Chapter 11: Vectors, Curves and Surfaces in Space

<u>Goals</u>: • want to study quantities involving several vars
- need to develop good notation, basic relations
- much extended in linear algebra

<u>Sections</u>: 11.1, 11.2, 11.3, 11.4, 11.8 (5 total)

<u>Motivation</u>: Orbits of planets. Trip to rare books library to see first editions. Took a long time to isolate concept of vectors in general
↳ Adv Reading: Quaternions, Gibbs

Section 11.1: Vectors in the Plane

- Generalizes readily to n-dimensional space
- Notation: $\mathbb{N} = \{0, 1, 2, \ldots\}$ natural numbers

  $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ integers (German zahl)

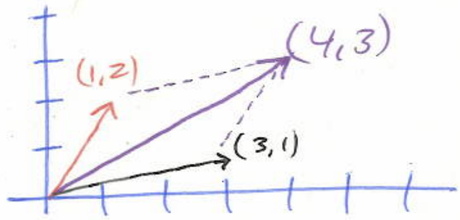
  $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z} \text{ and } q \neq 0\}$ rationals (quotients)

  $\mathbb{R}$ = Reals

  $\mathbb{C}$ = Complex

SEC 11.1: VECTORS IN THE PLANE (CONT)

Cartesian Coordinates



Vectors: magnitude and direction

Add componentwise!

$\vec{x} = (x_1, \dots, x_n)$

$\vec{y} = (y_1, \dots, y_n)$

$\vec{x} + \vec{y} = (x_1 + y_1, \dots, x_n + y_n)$

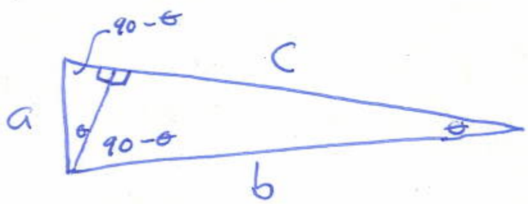
$\alpha \vec{x} = (\alpha x_1, \dots, \alpha x_n)$

Note: book uses bold for vectors and $\langle x_1, \dots, x_n \rangle$

Using $\langle\ \rangle$ is non-standard: I'll try but both equivalent.

KEY INPUT: PYTHAGOREAN THM

Advanced Proof (feel free to skip)



area is proportional to hypotenuse^2

say $f(\theta) \cdot (hyp)^2$, $f(\theta) \neq 0$

So $f(\theta) \cdot a^2 + f(\theta) b^2 = f(\theta) c^2$

$\Rightarrow a^2 + b^2 = c^2$

Unit analysis!

Very powerful in physics

BASIS:

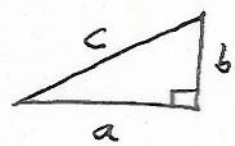
Standard: $\vec{i}, \vec{j}, \vec{k}$ or $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$

Sometimes use hat to indicate unit length: $\hat{i}, \hat{j}, \hat{k}$

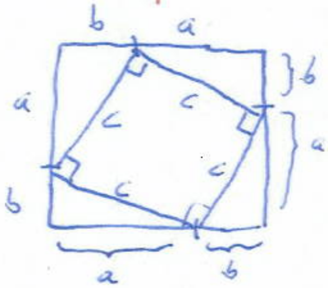
↳ How do we measure length?

PYTHAGOREAN THM

$$c^2 = a^2 + b^2$$



↳ So important we'll prove (Cleveland)



Thus $4 \cdot \frac{1}{2} ab + c^2 = (a+b)^2$

$2ab + c^2 = a^2 + 2ab + b^2$

$\Longrightarrow c^2 = a^2 + b^2$ ∎ QED

↳ Generalize to higher dimensions (Good Extra Credit: Prove!)

