MATH 105: PRACTICE PROBLEMS AND SOLUTIONS FOR CHAPTER 12: SPRING 2011

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Question 1: These problems deal with open sets. Open sets were covered in Spring 2010 but not that much in Spring 2011, so do not worry about these problems if you're in 105 in Spring 2011.

(1) Let $S = \{(x, y, z) : 3x^2 + 4y^2 + 5z^2 < 6\}$. Is S open?
Solution: Yes: This is an ellipsoid where we do not include the boundary as we have strictly less than 6. Given any point in the interior, we can find a sufficiently small radius.

(2) Let $S = \{(x, y) : x^2 - y^2 = 1\}$. Is S open?
Solution: No: The set is the two branches of a hyperbola. These are one-dimensional curve, and if we draw a ball about any point on either branch, most of the points in the ball will not be on the branch.

(3) Let $S = \{(x_1, \dots, x_n) : x_1^2 + \cdots + x_n^2 < 1\}$. Is S open?
Solution: Yes: This set is open, and in fact is just the n-dimensional sphere.

(4) Let $S = \{(x, y, z) : x^2 + y^2 \leq z\}$. Is S open?
Solution: No: This is a paraboloid where we include the boundary. If we take any point (x, y, z) such that $x^2 + y^2 - z$, then we see that any ball centered at such a point hits both points inside and outside our region.

(5) Let $S = \{(x, y) : xy = 1\}$. Is S open?
Solution: No: The reason is essentially the same as the reasoning in part (2). We have a two-dimensional object in the plane; if we draw a ball about any point on either branch of the hyperbola, we'll find many points not on the curve,

(6) Let $S = \{(x, y) : x^2 + y^2 > 1\}$. Is S open?
Solution: This set is open. It is all points more than 1 unit from the origin.

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Question 2: Compute the following limits (if they exist), or prove they do not. Remember $\log x$ means the logarithm of x base e.

(1) $\lim_{x\to 1}(x^4 - 2x^3 + 3x^2 + 4x - 5)$.
Solution: As $x \to 1$, the expression is just $1 - 2 + 3 + 4 - 5 = 1$, where we used the rules for limits of suns, differences and constant multiples.

(2) $\lim_{x\to 2} \sin(3x^2 - 12)$.
Solution: As $f(x) = \sin(3x^2 - 12)$ is continuous (the sine function is continuous), the limit is just $f(2)$, which is $\sin(12 - 12) = 0$.

(3) $\lim_{x\to 2} \frac{\sin(3x^2-12)}{x-2}$.
Solution: As we have $0/0$, we must resort to other methods than simply substituting. Using L'Hopital's rule, we find the limit is just $\lim_{x\to 2} \frac{6x(\cos(3x^2-12)}{1}$, which is 12. This is because the denominator is always 1, and as $x \to 2$ the numerator tends to $6 \cdot 2 \cdot \cos 0$, and $\cos 0 = 1$.

(4) $\lim_{x\to 0} \frac{\log x}{x}$.
Solution: It is natural to want to use L'Hopital's rule. Taking the derivatives, we would find it equals $\lim_{x\to 0} \frac{1/x}{1}$, which is undefined. Unfortunately, we don't have $0/0$ or ∞/∞. As $x \to 0$, $\log x \to -\infty$. Thus as $x \to 0$ through positive values, we have a very large negative number divided by a small positive number, which is an extremely large negative number. Thus the limit tends to $-\infty$ (or does not exist).

(5) $\lim_{x\to 0} \frac{x}{\log x}$.
Solution: This is the reciprocal of the previous problem, and hence tends to 0.

(6) $\lim_{(x,y)\to(0,0)}(4xy\cos(xy) + x^2 - y^3)$.
Solution: As each function is continuous, the limit is obtained by substituting $(0, 0)$ for (x, y); we may do this as we don't have $0/0$ or infinity anywhere. We find the limit equals $0 \cos 0 + 0 - 0$, which is 0.

(7) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2-1}{xy-1}$.
Solution: As the limit of the numerator is -1 and the limit of the denominator is -1, we may use the limit of a quotient is the quotient of the limits, and hence the answer is $-1/-1$ or 1.

