MATH 105: PRACTICE PROBLEMS FOR SERIES: SPRING 2011

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1. Problems

1.1. Sequences and Series.

Question 1: Let $a_n = \frac{1}{1+n+n^2}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Prove your claim.

Question 2: Let $a_n = \frac{n^4}{1+2^n+(-2)^n}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Prove your claim.

Question 3: Let $a_n = \frac{n^4}{6*n}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Prove your claim.

Question 4: Let $a_n = \frac{n}{n!}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Prove your claim.

Question 5: Let $a_n = \frac{n2^n}{3^n}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Prove your claim.

Question 6: Let $a_n = n^3/2^{n/2}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Prove your claim.

1.2. Taylor Series (one variable).

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Question 1.2. *Find the first three terms of the Taylor series for* $f(x) = x^8 + x^4 + 3$ *at* $x = 1$.

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Question 1.13. *Find the first two terms of the Taylor series for* $f(x) = \log((1-x) \cdot e^x) = \log((1-x) \cdot \exp(x))$ *at* $x = 0$.

Question 1.14. *Find the first three terms of the Taylor series for* $f(x) = \cos(x)\log(1+x)$ *at* $x = 0$.

Question 1.15. *Find the first two terms of the Taylor series for* $f(x) = \log(1+2x)$ *at* $x = 0$.

2. Solutions

2.1. Sequences and Series.

Question 1: Let $a_n = \frac{1}{1+n+n^2}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Prove your claim.

Solution: This series converges. Notice that for all $n \geq 1$, $1+n+n^2 > n^2$, so $1/(1+n+n^2) < 1/n^2$, meaning that each term of this series is strictly less than $1/n^2$. Since $\sum_{n=1}^{\infty} 1/n^2$ converges, this series converges as well.

Question 2: Let $a_n = \frac{n^4}{1+2^n+(-2)^n}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Prove your claim.

Solution: We claim that this series diverges. Remember that a necessary (but not sufficient!) condition for a series to converge is that $\lim_{n\to\infty} a_n = 0$. However, notice that the odd terms of this series are all growing! That is, if $n = 2k+1$ for some integer k, then we have

$$a_{2k+1} = \frac{(2k+1)^4}{1+2^{2k+1}+(-2)^{2k+1}} = (2k+1)^4 \tag{1}$$

because $2^{2k+1} + (-2)^{2k+1} = 0$. Therefore $\lim_{k\to\infty} a_{2k+1} = \infty$, so we cannot possibly have $\lim_{n\to\infty} a_n = 0$, so the series cannot converge.

Question 3: Let $a_n = \frac{n^4}{6*n}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Prove your claim.

Solution: This series also diverges. Notice that $n^4/(6n) = n^3/6$, so $\sum_{n=1}^{\infty} n^4/(6n) = \frac{1}{6}\sum_{n=1}^{\infty} n^3$. But $n^3 \geq n$ for all integer $n \geq 1$, so this series diverges by comparison to the sequence $a_n = n$.

If instead we had $n^4/6^n$, then it would converge by the ratio test.

Question 4: Let $a_n = \frac{n}{n!}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Prove your claim.

Solution: This series converges. Notice that $n/n! = 1/(n-1)!$, so shifting our index by 1 we have

$$\sum_{n=1}^{\infty} \frac{n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}, \tag{2}$$

recalling that $0! = 1$. We claim that for $n \geq 4$, $n! > n^2$. Notice that when $n = 4$, $n! = 24$, while $n^2 = 16$. We prove this claim using induction (notice we've already shown the base case). Suppose that for some $k \geq 4$, $k! > k^2$. We show that $(k+1)! > (k+1)^2$. We see $(k+1)! = (k+1)k! > (k+1)k^2$ by our inductive assumption. Thus if we can show $k^2 > k+1$, we'll be done. Consider the polynomial $f(x) = x^2 - x - 1$. We see that $f'(x) = 2x - 1 > 0$ for all $x > 1/2$. Therefore, since $k^2 - k - 1 > 0$ when $k = 4$, $k^2 - k - 1 > 0$ for all $k \geq 4$. Therefore $(k+1)! = (k+1)k! > (k+1)k^2 > (k+1)^2$, as we wanted to show. Therefore $1/n! < 1/n^2$, and the series converges by comparison to the sequence $a_n = 1/n^2$.

We can also solve this by using the ratio test.

