

# Math 150: Calculus III: Multivariable Calculus

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[https://web.williams.edu/Mathematics/sjmiller/public\\_html/150Sp22/](https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/)

**Lecture 27: 4-22-2022:** <https://youtu.be/o2ATZBoVxII>

slides:

[https://web.williams.edu/Mathematics/sjmiller/public\\_html/150Sp22/talks2022/Math150Sp22\\_lecture27.pdf](https://web.williams.edu/Mathematics/sjmiller/public_html/150Sp22/talks2022/Math150Sp22_lecture27.pdf)

## **Plan for the day: Lecture 27: April 22, 2022:**

### **Topics:**

**Root Test**

**Ratio Test**

**Taylor Series**

# Ratio Test

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

(S) (S)

$$\left\{ \begin{array}{l} < 1 \\ > 1 \\ = 1 \end{array} \right.$$

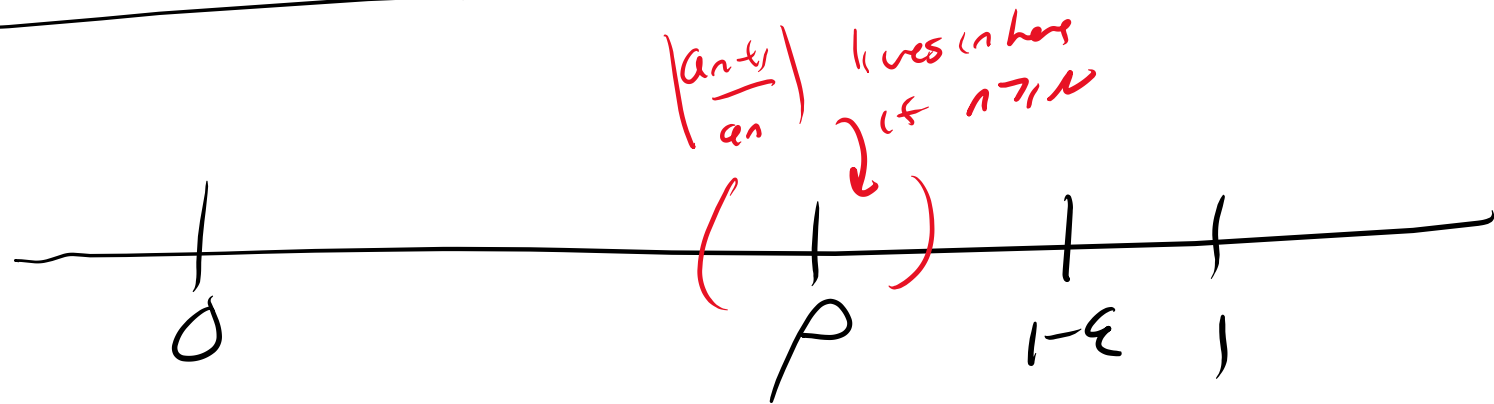
converges

diverges

no info

Ex:  $\rho = 1$  for  $a_n = 1/n$  and  $a_n = 1/n^2$

# Ratio compares to a geometric series



as  $\rho < 1$  There is an  $\epsilon > 0$  st  $\rho < 1 - \epsilon$   
 take  $\epsilon = \frac{1 - \rho}{1000}$

$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$  so  $\exists N$  such that  $\forall n > N$ ,  $\left| \frac{a_{n+1}}{a_n} \right| < 1 - \epsilon$   
 (There exists) (for all)

$|a_{n+1}| \leq |a_n| (1 - \epsilon)$   
 $|a_{n+2}| \leq |a_{n+1}| (1 - \epsilon) \leq |a_n| (1 - \epsilon)^2$   
 $|a_{n+3}| \leq |a_{n+2}| (1 - \epsilon) \leq |a_n| (1 - \epsilon)^3$   
 Geometric Series,  $r = 1 - \epsilon < 1$

$$\begin{aligned}
 \left| \sum_{n=N}^{\infty} a_n \right| &\leq \sum_{n=N}^{\infty} |a_n| \\
 &\leq \sum_{k=0}^{\infty} |a_n| (1 - \epsilon)^k = \frac{|a_n|}{1 - (1 - \epsilon)} \\
 &= \frac{1}{\epsilon} |a_n| < \infty \text{ converges}
 \end{aligned}$$

Ex:  $\exp(x) = e^x = \sum_{n=0}^{\infty} x^n / n! = a_0 + a_1 + a_2 + \dots$   
 or  $a_0(x) + a_1(x) + a_2(x) + \dots$

$$a_n = a_n(x) = x^n / n!$$

Compute  $\rho_x = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1} / (n+1)!}{x^n / n!} \right|$   
 $= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right|$  as  $(n+1)! = (n+1)n!$

as  $x$  fixed and  $n \rightarrow \infty$ , limit is zero!

$\Rightarrow$  converges for all  $x$ !