(8) $\lim_{(x,y)\to(1,1)} \frac{x^2y^2-1}{xy-1}$.
Solution: As we have $0/0$, we must be careful. We cannot use L'Hopital's rule as that is for one variable problems, and this has two. The easiest way to attack it is Thoreau it. Explicitly, notice that the numerator factors as $(xy - 1)(xy + 1)$, as it is a difference of two squares. We can thus cancel the factor $xy - 1$ in the numerator and the denominator, and our problem is the same as evaluating $\lim_{(x,y)\to(1,1)}(xy + 1)$, which is just 2. Note we could also have attacked the previous problem this way as well.

(9) $\lim_{(x,y)\to(0,0)} \frac{x^4-x^2y^2+y^4}{x^2+y^2+x^4y^4}$.

Solution: We have 0/0, so we have to be careful. If we use polar coordinates, we replace x with $r\cos\theta$, y with $r\sin\theta$, and then $(x,y)\to(0,0)$ becomes $r\to 0$ and θ does whatever it wants. Note that each term in the numerator is a multiple of r^4, while the denominator is $r^2+r^8\cos^4\theta\sin^4\theta$. Specifically, we have

$$\lim_{r\to 0}\frac{r^4(\cos^4\theta-\cos^2\theta\sin^2\theta+\sin^4\theta)}{r^2(1+r^6\cos^4\theta\sin^4\theta)} \;=\; \lim_{r\to 0}\frac{r^2(\cos^4\theta-\cos^2\theta\sin^2\theta+\sin^4\theta)}{1+r^6\cos^4\theta\sin^4\theta} \;=\; 0.$$

The reason the limit is zero is that we can now use the quotient rule – the limit of a quotient is the quotient of the limits, as the denominator tends to 1 as $r\to 0$. What is nice is that using polar coordinates allows us to check all possible paths of (x,y) tending to $(0,0)$.

(10) $\lim_{(x,y)\to(0,0)} x^2y^3\cos\left(\frac{1}{x^2+y^2}\right)$.

Solution: Remember that when we take limits, the point (x,y) is never $(0,0)$; thus the cosine term is always well defined, as we are never evaluating it at 1/0. The simplest way to determine the answer is to use the squeeze theorem. Note that for all choices of input, the absolute value of cosine is at most 1; however, as $(x,y)\to(0,0)$ we have x^2y^3 rapidly tending to 0. Thus we are taking the limit of a product, one term tending to zero and the other at most 1 in absolute value. Thus the product tends to 0.

Note that we cannot use the limit of a product is the product of the limits, as both limits do not exist (the cosine piece fluctuates between -1 and 1); however, we do not need the limit of each piece to exist, only the limit of the product.

(11) $\lim_{(x,y)\to(0,0)} \frac{x^2y^3\cos\left(\frac{1}{x^2+y^2}\right)}{x^2+y^2}$.

Solution: This problem is very similar to the previous. The only difference is that we now divide by x^2+y^2. The cosine piece is still at most 1 in absolute value; we now analyze the $x^2y^3/(x^2+y^2)$ term. Using polar coordinates, we find this piece is just $r^5\cos^2\theta\sin^3\theta/r^2$, which is just $r^3\cos^2\theta\sin^3\theta$. As $(x,y)\to(0,0)$, $r\to 0$ and hence this term tends to 0 as well. Thus, arguing similarly as the previous problem, we see that this limit is 0 as well.

(12) $\lim_{(x,y,z)\to(0,0,0)} \frac{x^3+y^3+z^3}{x^2+y^2+z^2}$.