$\vec{V} = (x_1, \ldots, x_n)$

Then length of $\vec{v}$, denoted $\|\vec{v}\|$, is

$$\|\vec{v}\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Say unit length if $\|\vec{v}\| = 1$

(Proof in general use induction)

Book uses $|\vec{v}|$ instead: I don't like this notation as overloads absolute value and can forget have a vector

PROPERTIES OF VECTOR ADDITION

Everything you would expect : $\vec{x}, \vec{y}, \vec{z}$ vectors, α, β scalars

↳ assoc : $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$

comm : $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

distributive: $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$

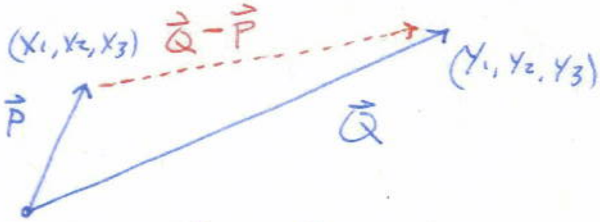
$(\alpha+\beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$

Special: $0 \cdot \vec{x} = \vec{0}, \quad 1 \cdot \vec{x} = \vec{x}$

USING A BASIS

$$\begin{aligned}(3, 4, 1701) &= 3\hat{i} + \cancel{3}4\hat{j} + 1701\hat{k} \\ &= 3\vec{i} + 4\vec{j} + 1701\vec{k} \\ &= 3\vec{e}_1 + 4\vec{e}_2 + 1701\vec{e}_3\end{aligned}$$

ADDING / SUBTRACTING VECTORS



$$\vec{Q} - \vec{P} = (y_1 - x_1, y_2 - x_2, y_3 - x_3)$$

Other notations: $\vec{P}_1 = (x_1, y_1, z_1)$

$\vec{P}_2 = (x_2, y_2, z_2)$

or Q is $\vec{P}' = (x_1', x_2', x_3')$ ...

SECTION 11-1 (CONT)

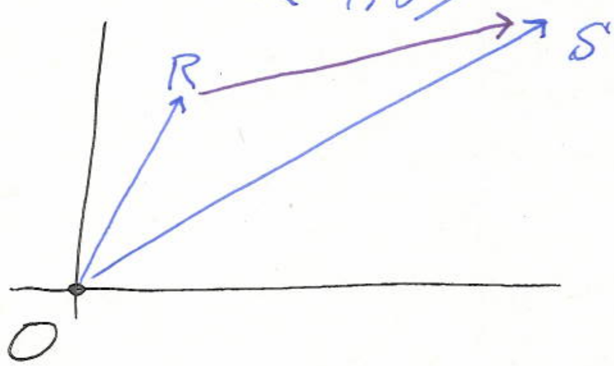
unit vectors have length 1.

<u>Ex:</u> $\vec{u} = (4,3) = \langle 4,3 \rangle$

Then $\|\vec{u}\| = \sqrt{4^2+3^2} = 5 = |\vec{u}|$

so unit vector is $\frac{\vec{u}}{|\vec{u}|} = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$

<u>Ex</u>: Given point $R = (1,6)$ and $S = (5,9)$, the vector from R to S is $\langle 5,9 \rangle - \langle 1,6 \rangle$ which is $\langle 4,3 \rangle$



<u>Homework</u>: Pg 823: #9, #18, #38, #42

<u>Additional (do not hand in)</u>: is #38 true for all points?
IE, if take any 3 points in the plane...

SECTION 11-2: Three Dimensional Vectors

- Other than more components, essentially the same
- If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ then

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

- If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ then

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \text{ and}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

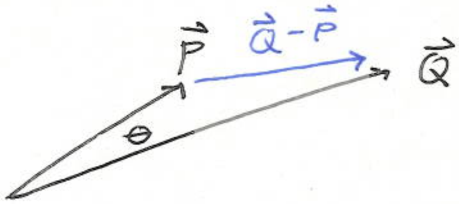
- <u>Do example</u>

DOT PRODUCT: $\vec{P} = \langle x_1, \dots, x_n \rangle$ and $\vec{Q} = \langle y_1, \dots, y_n \rangle$

Then $\vec{P} \cdot \vec{Q}$ is defined to be $x_1y_1 + \dots + x_ny_n$.

