MATH 105: PRACTICE PROBLEMS FOR CHAPTER 11 AND CALCULUS REVIEW: SPRING 2011

SOLUTION KEY (PLEASE REPORT ANY ERRORS TO ME)

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Question 1 : These problems deal with equations of lines.

(1) Find the equation of the line going through the points (2,3) and (4,9).
Solution: I prefer using the point-slope form; thus given two points we first find the slope. The slope is $m = \frac{9-3}{4-2} = \frac{6}{2} = 3$. As the equation of a line is $y - y_0 = m(x - x_0)$, we find $y-9 = 3(x-4)$; alternatively, we could use the other point and find $y-3 = 3(x-2)$.

(2) Find the equation of the line going through the points (2,3) and (-1,2).
Solution: Similar to the previous, we first compute the slope $m = \frac{2-3}{-1-2} = \frac{-1}{-3} = 1/3$, and thus the equation of the line is $y - 3 = \frac{1}{3}(x - 2)$.

(3) Find the equation of the line going through the point (2,3) with slope 3.
Solution: We can immediately use the point-slope form. We have $y - 3 = 3(x - 2)$.

(4) Find the equation of the line going through the points (2,3,4) and (4,9,16).
Solution: We first determine the direction of the line, which is $\overrightarrow{v} = (4, 9, 16)-(2, 3, 4) = (2, 6, 12)$. Thus the equation of the line is $(x, y, z) = (2, 3, 4) + t\overrightarrow{v} = (2, 3, 4) + t(2, 6, 12)$. We can expand this and write $(x, y, z) = (2 + 2t, 3 + 6t, 4 + 12t)$.

(5) Find the equation of the line going through the point (2,3,4) in the direction (2,6,12).
Solution: In the previous problem we had two points, and had to subtract in order to find the direction. This problem is simpler, and we can immediately use the point-direction formulation (which is the generalization of point-slope). The point is $(2, 3, 4)$ and the direction is $\overrightarrow{v} = (2, 6, 12)$, and thus the equation of the line is $(x, y, z) = (2, 3, 4)+t\overrightarrow{v}$, or $(x, y, z) = (2, 3, 4) + t(2, 6, 12)$. We can expand this and write $(x, y, z) = (2 + 2t, 3 + 6t, 4 + 12t)$. Note that this is the same answer as in the previous problem, which is of course as it should be since the two lines contain the same point and are in the same direction.

(6) Is the point (4,19,26) on the line going through the point (2,3,4) in the direction (2,6,12)?
Solution: The fastest way to see this is to note that if we set the direction vector to be $\overrightarrow{v} = (2, 6, 12)$, then $(2, 3, 4) + \overrightarrow{v} = (4, 9, 16)$; this is the only point on the line with x-coordinate 4, and thus the point is not on the line. Alternatively, the equation of the line is $(x, y, z) = (2, 3, 4) + t(2, 6, 12) = (2 + 2t, 3 + 6t, 4 + 12t)$. We want to see if the point $(4, 19, 26)$ is on this line; thus we are looking for a choice of t such that $(4, 19, 26) =$

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$(2+2t, 3+6t, 4+12t)$. This leads to three equations:

$$\begin{aligned} 4 &= 2+2t \\ 19 &= 3+6t \\ 26 &= 4+12t. \end{aligned}$$

The first equation forces t to equal 1, which does not solve the second or third equation. Thus there is no such point.

(7) Consider the lines in part (1) and part (2). Find all points on both lines.

Solution: The fastest way to solve this problem is to note that the two lines both contain the point $(2,3)$ and are in different directions. Thus there is only one point on both, namely $(2,3)$. In general two lines are either the same or intersect in just one point. Alternatively, we can look at the two lines and see if there is a point on both. The first is $y-3 = 3(x-2)$, the second is $y - 3 = \frac{1}{3}(x-2)$. We are trying to find a point (x,y) that satisfies both. Our problem is simplified by noting that both are of the form $y-3$ equals something, and thus these two somethings must be equal. We find $3(x-2) = \frac{1}{3}(x-2)$, which has the solution $x = 2$. When $x = 2$ we find $y = 3$. Thus the point is $(2,3)$.

Question 2 : These equations deal with vectors. For all problems below, let $\overrightarrow{P} = (1,2,3)$, $\overrightarrow{Q} = (4,9,6)$, $\overrightarrow{R} = (3,3,3)$, $\overrightarrow{v} = (3,7,3)$ and $\overrightarrow{w} = (2,1,0)$.