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$$

$$a_n(x) = \frac{x^n}{n^2 3^n} = \frac{(x/3)^n}{n^2}$$

$$\rho_x = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x/3)^{n+1} / (n+1)^2}{(x/3)^n / n^2} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(x/3)^{n+1}}{(x/3)^n} \cdot \frac{n^2}{(n+1)^2} \right| = \lim_{n \rightarrow \infty} \left| \left( \frac{x}{3} \right) \left( \frac{1}{1 + \frac{1}{n^2}} \right) \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x}{3} \right| \lim_{n \rightarrow \infty} \left| \frac{1}{1 + \frac{1}{n^2}} \right| = \left| \frac{x}{3} \right|$$

So if  $|x| < 3$  then  $\rho_x < 1$  and converges  
if  $|x| > 3$  then  $\rho_x > 1$  diverges

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$$

converges if  $|x| < 3$

diverges if  $|x| > 3$

if  $|x| = 3$

$$\sum \frac{1}{n^2} \text{ or } \sum \frac{(-1)^n}{n^2}$$

both converge

converges iff

$$-3 \leq x \leq 3$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n 3^n}$$

converges if  $|x| < 3$

diverges if  $|x| > 3$

if  $|x| = 3$

$$\sum \frac{1}{n} \text{ or } \sum \frac{(-1)^n}{n}$$

diverges

converges

converges iff

$$-3 \leq x < 3$$

Ex: Show  $\sum n^2 x^n / 3^n$

Converges iff  $|x| < 3$



Root Test:  $\sum_{n=0}^{\infty} a_n$

$\rho = \lim_{n \rightarrow \infty} |a_n|^{1/n}$  if this is  $\begin{cases} < 1 \\ > 1 \\ = 1 \end{cases}$   $\begin{matrix} \text{converges} \\ \text{diverges} \\ \text{no info} \end{matrix}$

Consider  $\sum_{n=0}^{\infty} x^n/n!$       $a_n(x) = x^n/n!$

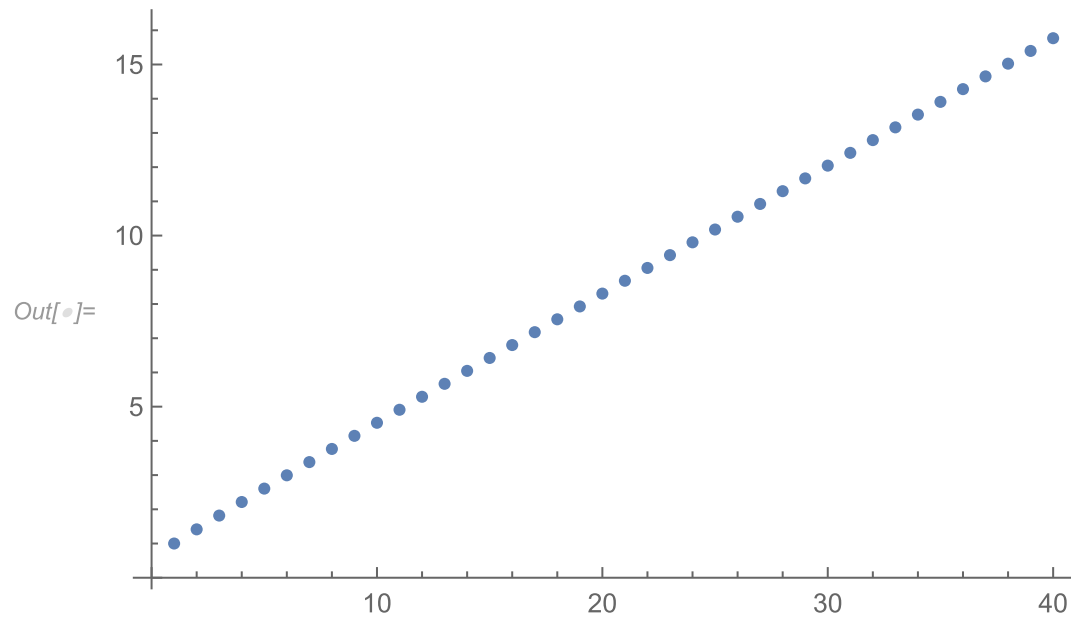
$$\rho_x = \lim_{n \rightarrow \infty} |a_n(x)|^{1/n} = \lim_{n \rightarrow \infty} |(x^n/n!)^{1/n}| = \lim_{n \rightarrow \infty} \frac{|x|}{n!^{1/n}}$$

$$= |x| / \lim_{n \rightarrow \infty} n!^{1/n}$$

Conjecture that

$$\lim_{n \rightarrow \infty} n!^{1/n} = \infty$$

```
In[•]:= list = {};  
For[n = 1, n ≤ 40, n++,  
  list = AppendTo[list, {n, n!^(1./n)}]]  
ListPlot[list]
```