> Skip everything on projections - This should be saved for a linear algebra class.

MAIN THM: $\vec{P} \cdot \vec{Q} = |\vec{P}|\,|\vec{Q}| \cos\theta$



Note good features: formula unchanged if send $\vec{P}$ to $\alpha\vec{P}$ and $\vec{Q}$ to $\alpha\vec{Q}$.

Will sketch proof later.

Ex: $\vec{P} = \langle 2, 0 \rangle$, $\vec{Q} = \langle 1, \sqrt{3} \rangle$

Then $|\vec{P}| = \sqrt{2^2 + 0^2} = 2$

$|\vec{Q}| = \sqrt{1^2 + \sqrt{3}^2} = 2$

$\vec{P} \cdot \vec{Q} = 2 \cdot 1 + 0 \cdot \sqrt{3} = 2$

so $\vec{P} \cdot \vec{Q} = |\vec{P}|\,|\vec{Q}| \cos\theta$

gives $\cos\theta = \dfrac{\vec{P} \cdot \vec{Q}}{|\vec{P}|\,|\vec{Q}|} = \dfrac{2}{2 \cdot 2} = \dfrac{1}{2}$

$\Rightarrow \theta = \cos^{-1}(1/2) = 60^\circ$ or $\pi/3$ radians

Homework: Page 833: #1, #39, and also: find the cosine of the angle between $\vec{a} = \langle 2, 5, -4 \rangle$ and $\vec{b} = \langle 1, -2, -3 \rangle$.

Suggested (Do not hand in): #59, #61

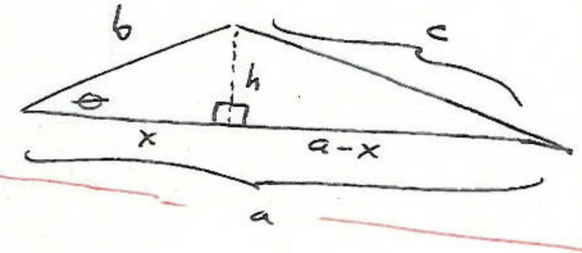
OPTIONAL, ADVANCED APPENDIX!!!

SECTION 11.2: INNER PRODUCT, LENGTH AND DISTANCE

GOAL: understand angle between vectors

LAW OF COSINES:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



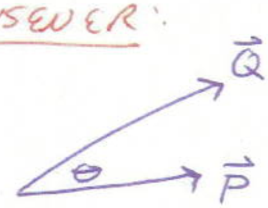
Proof: Drop auxiliary line h, two right triangles

$$h^2 = b^2 - x^2 = c^2 - (a-x)^2$$

Expanding $\Rightarrow$ $c^2 = a^2 + b^2 - 2ax$, but $x = b\cos\theta$ ∎

QUESTION: Given two vectors $\vec{P}$ and $\vec{Q}$, find angle b/w them in terms of coords

ANSWER:



$$\vec{P}\cdot\vec{Q} = \|\vec{P}\|\,\|\vec{Q}\|\cos\theta$$

with $\vec{P} = (p_1, \ldots, p_n)$ $\vec{Q} = (q_1, \ldots, q_n)$

and $\vec{P}\cdot\vec{Q} = p_1q_1 + \cdots + p_nq_n = \sum_{i=1}^{n} p_iq_i$

Call $\vec{P}\cdot\vec{Q}$ the dot product or the inner product

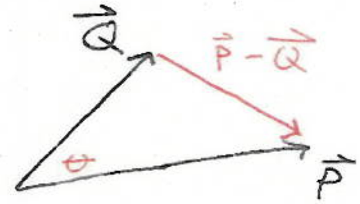
↳ EMPHASIZE GOOD FEATURES OF FORMULA *

↳ $\vec{P} \to \alpha\vec{P}$, $\vec{Q} \to \beta\vec{Q}$ doesn't change angle

OPTIONAL, ADVANCED APPENDIX!!!