(1) Find $\overrightarrow{P} + \overrightarrow{R}$, $4\overrightarrow{P} - 3\overrightarrow{Q} + 2\overrightarrow{R}$, $(\overrightarrow{P} + 2\overrightarrow{Q}) \cdot \overrightarrow{R}$, and $(\overrightarrow{P} \times \overrightarrow{Q}) \times \overrightarrow{R}$.

Solution: $\overrightarrow{P} + \overrightarrow{R} = (1+3, 2+3, 3+3) = (4,5,6)$, $4\overrightarrow{P} - 3\overrightarrow{Q} + 2\overrightarrow{R} = (-2,-13,0)$, and $(\overrightarrow{P} + 2\overrightarrow{Q}) \cdot \overrightarrow{R} = 132$. To see the last, note $\overrightarrow{P} + 2\overrightarrow{Q} = (9,20,15)$, and then the dot product is $9 \cdot 3 + 20 \cdot 3 + 15 \cdot 3 = 132$. Finally, $(\overrightarrow{P} \times \overrightarrow{Q}) \times \overrightarrow{R} = (15, 48, -63)$.

(2) Find the plane containing $\overrightarrow{P}$ with two directions $\overrightarrow{v}$ and $\overrightarrow{w}$.

Solution: The equation of the plane is $(x,y,z) = P + t\overrightarrow{v} + s\overrightarrow{w}$. AS $\overrightarrow{v} = (3,7,3)$ and $\overrightarrow{w} = (2,1,0)$, we have

$$(x,y,z) = (1,2,3) + t(3,7,3) + s(2,1,0).$$

(3) Find the equation of the plane containing the vectors $\overrightarrow{P}$, $\overrightarrow{Q}$ and $\overrightarrow{R}$.

Solution: The first direction is $\overrightarrow{v} = \overrightarrow{Q} - \overrightarrow{P} = (3,7,3)$, the second is $\overrightarrow{w} = \overrightarrow{R} - \overrightarrow{P} = (2,1,0)$. Thus the equation of the plane is $(x,y,z) = \overrightarrow{P} + t\overrightarrow{v} + s\overrightarrow{w}$, or

$$(x,y,z) = (1,2,3) + t(3,7,3) + s(2,1,0).$$

Note, of course, that this is the same as the previous problem!

(4) Find the equation of the plane containing the point $\overrightarrow{P}$ whose normal is in the direction $(-3,6,-11)$.

Solution: A plane can be regarded as the set of all points perpendicular to a given direction. Let $\overrightarrow{n} = (-3, 6, -11)$. We thus have $\left((x, y, z) - \overrightarrow{P}\right) \cdot \overrightarrow{n} = 0$, or $(x, y, z) \cdot \overrightarrow{n} = \overrightarrow{P} \cdot \overrightarrow{n}$, which means $-3x + 6y - 11z = -24$.

(5) Find the equation of the plane containing $\overrightarrow{Q}$ with two directions $\overrightarrow{v}$ and $\overrightarrow{w}$.
Solution: We can solve this using the same techniques as above; however, a faster way is to note that $\overrightarrow{Q} - \overrightarrow{v} = \overrightarrow{P}$. Thus the point $\overrightarrow{P}$ is also in the plane, and we have the same plane as before.
If we don't know it, then we can solve directly and find

$$(x, y, z) = \overrightarrow{Q} + t\overrightarrow{v} + s\overrightarrow{w} = (4 + 3t + 2s, 9 + 7t + S, 6 + 3t).$$

While this is the same plane as before, it looks different as we are using a different base point.

(6) Find the area of the parallelogram, two of whose sides are $\overrightarrow{v}$ and $\overrightarrow{w}$.
Solution: The solution is to use the cross product; the area is the length of $\overrightarrow{v} \times \overrightarrow{w} = (-3, 6, -11)$. The length is $\sqrt{(-3)^2 + 6^2 + (-11)^2} = \sqrt{166}$.

