Math 150: Calculus III: Multivariable Calculus

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https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/

Lecture 27: 4-22-2022: https://youtu.be/o2ATZBoVxII

slides:

https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture27.pdf

Plan for the day: Lecture 27: April 22, 2022:

Topics:

Root Test

Ratio Test

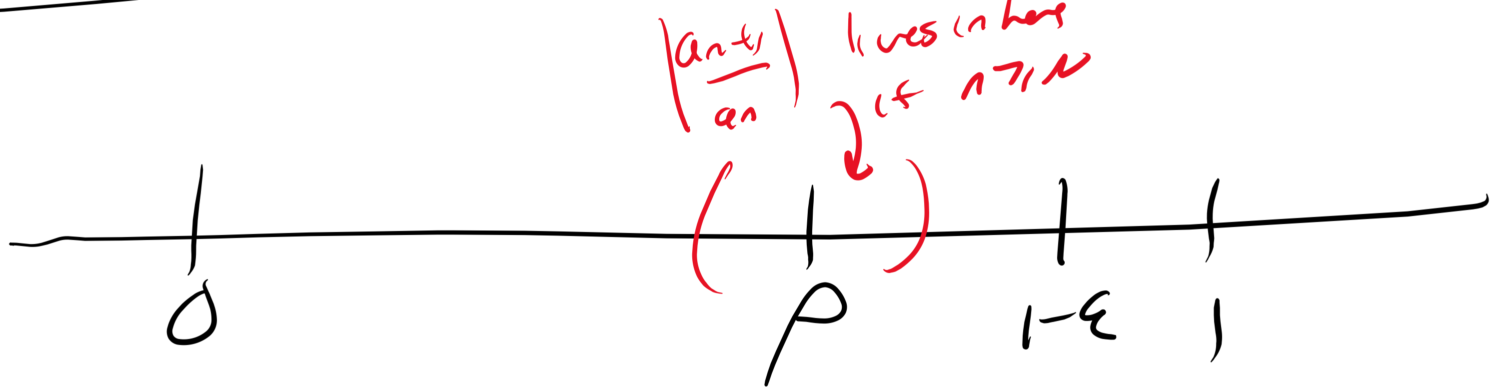
Taylor Series

Ratio Test

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \text{if is} \begin{cases} < 1 & \text{converges} \\ > 1 & \text{diverges} \\ = 1 & \text{no info} \end{cases}$$

Ex: $\rho = 1$ for $a_n = 1/n$ and $a_n = 1/n^2$

Ratio compares to a geometric series



as $\rho < 1$ there is an $\varepsilon > 0$ st

$\rho < 1-\varepsilon$

take $\varepsilon = \frac{1-\rho}{1000}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$ so $\exists$ (there exists) N such that $\forall$ (for all) $n \geq N$, $\left|\frac{a_{n+1}}{a_n}\right| < 1-\varepsilon$

$|a_{N+1}| \leq |a_N|(1-\varepsilon)$

$|a_{N+2}| \leq |a_{N+1}|(1-\varepsilon) \leq |a_N|(1-\varepsilon)^2$

$|a_{N+3}| \leq |a_{N+2}|(1-\varepsilon) \leq |a_N|(1-\varepsilon)^3$

Geometric Series, $r = 1-\varepsilon < 1$

$$\left|\sum_{n=N}^{\infty} a_n\right| \leq \sum_{n=N}^{\infty} |a_n|$$

$$\leq \sum_{k=0}^{\infty} |a_N|(1-\varepsilon)^k = \frac{|a_N|}{1-(1-\varepsilon)}$$

$$= \frac{1}{\varepsilon}|a_N| < \infty \text{ converges}$$

Ex: $\exp(x) = e^x = \sum_{n=0}^{\infty} x^n/n! = a_0 + a_1 + a_2 + \cdots$

or $a_0(x) + a_1(x) + a_2(x) + \cdots$

$a_n = a_n(x) = x^n/n!$

Compute $\rho_x = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right|$

$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right|$ as $(n+1)! = (n+1)\,n!$

as x fixed and $n \to \infty$, limit is zero!

$\Longrightarrow$ Converges for all x!

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$ $\qquad a_n(x) = \frac{x^n}{n^2 3^n} = \frac{(x/3)^n}{n^2}$

$$\rho_x = \lim_{n\to\infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \lim_{n\to\infty} \left| \frac{(x/3)^{n+1}/(n+1)^2}{(x/3)^n/n^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x/3)^{n+1}}{(x/3)^n} \cdot \frac{n^2}{(n+1)^2} \right| = \lim_{n\to\infty} \left| \left(\frac{x}{3}\right)\left(\frac{1}{1+\frac{1}{n^2}}\right) \right|$$

$$= \lim_{n\to\infty} \left|\frac{x}{3}\right| \lim_{n\to\infty} \left| \frac{1}{1+\frac{1}{n^2}} \right| = \left|\frac{x}{3}\right|$$

