something like "} \log_2(N) * \frac{1}{2} \text{"}$$

Similarly $\leq$ something like "$\log_2(N) * \frac{1}{2}$"

Assume Harmonic Sum Converges:

$$S = \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots\right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots\right) = S_{odd} + S_{even}$$

$S_{odd} > S_{even}$ as $\frac{1}{1} > \frac{1}{2}$ and $\frac{1}{3} > \frac{1}{4}$ and so on

S

$$S_{even} = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots\right) = \frac{1}{2} S$$

$$S = S_{odd} + S_{even} > S_{even} + S_{even} = 2 S_{even} = S$$

Assuming S is finite yields $S > S$!

Contradiction

Prove $\sum 1/n^2$ Converges

$$\underbrace{1}_{} + \underbrace{\frac{1}{2^2} + \frac{1}{3^2}}_{} + \underbrace{\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}}_{} + \underbrace{\frac{1}{8^2} + \frac{1}{9^2} + \cdots}_{}$$

Block k: go from $n = 2^k$ to $n = 2^{k} - 1$, # terms is 2^k

So $\frac{1}{n^2}$ goes from $\frac{1}{(2^k)^2}$ to $\frac{1}{(2^{k+1})^2} = \frac{1}{4}\frac{1}{(2^k)^2}$

$$\frac{1}{4}\frac{1}{2^{2k}} \leq \frac{1}{n^2} \leq \frac{1}{2^{2k}}$$

localizing

$$\sum_{k=0}^{\infty} \frac{2^k}{4\cdot 2^{2k}} \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le \underbrace{\sum_{k=0}^{\infty} \frac{2^k}{2^{2k}}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^k} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

Geometric, $r = 1/2$

Converges to $\frac{1}{1-1/2}$

or to 2

$$\frac{1}{4}2 \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le 2 \quad \text{or} \quad \frac{1}{2} \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le 2$$

Basel Problem; ans is $\pi^2/6$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq \sum_{k=1}^{\infty} \frac{2^k}{(2^k)^p} = \sum_{k=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^k$$

if $p > 1$ converges

if $p \leq 1$ diverges

Call this a p-series (power)

Comparison Test

Assume $0 \le a_n \le b_n$ or $0 \le c_n \le a_n$

If $\sum b_n$ converges then $\sum a_n$ converges

If $\sum c_n$ diverges then $\sum a_n$ diverges

$$x^2 - 7x + 12 = 0$$

1 · 12
2 · 6
3 · 4

$$(x - 3)(x - 4)$$

$$\frac{7 \pm \sqrt{49 - 48}}{2}$$

$$x^2 - 7x + 11 = 0$$

$$\frac{7 \pm \sqrt{49 - 4 \cdot 11}}{2}$$

Imagine $a_n \leq b_n$ and $\sum b_n$ converges.

Can $\sum_{n=1}^{\infty} a_n$ diverge?

Take $b_n = \frac{1}{n^2}$

Could take $\{a_n\} = \{-5, -5, -5, -5, -5, \ldots\}$

$-5 \leq \frac{1}{n^2}$

Imagine $c_n \le a_n$ and $\sum c_n$ diverges

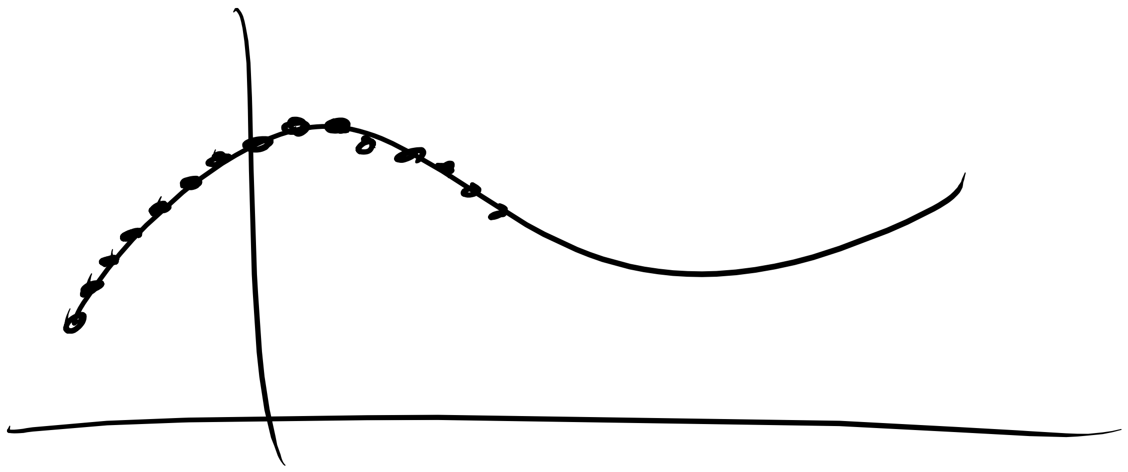
Does $\sum a_n$ have to diverge?

Try $c_n = -5$ for all n

Take $a_n = 1/n^2$

Care because of Taylor Series

$$(T_N f)(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(N)}(0)}{N!}x^N$$



Steganography:

Can you see the cat in the tree?

$$314 = 3 \cdot 10^2 + 1 \cdot 10^1 + 4 \cdot 10^0$$
$$= 1 \cdot 256 + 0 \cdot 128 + 0 \cdot 64 + 1 \cdot 32 + 1 \cdot 16 + 1 \cdot 8 + 0 \cdot 4 + 1 \cdot 2 + 0 \cdot 1$$
$$= 100111010_2$$

How to transmit an image?

- Have an $L \times W$ grid with LW pixels.
- Each pixel a triple, maybe (Red, Green, Blue).
- Often each value in $\{0, 1, 2, 3, \ldots, 2^n - 1\}$.
- $n = 8$ gives 256 choices for each, or 16,777,216 possibilities.

Steganography: Concealing a message in another message: https://en.wikipedia.org/wiki/Steganography.

Take one of the colors, say red, a number from 0 to 255.

Write in binary: $r_7 2^7 + r_6 2^6 + \cdots + r_1 2 + r_0$. $r_i \in \{0,1\}$

If change just the last or last two digits, very minor change to image.

Can hide an image in another.

If just do last, can hide a black and white image easily....

Can you see the cat in the tree?

Can you see the cat in the tree?