(7) Find the cosine of the angle between $\overrightarrow{v}$ and $\overrightarrow{w}$.
Solution: We have $\overrightarrow{v} \cdot \overrightarrow{w} = ||\overrightarrow{v}||\ ||\overrightarrow{w}||\ \cos\theta$. We have $\overrightarrow{v} \cdot \overrightarrow{w} = 13$, $||\overrightarrow{v}|| = \sqrt{67}$, $||\overrightarrow{w}|| = \sqrt{5}$; thus $\cos\theta = \frac{13}{\sqrt{67}\sqrt{5}} = \frac{13}{\sqrt{335}} = \frac{13\sqrt{335}}{335}$.

(8) Find the length of $\overrightarrow{v}$, and find a vector of unit length in the same direction as $\overrightarrow{v}$.
Solution: We have $||\overrightarrow{v}|| = \sqrt{3^2 + 7^2 + 3^2} = \sqrt{67}$. To find a vector of unit length in the same direction as $\overrightarrow{v}$, we simply divide $\overrightarrow{v}$ by its length. The solution is $\overrightarrow{u} = \overrightarrow{v}/||\overrightarrow{v}|| = (3/\sqrt{67}, 7/\sqrt{67}, 3/\sqrt{67})$.

(9) Find a vector perpendicular to both $\overrightarrow{v}$ and $\overrightarrow{w}$.
Solution: The cross product gives a vector perpendicular to the two vectors; thus a solution is $\overrightarrow{v} \times \overrightarrow{w} = (-3, 6, -11)$. Of course, any multiple of this works as well.

(10) Find a vector perpendicular to $\overrightarrow{v}$.
Solution: One way to solve this problem is to find a vector whose dot product with $\overrightarrow{v}$ is zero. We have $\overrightarrow{v} = (3, 7, 3)$, so if (x, y, z) is to be perpendicular to $\overrightarrow{v}$, we must have $3x + 7y + 3z = 0$. This is one equation with three unknowns; there are many solutions. One simple solution can be found by taking $y = 0$, which gives $3x + 3z = 0$, or $z = -x$. In particular, one solution is $(1, 0, -1)$. Note that even taking $y = 0$ still gives an infinitude of solutions; however, all of these solutions are in the direction of $(1, 0, -1)$. There is, of course, another way to find a vector perpendicular to $\overrightarrow{v}$. Take $\overrightarrow{u}$ to be *any* non-vector not in the direction of $\overrightarrow{v}$. Then $\overrightarrow{v} \times \overrightarrow{u}$ is perpendicular to $\overrightarrow{v}$. For definiteness, we could take as our second vector the vector $\overrightarrow{w}$, and thus use the solution from the earlier problem.

Question 3 : State the following results.

(1) The Pythagorean Formula.

Solution: If we have a right triangle with sides a and b and hypotenuse c, then $a^2 + b^2 = c^2$.

(2) The Law of Cosines.

Solution: If we have a triangle with sides a, b and c and θ is the angle opposite c, then $c^2 = a^2 + b^2 - 2ab\cos\theta$.

(3) The formula for the cosine of the angle between two vectors $\overrightarrow{P}$ and $\overrightarrow{Q}$.

Solution: We have $\overrightarrow{P} \cdot \overrightarrow{Q} = ||\overrightarrow{P}||\,||\overrightarrow{Q}||\,\cos\theta$, and thus $\cos\theta = \frac{\overrightarrow{P}\cdot\overrightarrow{Q}}{||\overrightarrow{P}||\,||\overrightarrow{Q}||}$.

(4) The formulas for the determinant of a 2×2 matrix A and a 3×3 matrix B, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Solution: We have $\det(A) = ad - bc$, while $\det(B) = aei + bfg + cdh - ceg - afh - bdi$.

(5) Give a reason why we care about determinants.

Solution: Determinants give the signed area in the plane or volume in three-space.

(6) Give the formula for the cross product of two vectors; specifically, what is the cross product $\overrightarrow{v} \times \overrightarrow{w}$, where $\overrightarrow{v} = (v_1, v_2, v_3)$ and $\overrightarrow{w} = (w_1, w_2, w_3)$. Give three properties of the cross product.

Solution: We have $\overrightarrow{v} \times \overrightarrow{w} = (-v3w2 + v2w3, v3w1 - v1w3, -v2w1 + v1w2)$. Some properties include $\overrightarrow{v} \times \overrightarrow{w} = -\overrightarrow{w} \times \overrightarrow{v}$, which of course implies $\overrightarrow{v} \times \overrightarrow{v} = \overrightarrow{0}$. Another property is $(\overrightarrow{v} + \overrightarrow{u}) \times \overrightarrow{w} = \overrightarrow{v} \times \overrightarrow{w} + \overrightarrow{u} \times \overrightarrow{w}$. Another is $(a\overrightarrow{v}) \times \overrightarrow{w} = a(\overrightarrow{v} \times \overrightarrow{w})$.