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\frac{(n+1)/(n+1)!}{n/n!} \\
&= \lim_{n\to\infty}\frac{n}{n+1}\frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty}\frac{n}{n+1}\frac{n!}{(n+1)n!} \\
&= \lim_{n\to\infty}\frac{n}{n+1}\frac{1}{n!} \\
&= \lim_{n\to\infty}\frac{n}{n+1}\lim_{n\to\infty}\frac{1}{n!} \;=\; 1\cdot 0 \;=\; 0.
\end{aligned}$$

Question 5: Let $a_n = \frac{n2^n}{3^n}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Prove your claim.

Solution: This series converges. Again, we claim that for sufficiently large n, $n(2/3)^n < 1/n^2$, which is equivalent to showing that $(3/2)^n/n > n^2$. Taking logs, we see this is equivalent to

$$n\log(3/2) - \log(n) > 2\log(n),$$

or $n\log(3/2) > 3\log(n)$, or $n/\log(n) > 3/\log(3/2)$ (assuming $n > 1$ so $\log(n) > 0$). Notice that the function $f(x) = x/\log(x)$ has derivative $(\log(x)-1)/(\log(x)^2)$, which is positive for all $n \geq e$. Therefore, since $f(e^{10}) = e^{10}/10 > 3/\log(3/2)$, $(3/2)^n/n > n^2$ for all $n \geq \lfloor e^{10} \rfloor$. Thus the series converges by comparison with the sequence $a_n = 1/n^2$.

Question 6: Let $a_n = n^3/2^{n/2}$. Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Prove your claim.

Solution: Let's use the Ratio test.

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\frac{(n+1)^3/2^{(n+1)/2}}{n^3/2^{n/2}} \\
&= \lim_{n\to\infty}\frac{(n+1)^3}{n^3}\frac{2^{n/2}}{2^{(n+1)/2}} \\
&= \lim_{n\to\infty}\frac{n+1}{n}\frac{n+1}{n}\frac{n+1}{n}\frac{2^{n/2}}{2^{n/2}2^{1/2}} \\
&= \lim_{n\to\infty}\frac{n+1}{n}\frac{n+1}{n}\frac{n+1}{n}\frac{1}{2^{1/2}} \\
&= \lim_{n\to\infty}\frac{n+1}{n}\lim_{n\to\infty}\frac{n+1}{n}\lim_{n\to\infty}\frac{n+1}{n}\lim_{n\to\infty}\frac{1}{2^{1/2}} \;=\; 1\cdot 1\cdot 1\cdot\frac{1}{2^{1/2}} \;<\; 1;
\end{aligned}$$

as this is less than 1, the series converges. Note the hardest part of this problem is doing the algebra well and canceling correctly.

2.2. **Taylor Series (one variable).**

Recall that the **Taylor series** of degree n for a function f at a point x_0 is given by

$$\begin{aligned} f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \end{aligned}$$

where $f^{(k)}$ denotes the k^{th} derivative of f. We can write this more compactly with summation notation as

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

where $f^{(0)}$ is just f. In many cases the point x_0 is 0, and the formulas simplify a bit to

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\ x^k \ = \ f(0) + f'(0)x + \frac{f''(0)}{2!}\ x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\ x^n.$$

The reason Taylor series are so useful is that they allow us to understand the behavior of a complicated function near a point by understanding the behavior of a related polynomial near that point; the higher the degree of our approximating polynomial, the smaller the error in our approximation. Fortunately, for many applications a first order Taylor series (ie, just using the first derivative) does a very good job. This is also called the **tangent line** method, as we are replacing a complicated function with its tangent line.

One thing which can be a little confusing is that there are $n + 1$ terms in a Taylor series of degree n; the problem is we start with the zeroth term, the value of the function at the point of interest. You should never be impressed if someone tells you the Taylor series at x_0 agrees with the function at x_0 – this is forced to hold from the definition! The reason is all the $(x - x_0)^k$ terms vanish, and we are left with $f(x_0)$, so of course the two will agree. Taylor series are only useful when they are close to the original function for x close to x_0.

