THE ARITHMETIC AND GEOMETRIC MEAN INEQUALITY

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ABSTRACT. We provide sketches of proofs of the Arithmetic Mean - Geometric Mean Inequality. These notes are based on discussions with Vitaly Bergelson, Eitan Sayag, and the students of Math 487 (Ohio State, Autumn 2003).

1. INTRODUCTION

Definition 1.1 (Arithmetic Mean)**.** *The Arithmetic Mean of* $a_1, \dots, a_n$ *is*

$$AM(a_1, \dots, a_n) \ = \ \frac{a_1 + \cdots + a_n}{n}. \tag{1.1}$$

Definition 1.2 (Geometric Mean)**.** *The Geometric Mean of* $a_1, \dots, a_n$ *is*

$$GM(a_1, \dots, a_n) \ = \ \sqrt[n]{a_1 \cdots a_n}. \tag{1.2}$$

Theorem 1.3 (Arithmetic Mean - Geometric Mean)**.** *Let* $a_1, \dots, a_n$ *be* n *positive numbers. Then*

$$AM(a_1, \dots, _n) \ \geq \ GM(a_1, \dots, a_n). \tag{1.3}$$

Remark 1.4. *Note the above trivially holds if* $a_1 = \cdots = a_n$*; in fact, equality holds if and only if all* a_i *are equal.*

Remark 1.5. *Note that if the Arithmetic Mean - Geometric Mean inequality holds for* $a_1, \dots, a_n$*, it holds for* $\alpha a_1, \dots, \alpha a_n$ *for any* $\alpha > 0$*. Thus, we can rescale the sum* $a_1 + \cdots + a_n$ *(assuming it is non-zero) to be whatever we want.*

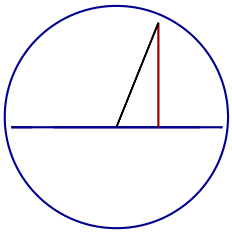
Remark 1.6. *Note the* $n = 2$ *case follows immediately from* $(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$.

2. GEOMETRIC PROOF WHEN $n = 2$

Without loss of generality, assume $a_1 > a_2$. Construct a circle with diameter $a_1 + a_2$, hence radius $\frac{a_1+a_2}{2}$. On the main diagonal, a_1 units from one end (a_2 from the other), draw the perpendicular bisector to the main diagonal, which hits the circle at some point, say P.

The length of the perpendicular bisector from the main diagonal to the circle is $\sqrt{a_1 a_2}$ – this can be shown by applications of the Pythagorean Theorem.

Form a triangle using this as one side, and with hypotenuse from the center of the circle to P. The hypotenuse will have length $\frac{a_1+a_2}{2}$, which must therefore be larger than the side $\sqrt{a_1 a_2}$.



Blue Line: Diameter of the Circle, Length $a_1 + a_2$
Black Line: Radius (Arithmetic Mean), Length $\frac{a_1+a_2}{2}$
Red Line: Altitude (Geometric Mean), Length $\sqrt{a_1 a_2}$

3. Multivariable Calculus Proof

Use Lagrange Multipliers, with

$$\begin{aligned} f(a_1, \dots, a_n) &= (a_1 \cdots a_n)^{\frac{1}{n}} \\ g(a_1, \dots, a_n) &= \frac{a_1 + \cdots + a_n}{n} - c. \end{aligned} \tag{3.4}$$

To find if the Lagrange Multipliers give a maximum or minimum, check at $1, 1, 1, \dots, 1, 2^n$.

4. Standard Induction Proof

We proceed by induction, the $n = 1$ and $n = 2$ cases already handled above.

We must show

$$\frac{a_1 + \cdots + a_n}{n} \ \geq \ \sqrt[n]{a_1 \cdots a_n}. \tag{4.5}$$

Without loss of generality, we may rescale the a_i so that $a_1 \cdots a_n = 1$. If all $a_i = 1$, the proof is trivial. Thus, assume at least one $a_i > 1$ and one $a_i < 1$; we assume $a_1 > 1$, $a_2 < 1$.

Thus, by the inductive assumption, we have

$$\frac{a_1 a_2 + a_3 + \cdots + a_n}{n-1} \ \geq \ \sqrt[n-1]{(a_1 a_2) a_3 \cdots a_n} \ = \ 1. \tag{4.6}$$

Thus, we have

$$a_1a_2 + a_3 + \cdots + a_n \ \geq\ n - 1. \tag{4.7}$$

We need to show

$$a_1 + a_2 + \cdots + a_n \ \geq\ n. \tag{4.8}$$

This would follow if $a_1 + a_2 - (a_1a_2 + 1) \geq 0$. But

$$a_1 + a_2 - (a_1a_2 + 1) \ =\ (a_1 - 1)(1 - a_2) \ \geq\ 1, \tag{4.9}$$

proving the claim.