(7) Give the formula for the inner (or dot) product of two vectors; specifically, what is $\overrightarrow{v} \cdot \overrightarrow{w}$ where $\overrightarrow{v} = (v_1, \dots, v_n)$ and $\overrightarrow{w} = (w_1, \dots, w_n)$. Give three properties of the inner product.

Solution: We have $\overrightarrow{v} \cdot \overrightarrow{w} = v_1w_1 + \cdots + v_nw_n = \sum_{i=1}^n v_iw_i$. Three properties are $\overrightarrow{v} \cdot \overrightarrow{w} = \overrightarrow{w} \cdot \overrightarrow{v}$, $(\overrightarrow{v} + \overrightarrow{u}) \cdot \overrightarrow{w} = \overrightarrow{v} \cdot \overrightarrow{w} + \overrightarrow{u} \cdot \overrightarrow{w}$. Another is $(a\overrightarrow{v}) \cdot \overrightarrow{w} = a(\overrightarrow{v} \cdot \overrightarrow{w})$.

(8) Explain what the phrase *right hand screw rule* means, and why it is useful.

Solution: The phrase tells us what direction the cross product of two vectors points. If we have $\overrightarrow{v} \times \overrightarrow{w}$, if we take our right hand and curl our fingers from $\overrightarrow{v}$ towards $\overrightarrow{w}$, then our thumb points in the direction of their cross product. In particular, this tells us how to orient the x, y and z-axes.

(9) Prove the triple product formula; specifically, if $\overrightarrow{A} = (a_1, a_2, a_3)$, $\overrightarrow{B} = (b_1, b_2, b_3)$ and $\overrightarrow{C} = (c_1, c_2, c_3)$ then

$$(\overrightarrow{A} \times \overrightarrow{B}) \cdot \overrightarrow{C} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Solution: One proof is by brute force, which we leave as an exercise. In other words, compute both sides and see that they are equal.

Question 4 : (**Calculus Review**) Find the maximum and minimum values for $f(x) = \frac{1}{3}x^3 - 9x^2 + 80x + 1$ when $-20 \leq x \leq 40$. Use the first and second derivative tests to classify the local maximum and minimums, and sketch the curve.

Solution: To find maxima and minima, we check the critical points and the end points. For the endpoints, we have $f(-20) = -7865.\overline{6}$ (which means 6 repeating) and $f(40) = 10134.\overline{3}$. The critical points are found by solving $f'(x) = 0$, or finding all x such that $f'(x) = x^2 - 18x + 80 = 0$. We can use the quadratic formula to solve this[1], or we could note that $x^2 - 18x + 80 = (x-8)(x-10)$. Thus the two critical points are $x = 8$ and $x = 10$. The function at the critical points is $f(8) = 235.\overline{6}$ and $f(10) = 234.\overline{3}$. Thus the maximum value of $10134.\overline{3}$ is attained at 40 and the minimum value of $-7865.\overline{6}$ at -20 in this interval. We plot our function below, with a zoom of the region near the two critical points. One way to plot functions such as this easily (without resorting to computers, of course!) is to use information from the first and second derivatives, evaluate the function at these key points (endpoints, critical points and points of inflection), and then use the knowledge of the various derivatives to connect the points with the proper concavity. (I'm not expecting you to do this on exams.)

Question 5 : (**Calculus Review**) Consider all rectangles with perimeter 100. Find the rectangle with largest area.

Solution: Let x and y be the two sides. Then our goal is to maximize xy subject to $2x+2y = 100$ and $x, y \geq 0$. In multivariable calculus we will learn how to optimize functions of several variables; however, in Calculus I we don't know such techniques, and thus must convert everything to a function of one variable. Fortunately we may use the perimeter relation to write $x + y = 50$ or $y = 50 - x$. Substituting into the area formula we find that our problem is equivalent to maximizing $A(x) = x(50 - x)$ subject to $0 \leq x \leq 50$. The two endpoints give 0, and thus our answer will be the critical point. We could differentiate $A(x)$ by using the product rule, but it is faster to *Thoreau* it. In other words, if we expand we find $A(x) = 50x - x^2$, and thus there is no need to use the product rule. We find $A'(x) = 50 - 2x$, so $A'(x) = 0$ means $50 - 2x = 0$ or $x = 25$. Substituting

[1] If we have the polynomial $ax^2 + bx + c$, the roots are $(-b \pm \sqrt{b^2 - 4ac})/2a$.