SECTION 11.2 (CONT)

PROOF OF ANGLE FORMULA



$$\|\vec{P}\|^2 = p_1^2 + \cdots + p_n^2 = \vec{P}\cdot\vec{P}$$

$$\|\vec{Q}\|^2 = q_1^2 + \cdots + q_n^2 = \vec{Q}\cdot\vec{Q}$$

Law of Cosines: $\|\vec{P}-\vec{Q}\|^2 = \|\vec{P}\|^2 + \|\vec{Q}\|^2 - 2\|\vec{P}\|\,\|\vec{Q}\|\cos\theta$

so $\|(p_1-q_1, \dots, p_n-q_n)\|^2 = \|\vec{P}\|^2 + \|\vec{Q}\|^2 - 2\|\vec{P}\|\,\|\vec{Q}\|\cos\theta$

$$\sum_{i=1}^{n} (p_i - q_i)^2 = \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} q_i^2 - 2\|\vec{P}\|\,\|\vec{Q}\|\cos\theta$$

$$\sum_{i=1}^{n}(\cancel{p_i^2} - \cancel{2}p_iq_i + \cancel{q_i^2}) = \cancel{\sum_{i=1}^{n} p_i^2} + \cancel{\sum_{i=1}^{n} q_i^2} - \cancel{2}\|\vec{P}\|\,\|\vec{Q}\|\cos\theta$$

$$\Rightarrow \sum_{i=1}^{n} p_iq_i = \|\vec{P}\|\,\|\vec{Q}\|\cos\theta$$

or $\vec{P}\cdot\vec{Q} = \|\vec{P}\|\,\|\vec{Q}\|\cos\theta$ $\blacksquare$

↳ "Faster" Proof

$$\|\vec{P}-\vec{Q}\|^2 = (\vec{P}-\vec{Q})\cdot(\vec{P}-\vec{Q}) = \|\vec{P}\|^2 - 2\,\vec{P}\cdot\vec{Q} + \|\vec{Q}\|^2$$

↳ Corollary: Cauchy-Schwarz Ineq

$$|\vec{P}\cdot\vec{Q}| \le \|\vec{P}\|\,\|\vec{Q}\|$$

↳ Triangle Inequality

$\|\vec{P}+\vec{Q}\| \le \|\vec{P}\| + \|\vec{Q}\|$: Proof: square both sides and compare

note $|\cos\theta| \le 1$

Section 11.3: Matrices, Determinants and the Cross Product

Need linear algebra for $n \geq 4$, not too bad if $n \leq 3$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Then $\operatorname{Det}(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ Then $\operatorname{Det}(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb$

↳ Trick:

$$\begin{array}{ccccc} & & - & - & - \\ \left|\begin{matrix} a \\ d \\ g \end{matrix}\right. & \begin{matrix} b \\ e \\ h \end{matrix} & \left.\begin{matrix} c \\ f \\ i \end{matrix}\right| & \begin{matrix} a \\ d \\ g \end{matrix} & \begin{matrix} b \\ e \\ h \end{matrix} \\ & & + & + & + \end{array}$$

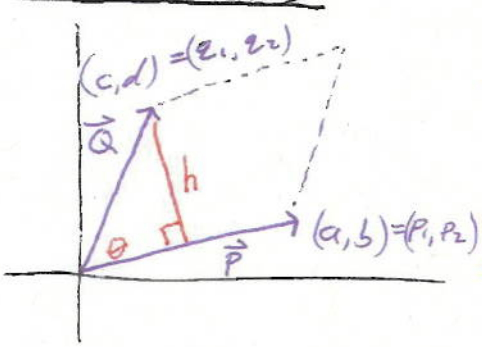
copy first two columns
6 products to combine
↳ Three with + sign
Three with − sign

↳ **WARNING: ONLY WORKS FOR $n=3$!!**

Geometric Interpretation

$A = \begin{pmatrix} \cdots \vec{v}_1 \cdots \\ \vdots \\ \cdots \vec{v}_n \cdots \end{pmatrix}$, determinant gives signed volume for generalized parallelogram spanned by $\vec{v}_1, \ldots, \vec{v}_n$.