5. Induction by Powers of 2

We first show if the Arithmetic Mean - Geometric Mean Inequality holds for $n = 2^{k-1}$, then it holds for $n = 2^k$. We then show how to handle n that are not powers of 2.

Lemma 5.1. *If the AM - GM Inequality holds for $n = 2^{k-1}$, it holds for $n = 2^k$.*

Proof. We assume the case $n = 2$ has already been done, and is available for use below. namely for any $c_1, c_2 > 0$,

$$\frac{c_1 + c_2}{2} \ \geq\ \sqrt{c_1c_2} \ =\ 1. \tag{5.10}$$

Without loss of generality, rescale so that $a_1 \cdots a_{2^k} = 1$. Let $b_1 = a_1 + \cdots + a_{2^{k-1}}$ and $b_2 = a_{2^{k-1}+1} + \cdots + a_{2^k}$.

By induction, we can apply the AM-GM to b_1 and b_2 and we find

$$\frac{b_1}{2^{k-1}} \ =\ \frac{a_1 + \cdots + a_{2^{k-1}}}{2^{k-1}} \ \geq\ \sqrt[2^{k-1}]{a_1 \cdots a_{2^{k-1}}} \tag{5.11}$$

and

$$\frac{b_2}{2^{k-1}} \ =\ \frac{a_{2^{k-1}+1} + \cdots + a_{2^k}}{2^{k-1}} \ \geq\ \sqrt[2^{k-1}]{a_{2^{k-1}+1} \cdots a_{2^k}}. \tag{5.12}$$

Combining yields

$$\frac{b_1 + b_2}{2^k} \ =\ \frac{a_1 + \cdots + a_{2^k}}{2^k} \ \geq\ \frac{\sqrt[2^{k-1}]{a_1 \cdots a_{2^{k-1}}} + \sqrt[2^{k-1}]{a_{2^{k-1}+1} \cdots a_{2^k}}}{2}. \tag{5.13}$$

Applying the $n = 2$ case to the right hand side yields

$$\frac{a_1 + \cdots + a_{2^k}}{2^k} \ \geq\ \sqrt{\sqrt[2^{k-1}]{a_1 \cdots a_{2^{k-1}}} \cdot \sqrt[2^{k-1}]{a_{2^{k-1}+1} \cdots a_{2^k}}} \ =\ 1, \tag{5.14}$$

as the right hand side is now just $\sqrt[2^k]{a_1 \cdots a_n} = 1$.

$\square$

We now prove the AM - GM Inequality for any n. Choose k so that $2^{k-1} < n < 2^k$. Then we need to add $2^k - n$ terms to have a power of 2. As always, we may assume $a_1 \cdots a_n = 1$.

Let $A_n = \frac{a_1 + \cdots + a_n}{n}$. Consider the sequence $a_1, \dots, a_n, A_n, \dots, A_n$, where we have A_n a total of $2^k - n$ times.

Exercise 5.2. *Show the Arithmetic Mean of these 2^k numbers is still A_n.*

Exercise 5.3. *Show the Geometric Mean of these 2^k numbers is*

$$\sqrt[2^k]{a_1 \cdots a_n A_n^{2^k - n}} = \left(\frac{a_1 + \cdots + a_n}{n}\right)^{1 - \frac{n}{2^k}}; \tag{5.15}$$

remember we have rescaled so that $a_1 \cdots a_n = 1$.

As we have 2^k numbers, we may apply the AM - GM Inequality, and we obtain

$$\begin{aligned}
\frac{a_1 + \cdots a_n + (2^k - n) \cdot \frac{a_1 + \cdots + a_n}{n}}{2^k} &\geq \sqrt[2^k]{a_1 \cdots a_n \left(\frac{a_1 + \cdots + a_n}{n}\right)^{2^k - n}} \\
\frac{a_1 + \cdots + a_n}{n} &\geq \left(\frac{a_1 + \cdots + a_n}{n}\right)^{1 - \frac{n}{2^k}} \\
\left(\frac{a_1 + \cdots + a_n}{n}\right)^{\frac{2^k}{n}} &\geq 1 \\
\frac{a_1 + \cdots + a_n}{n} &\geq 1, \qquad (5.16)
\end{aligned}$$

as claimed, completing the proof.

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