So if $|x| < 3$ then $\rho_x < 1$ and converges

if $|x| > 3$ then $\rho_x > 1$ diverges

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$$

Converges if $|x|<3$

diverges if $|x|>3$

if $|x|=3$

$$\sum \frac{1}{n^2} \text{ or } \sum \frac{(-1)^n}{n^2}$$

both converge

Converges iff

$$-3 \leq x \leq 3$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n 3^n}$$

converges $|x|<3$

diverges $|x|>3$

if $|x|=3$

$$\sum \frac{1}{n} \text{ or } \sum \frac{(-1)^n}{n}$$

diverges converges

Converges iff

$$-3 \leq x < 3$$

Ex: Show $\sum n^2 x^n / 3^n$

Converges iff $|x| < 3$

Root Test: $\sum_{n=0}^{\infty} a_n$

$$\rho = \lim_{n\to\infty} |a_n|^{1/n} \quad \text{if this is} \begin{cases} <1 & \text{converges} \\ >1 & \text{diverges} \\ =1 & \text{no info} \end{cases}$$

Consider $\sum_{n=0}^{\infty} x^n/n!$ $\qquad a_n(x) = x^n/n!$

$$\rho_x = \lim_{n\to\infty} |a_n(x)|^{1/n} = \lim_{n\to\infty} \left|\left(x^n/n!\right)^{1/n}\right| = \lim_{n\to\infty} \frac{|x|}{n!^{1/n}}$$

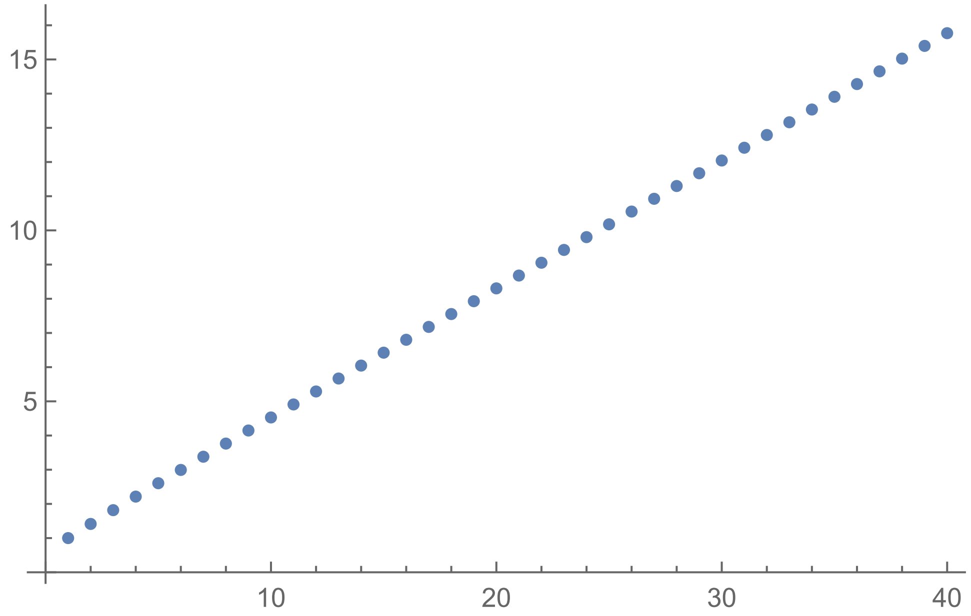
$$= |x| \Big/ \lim_{n\to\infty} n!^{1/n}$$

Conjecture that

$$\lim_{n\to\infty} n!^{1/n} = \infty$$

In[•]:=
```
list = {};
For[n = 1, n ≤ 40, n++,
 list = AppendTo[list, {n, n!^(1./n)}]]
ListPlot[list]
```

Out[•]=



Study $\lim_{n \to \infty} n!^{1/n}$

Expect limit is ∞ so find a lower bound going to ∞

$$n! = 1 \cdot 2 \cdots \frac{n}{2} \cdot \underbrace{\left(\frac{n}{2}+1\right) \cdots n}_{\text{replace with } n/2} \geq \left(\frac{n}{2}\right)^{n/2}$$

$$n!^{1/n} \geq \left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right)^{\frac{1}{n}} = \left(\frac{n}{2}\right)^{\frac{n}{2} \cdot \frac{1}{n}} = \left(\frac{n}{2}\right)^{\frac{1}{2}} = \sqrt{\frac{n}{2}}$$

Goes to infinity!