Solution: This problem requires spherical coordinates, which are discussed in Section 1.4 (page 69). Sometimes physicists and mathematicians have different conventions for spherical coordinates. Using the book's convention, we have

$$x \;=\; \rho\sin\phi\cos\theta, \quad y \;=\; \rho\sin\phi\sin\theta, \quad z \;=\; \rho\cos\phi.$$

As $(x,y,z)\to(0,0,0)$ we have $\rho\to 0$ and θ,ϕ vary however they want. Thus our limit becomes

$$\lim_{\rho\to 0}\frac{\rho^3\sin^3\phi\cos^3\theta+\rho^3\sin^3\phi\sin^3\theta+\rho^3\cos^3\phi}{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+\rho^2\cos^2\phi}.$$

A little algebra shows the denominator is just ρ^2, while the numerator is a multiple of ρ^3. We thus factor out a ρ^2 and find our limit equals

$$\lim_{\rho \to 0} \rho \left(\sin^3 \phi \cos^3 \theta + \sin^3 \phi \sin^3 \theta + \cos^3 \phi \right).$$

As the trig piece is at most 3 in absolute value (each term is at most 1) and $\rho \to 0$, the product tends to 0 and thus the limit is 0.

FIGURE 1. Level sets and plot of $f(x,y) = \sin(x+y)$.

FIGURE 2. Level sets of $f(x,y) = x^2 - 4y^2$.

FIGURE 3. Level sets of $f(x,y) = x^2 + 4y$.

Question 3: Plot the level sets of value c for each function below (do enough values of c so you can recognize the result).

(1) $f(x,y) = \sin(x+y)$.
Solution: The level sets are where $x+y$ is constant. If we want to find the level set of value c, we must find all (x,y) such that $\sin(x+y) = c$, or equivalently all (x,y) such that $x+y = \arcsin(c)$. Note that this is the equation of a line, namely $y = -x + \arcsin(c)$. Of course, not all c are permissible; as the sine of any quantity is between -1 and 1, the only values of c leading to non-empty level sets are when $-1 \le c \le 1$. Note that the level sets are periodic. For example, for $c = 0$ we have $\arcsin(0) = 0, \pm\pi, \pm 2\pi, \ldots$; for $c = \sqrt{2}/2$ we have $\arcsin(\sqrt{2}/2) = \pi/4, \pi/4 \pm 2\pi, \pi/4 \pm 4\pi$ as well as $3\pi/4, 3\pi/4 \pm 2\pi, 3\pi/4 \pm 4\pi, \ldots$. See Figure 1.

(2) $f(x,y) = (x+y)\sin(x+y)$.
Solution: This problem is similar to the previous; we will still have the function constant whenever $x+y$ is constant.

(3) $f(x,y) = x^2 - 4y^2$.
Solution: The level sets are hyperbolas. See Figure 2.

(4) $f(x,y) = x^2 + 4y$.
Solution: The level sets are parabolas. We have $x^2 + 4y = c$, which implies $y = -\frac{x^2}{4} + \frac{c}{4}$. Thus the level set of value c is a downward pointing parabola with y-intercept $c/4$. See Figure 3.

(5) $f(x,y) = e^{\cos x}$.
Solution: Note there is no y-dependence in the function. We want $e^{\cos x} = c$, which means $\cos x = \log c$, or $x = \arccos(\log c)$. Of course, we need to be careful and see which values of c are permissible. As the exponential of any number is positive, the level set is empty if $c \le 0$. Further, we know that the cosine is always between -1 and 1, and thus we only have non-empty level sets for $1/e \le c \le e$. The answer will be a series of parallel lines. Specifically, y is arbitrary, which gives us a line. The reason

FIGURE 4. Level sets of $f(x, y) = e^{\cos x}$.

we have a series of parallel lines is that if we increase x by 2π we do not change the value of its cosine. See Figure 4.

Question 4: Find the gradients of the following functions:

(1) $f(x,y,z) = xy + yz + zx$.

Solution: As

$$\mathrm{grad}(f) \ = \ \nabla f \ = \ \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

we have

$$\nabla f \ = \ (y+z, x+z, x+y).$$

(2) $f(x,y) = x\cos(y) + y\cos(x)$.

Solution: As $f : \mathbb{R}^2 \to \mathbb{R}$, now the gradient is

$$\mathrm{grad}(f) \ = \ \nabla f \ = \ \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

Differentiating our function yields

$$\nabla f \ = \ (\cos y - y\sin x, -x\sin y + \cos x).$$

(3) $f(x_1, \dots, x_n) = x_1 x_2 \cdots x_n$.