↳ 2 Dimensions



Area is base $*$ height (prove if desired)
base is $\|\vec{P}\|$, $h = \|\vec{Q}\| \sin\theta$ ~~cos~~
now $\cos\theta = \vec{P}\cdot\vec{Q} / \|\vec{P}\|\,\|\vec{Q}\|$
so Area = ?

SECTION 11.3 (CONT)

GEOMETRIC INTERPRETATION (CONT)

Area is $\|\vec{P}\| \, \|\vec{Q}\| \sin\theta$, $\sin\theta = (1-\cos^2\theta)^{1/2}$

$$\text{Area}^2 = \|\vec{P}\|^2 \|\vec{Q}\|^2 (1-\cos^2\theta)$$

$$= \|\vec{P}\|^2 \|\vec{Q}\|^2 - \|\vec{P}\|^2 \|\vec{Q}\|^2 \left(\frac{\vec{P}\cdot\vec{Q}}{\|\vec{P}\|\,\|\vec{Q}\|}\right)^2$$

$$= \|\vec{P}\|^2 \|\vec{Q}\|^2 - (\vec{P}\cdot\vec{Q})^2$$

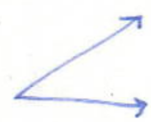
$$= (p_1^2+p_2^2)(q_1^2+q_2^2) - (p_1q_1+p_2q_2)^2$$

algebra

$$= (p_1q_2 - p_2q_1)^2 \;!$$

unenlightening! algebra

↳ Gain intuition from special cases:



This proof is entirely optional, and is meant to motivate our defn of the determinant which is done in detail in Lin Algebra.

CROSS PRODUCT

Only in $\mathbb{R}^3$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = (p_2q_3 - p_3q_2)\hat{i} - (p_1q_3 + p_3q_1)\hat{j} + (p_1q_2 - p_2q_1)\hat{k}$$

$$= \vec{P} \times \vec{Q} \qquad = (p_2q_3 - p_3q_2,\; p_3q_1 - p_1q_3,\; p_1q_2 - p_2q_1)$$

Ex: $\vec{v} = \langle 1, 2, 3 \rangle$

$\vec{w} = \langle 2, -4, 0 \rangle$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & -4 & 0 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ 1 & 2 \\ 2 & -4 \end{matrix}$$

$$= \hat{i}(2\cdot 0) + \hat{j}(3\cdot 2) + \hat{k}(1\cdot(-4)) - \hat{k}(2\cdot 2) - \hat{i}(-4\cdot 3) - \hat{j}(0\cdot 1)$$

$$= (0+12)\hat{i} + (6+0)\hat{j} + (-4-4)\hat{k}$$

$$= 12\hat{i} + 6\hat{j} - 8\hat{k} = \langle 12, 6, -8 \rangle$$

Note: $\vec{v}\cdot(\vec{v}\times\vec{w}) = 1\cdot 12 + 2\cdot 6 + 3\cdot(-8) = 0$

$\vec{w}\cdot(\vec{v}\times\vec{w}) = 2\cdot 12 - 4\cdot 6 + 0\cdot(-8) = 0$

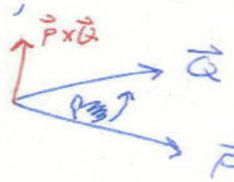
Hmm... $\vec{v}\times\vec{w}$ appears to be perpendicular to $\vec{v}$ and $\vec{w}$. This is a key fact.