Question 2.1. *Find the first five terms of the Taylor series for $f(x) = x^8 + x^4 + 3$ at $x = 0$.*

Solution: To find the first five terms requires evaluating the function and its first four derivatives:

$$\begin{array}{rcl} f(0) & = & 3 \\ f'(x) & = & 8x^7 + 4x^3 \quad \Rightarrow \quad f'(0) \ = \ 0 \\ f''(x) & = & 56x^6 + 12x^2 \quad \Rightarrow \quad f''(0) \ = \ 0 \\ f'''(x) & = & 336x^5 + 24x \quad \Rightarrow \quad f'''(0) \ = \ 0 \\ f^{(4)}(x) & = & 1680x^4 + 24 \quad \Rightarrow \quad f^{(4)} \ = \ 24. \end{array}$$

Therefore the first five terms of the Taylor series are

$$f(0) + f'(0)x + \cdots + \frac{f^{(4)}(0)}{4!}\ x^4 \ = \ 3 + \frac{24}{4!}\ x^4 \ = \ 3 + x^4.$$

This answer shouldn't be surprising as we can view our function as $f(x) = 3 + x^4 + x^8$; thus our function is presented in such a way that it's easy to see its Taylor series about 0. If we wanted the first six terms of its Taylor series expansion about 0, the answer would be the same. We won't see anything new until we look at the degree 8 Taylor series (ie, the first nine terms), at which point the x^8 term appears. $\square$

Question 2.2. *Find the first three terms of the Taylor series for $f(x) = x^8 + x^4 + 3$ at $x = 1$.*

Solution: We can find the expansion by taking the derivatives and evaluating at 1 and not 0. We have

$$\begin{array}{rclcrcl} f(x) & = & x^8 + x^4 + 3 & \Rightarrow & f(1) & = & 5 \\ f'(x) & = & 8x^7 + 4x^3 & \Rightarrow & f'(1) & = & 12 \\ f''(x) & = & 56x^6 + 12x^2 & \Rightarrow & f''(1) & = & 68. \end{array}$$

Therefore the first three terms gives

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \ = \ 5 + 12(x-1) + 34(x-1)^2.$$

$\square$

Important Note: Another way to do this problem is one of my favorite tricks, namely converting a Taylor expansion about one point to another. We write x as $(x-1)+1$; we have just added zero, which is one of the most powerful tricks in mathematics. We then have

$$x^8 + x^4 + 3 \ = \ ((x-1)+1)^8 + ((x-1)+1)^4 + 3;$$

we can expand each term by using the Binomial Theorem, and after some algebra we'll find the same answer as before. For example, $((x-1)+1)^4$ equals

$$\binom{4}{0}(x-1)^4 1^0 + \binom{4}{1}(x-1)^3 1^1 + \binom{4}{2}(x-1)^2 1^2 + \binom{4}{3}(x-1)^1 1^3 + \binom{4}{4}(x-1)^0 1^5.$$

In this instance, it is not a good idea to use this trick, as this makes the problem more complicated rather than easier; however, there are situations where this trick does make life easier, and thus it is worth seeing. We'll see another trick in the next problem (and this time it *will* simplify things).

Question 2.3. *Find the first three terms of the Taylor series for $f(x) = \cos(5x)$ at $x = 0$.*

Solution: The standard way to solve this is to take derivatives and evaluate. We have

$$\begin{array}{rclcrcl} f(x) & = & \cos(5x) & \Rightarrow & f(0) & = & 1 \\ f'(x) & = & -5\sin(5x) & \Rightarrow & f'(0) & = & 0 \\ f''(x) & = & -25\cos(5x) & \Rightarrow & f''(0) & = & -25. \end{array}$$

Thus the answer is

$$f(0) + f'(0)x + \frac{f''(0)}{2}\,x^2 \ = \ 1 - \frac{25}{2}x^2.$$

$\square$

Important Note: We discuss a faster way of doing this problem. This method assumes we know the Taylor series expansion of a related function, $g(u) = \cos(u)$. This is one of the three standard Taylor series expansions one sees in calculus (the others being the expansions for $\sin(u)$ and $\exp(u)$; a good course also does $\log(1 \pm u)$). Recall

$$\cos(u) \;=\; 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k}}{(2k)!}.$$

If we replace u with $5x$, we get the Taylor series expansion for $\cos(5x)$:

$$\cos(5x) \;=\; 1 - \frac{(5x)^2}{2!} + \frac{(5x)^4}{4!} - \frac{(5x)^6}{6!} + \cdots .$$

As we only want the first three terms, we stop at the x^2 term, and find it is $1 - 25x^2/2$. The answer is the same as before, but this seems much faster. Is it? At first it seems like we avoided having to take derivatives. We haven't; the point is we took the derivatives years ago in Calculus when we found the Taylor series expansion for $\cos(u)$. We now use that. We see the advantage of being able to recall previous results – we can frequently modify them (with very little effort) to cover a new situation; however, we can of course only do this if we remember the old results!

