$\vec{v}$ and $\vec{w}$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n$$

$$\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = v_1 w_1 + \cdots + v_n w_n$$

$\vec{v}^T \vec{w}$ $\quad$ $\vec{w}$ column vector
$\vec{v}$ row vector

More generally, consider a ~~2~~ matrix with 2 rows and $n$ columns:

$$\begin{pmatrix} \leftarrow \vec{u} \rightarrow \\ \leftarrow \vec{v} \rightarrow \end{pmatrix} = \begin{pmatrix} u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{pmatrix}$$

We define the product of this with $\vec{w}$ by

$$\begin{pmatrix} u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} \vec{u} \cdot \vec{w} \\ \vec{v} \cdot \vec{w} \end{pmatrix} = \begin{pmatrix} u_1 w_1 + \cdots + u_n w_n \\ v_1 w_1 + \cdots + v_n w_n \end{pmatrix}$$

For our problem

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1\cdot 3 + 2\cdot 4 \\ 2\cdot 3 + 4\cdot 4 \\ 3\cdot 3 + 6\cdot 4 \end{pmatrix} = \begin{pmatrix} 11 \\ 22 \\ 33 \end{pmatrix}$$

view this as $(1,2)\cdot(3,4) = 11$

$$(2,4)\cdot(3,4) = 22$$

$$(3,6)\cdot(3,4) = 33$$

Motivation: $A$ has $m$ rows and $n$ columns

$$A : \mathbb{R}^m \to \mathbb{R}^n$$

Ex: $A : \mathbb{R}^2 \to \mathbb{R}^3$ by $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1x + 2y \\ 3x + 4y \\ 5x + 6y \end{pmatrix}$$

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Consider $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$A : \mathbb{R}^2 \to \mathbb{R}^2$
$B : \mathbb{R}^2 \to \mathbb{R}^2$

Let $C = BA$

So $C \begin{pmatrix} x \\ y \end{pmatrix} = B \left( A \begin{pmatrix} x \\ y \end{pmatrix} \right)$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix}$

$$A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}$$

$$A\begin{pmatrix}x\\y\end{pmatrix} = A\left[x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$= xA\begin{pmatrix}1\\0\end{pmatrix} + yA\begin{pmatrix}0\\1\end{pmatrix}$$ Calculation

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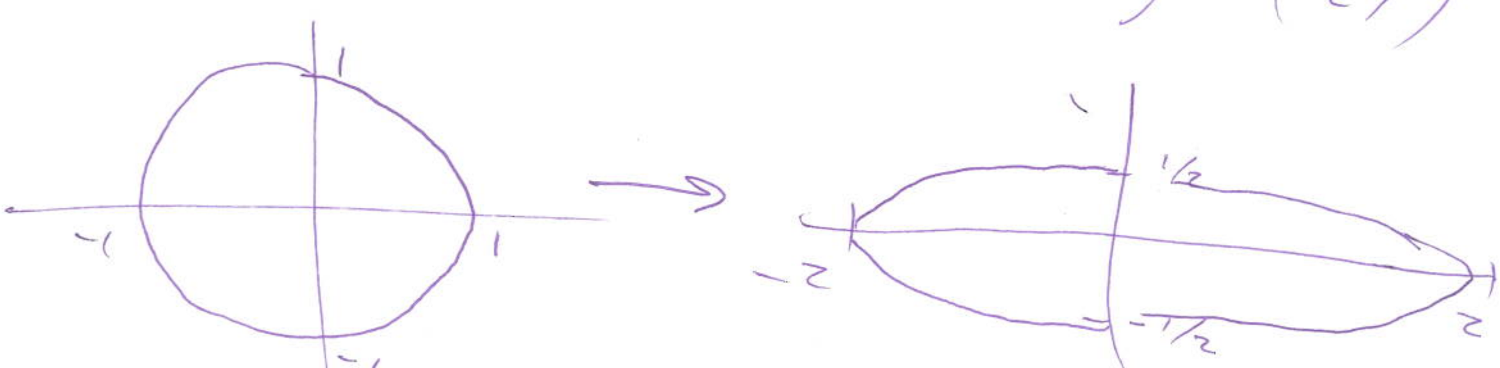
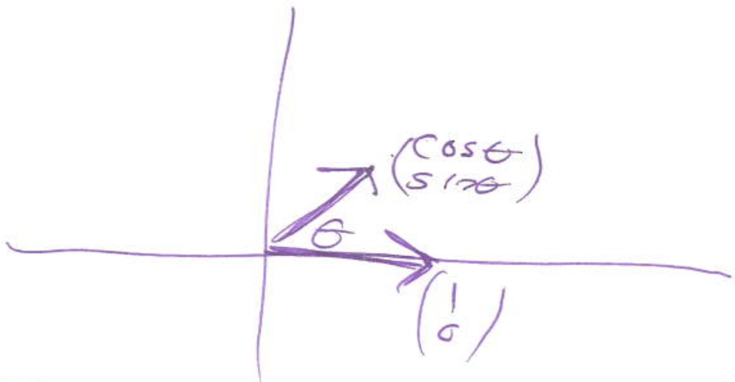
Stretch x-dir by 2, compress y-dir by 2

$$\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x + 0y\\0x + \frac{1}{2}y\end{pmatrix} = \begin{pmatrix}2x\\\frac{1}{2}y\end{pmatrix}$$



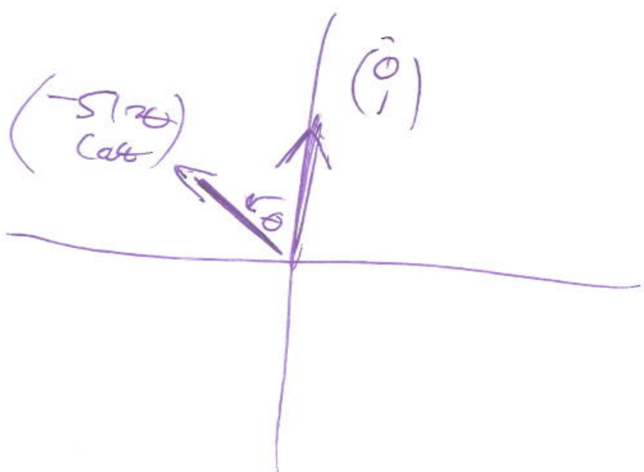
Conside $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\hookrightarrow R(\theta)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$




Rotates by $\theta$ degrees

$R(\theta)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$



$R(\theta)$ rotates by $\theta$ radians counter-clock
$R(\varphi)$ " " $\varphi$ " "

$R(\varphi)R(\theta)$ rotates $\theta+\varphi$ radians counter-clock

<u>Claim</u>: $R(\varphi)R(\theta) = R(\varphi+\theta)$

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\stackrel{?}{=} \begin{pmatrix} \cos(\varphi+\theta) & -\sin(\varphi+\theta) \\ \sin(\varphi+\theta) & \cos(\varphi+\theta) \end{pmatrix}$$

$$\cos(\varphi+\theta) = \cos\varphi\cos\theta - \sin\varphi\sin\theta$$

$$\sin(\varphi+\theta) = \sin\varphi\cos\theta + \cos\varphi\sin\theta$$

This is why we multiply matrices as we do.