$x = 25$, we find the maximum area is $25^2 = 625$.

Question 6 : State the fundamental theorem of calculus (FTC). (1) Use the FTC to calculate the area under the curve $f(x) = x^2 + 2x + 1$ from $x = 1$ to $x = 4$; (2) use the FTC to calculate the area under the curve of $f(x) = \sin(x)$ from $x = 0$ to $x = \pi/2$. Note we may denote these areas by $\int_1^4 (x^2 + 2x + 1)dx$ and $\int_0^{\pi/2} \sin(x)dx$.

Solution: The FTC states that the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is $F(b) - F(a)$, where F is any function such that $F'(x) = f(x)$. For the first, we have

$$\int_1^4 (x^2 + 2x + 1)dx \ = \ \left(\frac{x^3}{3} + x^2 + x\right)\Bigg|_1^4 \ = \ \left(\frac{64}{3} + 16 + 4\right) - \left(\frac{1}{3} + 1 + 1\right) \ = \ 39.$$

For the second, we have

$$\int_0^{\pi/2} \sin(x)dx \ = \ -\cos(x)\Bigg|_0^{\pi/2} \ = \ -\cos(\pi/2) + \cos(0) \ = \ 1$$

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Question 7 : Find *all* the anti-derivatives of the following: (1) x^4; (2) $x^4 + 3x^5$; (3) $(x + 6)^8$; (4) $(x^3 + 4x^2 + 1)^7 \cdot (3x^2 + 8x)$; (5) $\sin(x) - \cos(x) + e^x$.

Solution: For (1) it is $x^5/5 + C$, for (2) it is $x^5/5 + x^6/2 + C$ and for (3) it is $(x + 6)^9/9$. Part (4) is a little harder. It isn't the product rule, but rather the chain rule. Note the second factor is the derivative of the first, and hence the answer is just $(x^3 + 4x^2 + 1)^8/8 + C$, where we are using the derivative of $f(g(x))$ is $f'(g(x))g'(x)$. Finally, (5) is simply $-\cos(x) - \sin(x) + e^x + C$.

Question 8 : State L'Hopital's rule. Determine

$$\lim_{x\to 0} \frac{\sin(x)}{x}, \quad \lim_{x\to 0} \frac{\sin(x)\cos(x) - x}{x^2}, \quad \lim_{x\to 2} \frac{x^2 - 4}{(x-2)\sin(x)}.$$

Solution: L'Hopital's rule says that if f and g are differentiable at x_0 and $f(x_0) = g(x_0) = 0$ (or also if both equal $\pm\infty$) then $\lim_{x\to x_0} f(x)/g(x) = \lim_{x\to x_0} f'(x)/g'(x)$. If $g'(x_0)$ is not zero or $\pm\infty$, the last limit is simply $f(x_0)/g(x_0)$.

Note that in all three cases we have 0/0, and thus we may use L'Hopital's rule. The third is the simplest. While we may use L'Hopital's rule, we can avoid it by *Thoreauing* it. Namely, we may remove a factor of $x - 2$ from both the numerator and the denominator, and find that we have $\lim_{x\to 2} \frac{x+2}{\sin(x)}$. This limit is not 0/0 and may be evaluated directly as $4/\sin(2)$.

For the first, the derivative of $\sin(x)$ is $\cos(x)$ and the derivative of x is 1. Thus the first limit is simply $\cos(0)/1 = 1$.

For the second, the derivative of the numerator is $\cos^2(x) - \sin^2(x) - 1$ and the derivative of the denominator is $2x$. Unfortunately when we evaluate both of these at 0 we obtain 0/0, and thus we need to apply L'Hopital's rule again. The second derivative of the numerator is, after a little algebra, $-4\cos(x)\sin(x)$ while the denominator's second derivative is 2. We no longer have 0/0, and thus the answer is $-4\cos(0)\sin(0)/2 = 0$.