Question 2.4. *Find the first five terms of the Taylor series for $f(x) = \cos^3(5x)$ at $x = 0$.*

Solution: Doing (a lot of!) differentiation and algebra leads to

$$1 - \frac{75}{2}\, x^2 + \frac{4375}{8}\, x^4 - \frac{190625}{48}\, x^6;$$

we calculated more terms than needed because of the comment below. Note that $f'(x) = -15\cos^2(5x)\sin(5x)$. To calculate $f''(x)$ involves a product and a power rule, and we can see that it gets worse and worse the higher derivative we need! It is worth doing all these derivatives to appreciate the alternate approach given below. □

Important Note: There is a faster way to do this problem. From the previous exercise, we know

$$\cos(5x) \;=\; 1 - \frac{25}{2}\, x^2 + \text{terms of size } x^3 \text{ or higher}.$$

Thus to find the first five terms is equivalent to just finding the coefficients up to x^4. Unfortunately our expansion is just a tad too crude; we only kept up to x^2, and we need to have up to x^4. So, let's spend a little more time and compute the Taylor series of $\cos(5x)$ of degree 4: that is

$$1 - \frac{25}{2}\, x^2 + \frac{625}{24}\, x^4.$$

If we cube this, we'll get the first six terms in the Taylor series of $\cos^3(5x)$. In other words, we'll have the degree 5 expansion, and all our terms will be correct up to the x^6 term. The reason is when we cube, the only way we can get a term of degree 5 or less is covered. Thus we need to compute

$$\left(1 - \frac{25}{2}\, x^2 + \frac{625}{24}\, x^4\right)^3;$$

however, as we only care about the terms of x^5 or lower, we can drop a lot of terms in the product. For instance, one of the factors is the x^4 term; if it hits another x^4 term or an x^2 it will give an x^6 or higher term, which we don't care about. Thus, taking the cube but only keeping terms like x^5 or lower degree, we get

$$1 + \binom{3}{1} 1^2 \left(-\frac{25}{2}\,x^2\right) + \binom{3}{2} 1 \left(-\frac{25}{2}\,x^2\right)^2 + \binom{3}{1} 1^2 \left(\frac{625}{24}\,x^4\right).$$

After doing a little algebra, we find the same answer as before.

So, was it worth it? To each his own, but again the advantage of this method is we reduce much our problem to something we've already done. If we wanted to do the first seven terms of the Taylor series, we would just have to keep a bit more, and expand the original function $\cos(5x)$ a bit further. As mentioned above, to truly appreciate the power of this method you should do the problem the long way (ie, the standard way).

Question 2.5. *Find the first two terms of the Taylor series for $f(x) = e^x$ at $x = 0$.*

Solution: This is merely the first two terms of one of the most important Taylor series of all, the Taylor series of e^x. As $f'(x) = e^x$, we see $f^{(n)}(x) = e^x$ for all n. Thus the answer is

$$f(0) + f'(0)x \;=\; 1 + x.$$

More generally, the full Taylor series is

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

□

Question 2.6. *Find the first six terms of the Taylor series for $f(x) = e^{x^8} = \exp(x^8)$ at $x = 0$.*

Solution: The first way to solve this is to keep taking derivatives using the chain rule. Very quickly we see how tedious this is, as $f'(x) = 8x^7\exp(x^8)$, $f''(x) = 64x^{14}\exp(x^8) + 56x^6\exp(x^8)$, and of course the higher derivatives become even more complicated. We use the faster idea mentioned above. We know

$$e^u \;=\; 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots \;=\; \sum_{n=0}^{\infty} \frac{u^n}{n!},$$

so replacing u with x^8 gives

$$e^{x^8} \;=\; 1 + x^8 + \frac{(x^8)^2}{2!} + \cdots.$$

As we only want the first six terms, the highest term is x^5. Thus the answer is just 1 – we would only have the x^8 term if we wanted at least the first nine terms! For this problem, we see how much better this approach is; knowing the first two terms of the Taylor series expansion of e^u suffice to get the first six terms of e^{x^8}. This is magnitudes easier than calculating all those derivatives. Again, we see the advantage of being able to recall previous results.