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but Ratio Test is Useless:

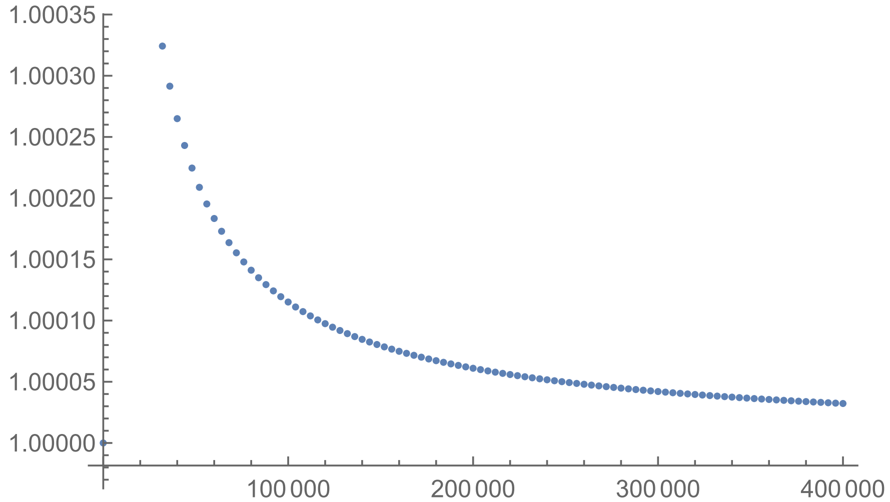
gives $\rho = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to\infty} \frac{n+1}{n} = 1$

Try Root:

$\rho = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1 \Big/ \lim_{n\to\infty} n^{1/n}$

```
In[•]:= list = {};
For[n = 1, n ≤ 400001, n = n + 4000,
 list = AppendTo[list, {n, n^(1./n)}]]
ListPlot[list]
```

Out[•]=



Suggests limit is 1

Thus would get no information

Prove $\lim_{n\to\infty} n^{1/n} = 1$

See product, think logarithm

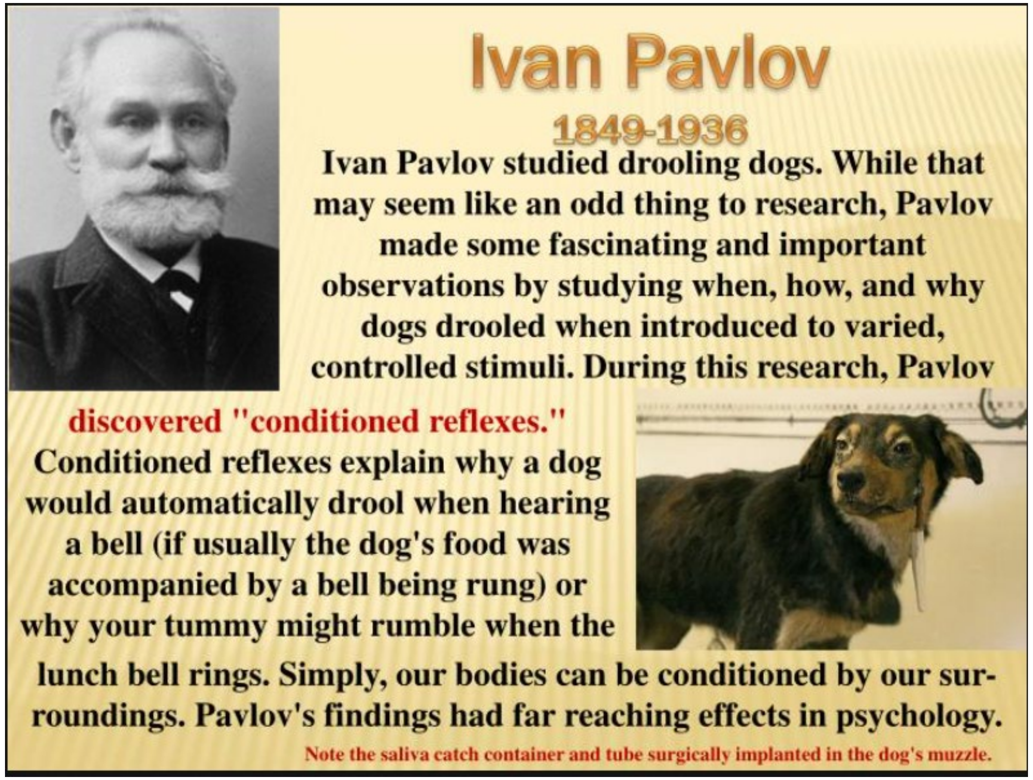
Study lim of $\log(n^{1/n})$ and then exponentiate, solves the problem

$$\lim_{n\to\infty} \log(n^{1/n}) = \lim_{n\to\infty} \frac{\log n}{n}$$

L'Hopital

$$= \lim_{n\to\infty} \frac{1/n}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

So $\log(n^{1/n}) \to 0$ so $n^{1/n} \to 1$



Intuition: $n = 1024 = 2^{10}$

$$n^{1/n} = (2^{10})^{\frac{1}{2^{10}}}$$

10^{th} root gets us to 2

20^{th} root gets us to $\sqrt{2}$

40^{th} root is $\sqrt[4]{2}$

80^{th} root is $\sqrt[8]{2}$