Study  $\lim_{n \rightarrow \infty} n!^{1/n}$

Expect limit is  $\infty$  so find a lower bound going to  $\infty$

$$n! = 1 \cdot 2 \cdot \dots \cdot \frac{n}{2} \cdot \underbrace{\left(\frac{n}{2} + 1\right) \cdot \dots \cdot n}_{\text{replace with } n/2} \geq \left(\frac{n}{2}\right)^{n/2}$$

$$n!^{1/n} \geq \left(\left(\frac{n}{2}\right)^{n/2}\right)^{1/n} = \left(\frac{n}{2}\right)^{n/2 \cdot 1/n} = \left(\frac{n}{2}\right)^{1/2} = \sqrt{\frac{n}{2}}$$

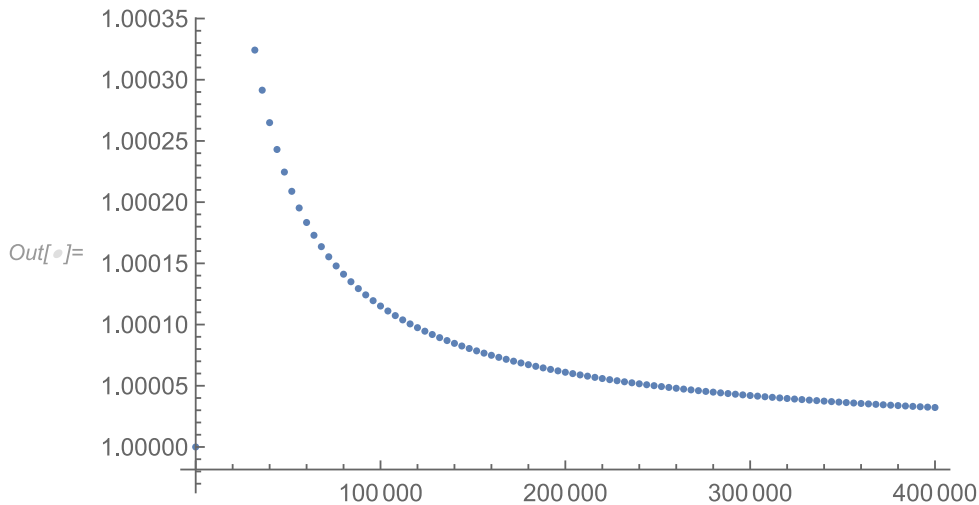
Goes to infinity!

$\sum_{n=1}^{\infty} \frac{1}{n}$  diverges but Ratio Test is useless!  
 gives  $\rho = \lim_{n \rightarrow \infty} \frac{\sqrt[n+1]}{\sqrt[n]} = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$

Try Root:

$$\rho = \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right)^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{n^{1/n}} = \lim_{n \rightarrow \infty} n^{1/n}$$

```
In[ ]:= list = {};
For[n = 1, n <= 400001, n = n + 4000,
  list = AppendTo[list, {n, n^(1./n)}]]
ListPlot[list]
```



Suggests  
 limit is 1

This would  
 get no  
 information

Prove  $\lim_{n \rightarrow \infty} n^{1/n} = 1$

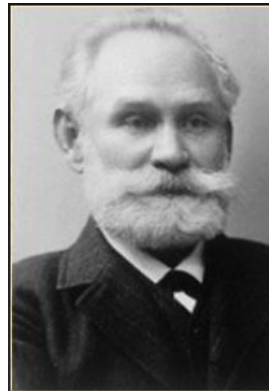
See product, think logarithm  
Study  $\lim$  of  $\log(n^{1/n})$  and then  
exponentiate, solves the problem

$$\lim_{n \rightarrow \infty} \log(n^{1/n}) = \lim_{n \rightarrow \infty} \frac{\log n}{n}$$

L'Hopital  $\downarrow$

$$= \lim_{n \rightarrow \infty} \frac{1/n}{1} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

So  $\log(n^{1/n}) \rightarrow 0$  so  $n^{1/n} \rightarrow 1$



## Ivan Pavlov

1849-1936

Ivan Pavlov studied drooling dogs. While that may seem like an odd thing to research, Pavlov made some fascinating and important observations by studying when, how, and why dogs drooled when introduced to varied, controlled stimuli. During this research, Pavlov

**discovered "conditioned reflexes."**

Conditioned reflexes explain why a dog would automatically drool when hearing a bell (if usually the dog's food was accompanied by a bell being rung) or why your tummy might rumble when the



lunch bell rings. Simply, our bodies can be conditioned by our surroundings. Pavlov's findings had far reaching effects in psychology.

Note the saliva catch container and tube surgically implanted in the dog's muzzle.

Factoring:  $n = 1024 = 2^{10}$

$$n^{1/n} = (2^{10})^{\frac{1}{2^{10}}}$$

10<sup>th</sup> root gets us to 2

20<sup>th</sup> root gets us to  $\sqrt{2}$

40<sup>th</sup> root is  $\sqrt[4]{2}$

80<sup>th</sup> root is  $\sqrt[8]{2}$