Question 2.7. *Find the first four terms of the Taylor series for $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} = \exp(-x^2/2)/\sqrt{2\pi}$ at $x = 0$.*

Solution: The answer is
$$\frac{1}{2\pi} - \frac{x^2}{4\pi}.$$
We can do this by the standard method of differentiating, or we can take the Taylor series expansion of e^u and replace u with $-x^2/2$. $\square$

Question 2.8. *Find the first three terms of the Taylor series for $f(x) = \sqrt{x}$ at $x = \frac{1}{3}$.*

Solution: If $f(x) = x^{1/2}$, $f'(x) = \frac{1}{2}x^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2}$. Evaluating at 1/3 gives
$$\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{2}\left(x - \frac{1}{3}\right) - \frac{3\sqrt{3}}{8}\left(x - \frac{1}{3}\right)^2.$$
$\square$

Question 2.9. *Find the first three terms of the Taylor series for $f(x) = (1+x)^{1/3}$ at $x = \frac{1}{2}$.*

Solution: Doing a lot of differentiation and algebra yields
$$\left(\frac{3}{2}\right)^{1/3} + \frac{1}{3}\left(\frac{2}{3}\right)^{2/3}\left(x - \frac{1}{2}\right) - \frac{2}{27}\left(\frac{2}{3}\right)^{2/3}\left(x - \frac{1}{2}\right)^2.$$
$\square$

Question 2.10. *Find the first three terms of the Taylor series for $f(x) = x\log x$ at $x = 1$.*

Solution: One way is to take derivatives in the standard manner and evaluate; this gives
$$(x-1) + \frac{(x-1)^2}{2}.$$
$\square$

Important Note: Another way to do this problem involves two tricks we've mentioned before. The first is we need to know the series expansion of $\log(x)$ about $x = 1$. One of the the most important Taylor series expansions, which is often done in a Calculus class, is
$$\log(1+u) \;=\; u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots \;=\; \sum_{n=1}^{\infty} \frac{(-1)^{k+1}u^k}{k}.$$
We then write
$$x\log x \;=\; ((x-1)+1)\cdot\log\left(1+(x-1)\right);$$
we can now grab the Taylor series from
$$((x-1)+1)\cdot\left((x-1) - \frac{(x-1)^2}{2}\right) \;=\; (x-1) + \frac{(x-1)^2}{2} + \cdots.$$

Question 2.11. *Find the first three terms of the Taylor series for $f(x) = \log(1+x)$ at $x = 0$.*

Question 2.12. *Find the first three terms of the Taylor series for $f(x) = \log(1-x)$ at $x = 1$.*

Solution: The expansion for $\log(1-x)$ is often covered in a Calculus class; equivalently, it can be found from $\log(1+u)$ by replacing u with $-x$. We find
$$\log(1-x) \;=\; -\left(x + \frac{x}{2} + \frac{x}{3} + \cdots\right) \;=\; -\sum_{n=1}^{\infty}\frac{x^n}{n}.$$
For this problem, we get $x + \frac{x^2}{2}$. $\square$

Question 2.13. *Find the first two terms of the Taylor series for* $f(x) = \log((1-x)\cdot e^x) = \log((1-x)\cdot \exp(x))$ *at* $x = 0$.

Solution: Taking derivatives and doing the algebra, we see the answer is just zero! The first term that has a non-zero coefficient is the x^2 term, which comes in as $-x^2/2$. A better way of doing this is to simplify the expression before taking the derivative. As the logarithm of a product is the sum of the logarithms, we have $\log((1-x)\cdot e^x)$ equals $\log(1-x) + \log e^x$. But $\log e^x = x$, and $\log(1-x) = -x - x^2/2 - \cdots$. Adding the two expansions gives $-x^2/2 - \cdots$, which means that the first two terms of the Taylor series vanish. □

Question 2.14. *Find the first three terms of the Taylor series for* $f(x) = \cos(x)\log(1+x)$ *at* $x = 0$.

Solution: Taking derivatives and doing the algebra gives $x - x^2/2$. □

Important Note: A better way of doing this is to take the Taylor series expansions of each piece and then multiply them together. We need only take enough terms of each piece so that we are sure that we get the terms of order x^2 and lower correct. Thus

$$\cos(x)\log(1+x) \;=\; \left(1 - \frac{x^2}{2} + \cdots\right)\cdot\left(x - \frac{x^2}{2} + \cdots\right) \;=\; x - \frac{x^2}{2} + \cdots.$$

Question 2.15. *Find the first two terms of the Taylor series for* $f(x) = \log(1+2x)$ *at* $x = 0$.

Solution: The fastest way to do this is to take the Taylor series of $\log(1+u)$ and replace u with $2x$, giving $2x$. □