SECTION 11.3 (CONT)

PROPERTIES OF THE CROSS PRODUCT

- $\|\vec{P} \times \vec{Q}\| = \|\vec{P}\| \cdot \|\vec{Q}\| \cdot \sin\theta$ = area of parallelogram
- $\vec{P} \times \vec{P} = \vec{0}$
- $\vec{P} \times \vec{Q} = -\vec{Q} \times \vec{P}$
- $\vec{P} \times \vec{Q} = \vec{0}$ if and only if $\vec{P}$ and $\vec{Q}$ parallel
- $\alpha(\vec{P} \times \vec{Q}) = \alpha\vec{P} \times \vec{Q} = \vec{P} \times \alpha\vec{Q}$
- $\vec{P} \times (\vec{Q} + \vec{R}) = \vec{P} \times \vec{Q} + \vec{P} \times \vec{R}$
- $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

↳ Right hand rule



~~EXTRA CREDIT: IS THE CROSS PRODUCT ASSOCIATIVE? DOES $\vec{P} \times (\vec{Q} \times \vec{R}) = (\vec{P} \times \vec{Q}) \times \vec{R}$? PROVE OR DISPROVE~~

- TRIPLE PRODUCT: $(\vec{A} \times \vec{B}) \cdot \vec{C}$ =algebra⟹ $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

- NOTE $\vec{P}$ and $\vec{Q}$ are perpendicular to $\vec{P} \times \vec{Q}$

Homework: Pg 842: #1, #5, #11, #12

Suggested: #7, #17a

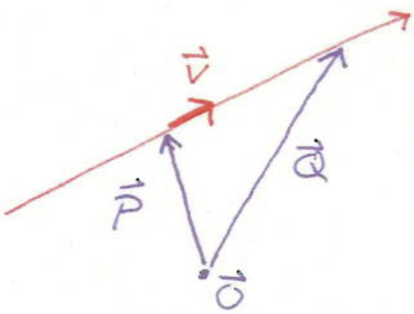
SECTION 11.4: LINES AND PLANES IN SPACE

EQUATION OF A LINE

Always start in one-dim and try to generalize

$y = mx + b \qquad \dfrac{y - y_1}{x - x_1} = m \iff y = y_1 + m(x - x_1)$

$$y - y_1 = m(x - x_1)$$



Want $\vec{Q} - \vec{P}$ to be a multiple of $\vec{v}$

Line is $\vec{Q} = \vec{P} + t\vec{v}$

Given point $\vec{P}$ on line ℓ with direction $\vec{v}$

↳ $\boxed{\ell(t) = \vec{P} + t\vec{v}}$

↳ if $\vec{P} = (P_1, \ldots, P_n)$ and $\vec{v} = (v_1, \ldots, v_n)$ Then

$$x_1 = P_1 + t v_1$$
$$\vdots$$
$$x_n = P_n + t v_n$$

In $\mathbb{R}^3$ often write $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} P_1 + t v_1 \\ P_2 + t v_2 \\ P_3 + t v_3 \end{pmatrix}$ or $\begin{pmatrix} x_1 + t v_1 \\ x_2 + t v_2 \\ x_3 + t v_3 \end{pmatrix}$

Given two points $\vec{P}, \vec{Q}$ on line, have

$$\boxed{\ell(t) = \vec{P} + (\vec{Q} - \vec{P})t}$$

EXAMPLE:

$\vec{P} = (1,2,3) \quad \vec{v} = (0,1,-1)$

$\ell(t) = (1,2,3) + t(0,1,-1)$

$= (1, 2+t, 3-t)$

so $X = x(t) = 1$

$y = y(t) = 2+t$

$z = z(t) = 3-t$

$\vec{P} = (1,2,3) \quad \vec{Q} = (1,4,1)$

$\vec{Q} - \vec{P} = (1,4,1) - (1,2,3)$

$= (1-1, 4-2, 1-3)$

$= (0, 2, -2)$

$\ell(t) = \vec{P} + \cancel{t}\,s(\vec{Q} - \vec{P})$

$= (1,2,3) + \cancel{t}\,s(0,2,-2)$

$= 1, 2+2\cancel{t}\,s, 3-2s)$

so $X = x(s) = 1$

$y = y(s) = 2+2s$

$z = z(s) = 3-2s$

"look" different, but same line (send/replace s with $2t$)

↳ changes how fast travel

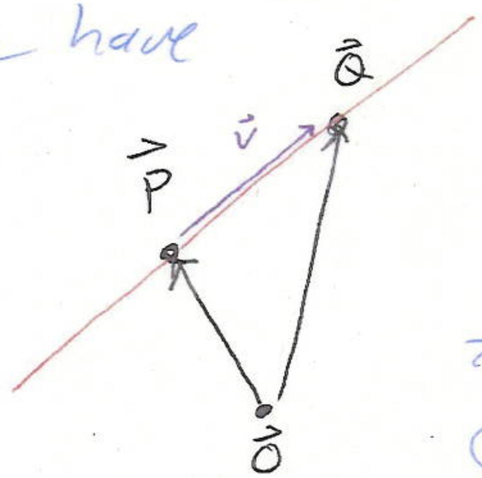
↳ could make look really different

↳ pattern recognition crucial

↳ telescoping sums example from FTC

OPTIONAL APPENDIX
MATH 105: LINES IN SPACES

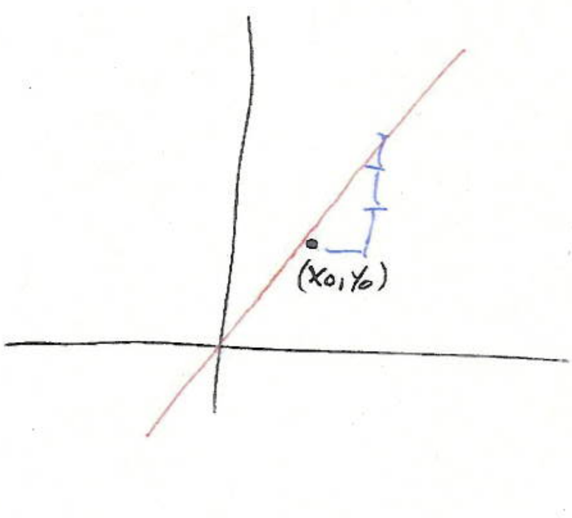
The following might help look at generalizing equations of lines. In three dimensional space we have



The direction of the line is $\vec{v} = \vec{Q} - \vec{P}$

The equation of the line is
$(x, y, z) = \vec{P} + t\vec{v}$

Some people have had some trouble seeing this as a generalization of the standard line in the plane, so I thought I'd provide another attempt at explaining it.



Here is a line going through the point (x_0, y_0) with slope 3. We may write this as
$y - y_0 = 3(x - x_0)$

OPTIONAL APPENDIX (CONT)

We note that this line is in the direction $(1,3)$; for every one unit of x we move, we go three units in the y-direction.

In our notation, we have the anchor point (x_0, y_0) with direction $(1,3)$ (which is like a slope of 3)

$$(x,y) = (x_0, y_0) + t(1,3)$$

$$(x,y) = (x_0 + t,\ y_0 + 3t)$$

Note: two equations: $x = x_0 + t$

$$y = y_0 + 3t$$

subtracting: $x - x_0 = t$

$$y - y_0 = 3t$$

Thus $y - y_0 = 3(x - x_0)$,

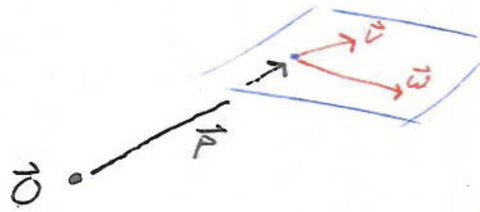
the old equation for the line!

Section 11.4 (cont)

Planes

Generalize what did for lines

Input: point and two directions



Plane is all Q satisfying $\vec{Q} - \vec{P} = t\vec{v} + s\vec{w}$ for some t, s

$$\text{Plane}(t,s) = \{\vec{Q} : \vec{Q} = \vec{P} + t\vec{v} + s\vec{w},\ t, s \in \mathbb{R}\}$$

Say plane is spanned by $\vec{v}$ and $\vec{w}$ (do more in lin alg)

Ex: $\{(2,1,0) + t(1,0,-1) + s(2,4,6)\}$

~~HW: #4, #7, #13, #16, #22~~

~~Suggested: #9, #19, #28, #30~~ ~~[illegible]~~

SECTION 11.4 (CONT)

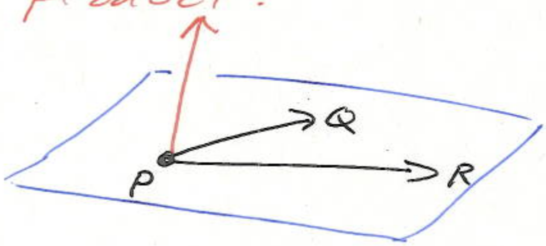
EQ OF PLANES

Given point $\vec{P}$ and normal direction $\vec{n}$, plane is all $\vec{Q}$ such that $(\vec{Q}-\vec{P})\cdot\vec{n}=0$

Ex: $\vec{P}=(x_0,y_0,z_0)$, $\vec{n}=(a,b,c)$, $Q=(x,y,z)$

Equation is $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$.

Key fact/shortcut: if have two vectors in the plane, the normal is parallel to the cross product!



normal is parallel to $\vec{PQ}\times\vec{PR}$

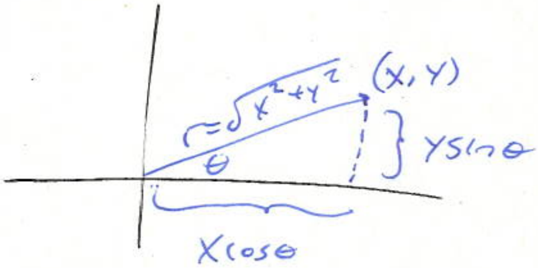
Homework: Page 849: #1, #2, #3, #22

Suggested Problems: #25, #54, #58, #60

SECTION 11.8: CYLINDRICAL AND SPHERICAL COORDS

- Polar Coords: $X = r\cos\theta$ $\quad$ $r = \sqrt{x^2+y^2}$

  $Y = r\sin\theta$ $\quad$ $\theta = \arctan(Y/X)$



- Cylindrical Coords: $X = r\cos\theta$ $\quad$ $r = \sqrt{x^2+y^2}$

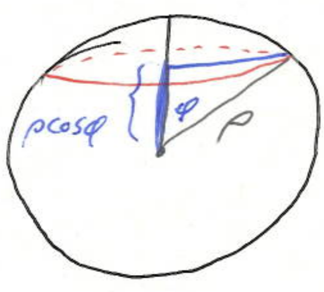
  $Y = r\sin\theta$ $\quad$ $\theta = \arctan(Y/X)$

  $Z = Z$ $\quad$ $Z = Z$

- Spherical Coords: $X = \rho\sin\varphi\cos\theta$ $\quad$ $\rho = \sqrt{x^2+y^2+z^2}$

  $Y = \rho\sin\varphi\sin\theta$ $\quad$ $\varphi \in [0, \pi]$

  $Z = \rho\cos\varphi$ $\quad$ $\theta \in [0, 2\pi)$



see have polar coords with radius $r = \rho\sin\varphi$ and angle θ

Homework: Pg 893: #1, #26, Extra Credit: #55

Additional: #33, #53