Solution: As $f : \mathbb{R}^n \to \mathbb{R}$, now the gradient is

$$\mathrm{grad}(f) \ = \ \nabla f \ = \ \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right).$$

Differentiating yields

$$\nabla f \ = \ (x_2 x_3 \cdots x_n, x_1 x_3 \cdots x_n, \dots, x_1 \cdots x_{n-1}).$$

A particularly nice way of writing this is to note that

$$\frac{\partial f}{\partial x_i} \ = \ x_1 \cdots x_{i-1} x_{i+1} \cdots x_n \ = \ \frac{x_1 \cdots x_n}{x_i} \ = \ \frac{f(x_1, \dots, x_n)}{x_i}.$$

Thus

$$\nabla f \ = \ f(x_1, \dots, x_n)\left(\frac{1}{x_1}, \dots, \frac{1}{x_n}\right).$$

(4) $f(x,y,z) = 1701 x^{24601} \log(1793 x^5 y^4)$.

Solution: We could differentiate directly, but it is much easier to Thoreau the problem first and simplify Note

$$\begin{aligned} f(x,y,z) \ &= \ 1701 x^{24601} (\log 1793 + 5\log x + 4\log y) \\ &= \ (1701 \log 1793) x^{24601} + 8505 x^{24601} \log x + 6804 x^{24601} \log y. \end{aligned}$$

There is no z dependence, so $\frac{\partial f}{\partial z} = 0$. To find $\frac{\partial f}{\partial x}$, it is best not to completely expand. It does help a bit to expand the logarithm term, but not to multiply everything out (though of course it is not wrong to do so). We thus have

$$\nabla f = \left(1701 \cdot 24601 x^{24600} \left(\log 1793 + 5 \log x + 4 \log y\right) + 1701 x^{24601} \cdot \frac{5}{x}, 1701 x^{24601} \cdot \frac{4}{y}, 0\right).$$

(5) $f(x, y) = \sin(x^2 + y^2)$.

Solution: Using the chain rule, we have

$$\nabla f = \left(\cos(x^2 + y^2) \cdot 2x, \cos(x^2 + y^2) \cdot 2y\right) = 2\cos(x^2 + y^2)\,(x, y)\,;$$

of course there is no need to simplify, but by pulling out these pieces we see the gradient is in the direction (x, y), which is hidden at first.

Question 5: Determine which functions below are differentiable. Determine which functions below are differentiable. To be differentiable the tangent plane is supposed to do an excellent job approximating the function. A sufficient condition to ensure the function is differentiable is that the partial derivatives all exist and are continuous. This concept was covered more in Spring 2010 than Spring 2011, so if you are taking this in Spring 2011 do not worry as much about this problem.

We constantly use the result that if the partial derivatives exist and are continuous then the function is differentiable. Recall $\text{grad}(f) = \nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$.

(1) $f(x, y, z) = (xyz)^{4/3}$.
Solution: Taking the derivatives, we find

$$\nabla f \;=\; \left(\frac{4}{3}x^{1/3}(yz)^{4/3}, \frac{4}{3}y^{1/3}(xz)^{4/3}, \frac{4}{3}z^{1/3}(xy)^{4/3}\right).$$

Note the partial derivatives exist and are continuous, thus the function is differentiable.

(2) $f(x, y) = (xy)^{2/3}$.
Solution: This is a slight modification of the problem from class, where we had $(xy)^{1/3}$. A similar calculation (using the definition of the derivative) gives

$$\frac{\partial f}{\partial x}(0,0) \;=\; 0, \quad \frac{\partial f}{\partial y}(0,0) \;=\; 0.$$

If $x \neq 0$ we have $\frac{\partial f}{\partial x} = \frac{2}{3}x^{-1/3}y^{2/3}$, and if $y \neq 0$ we have $\frac{\partial f}{\partial y} = \frac{2}{3}x^{2/3}y^{-1/3}$. Thus the partial derivatives are not continuous, and we cannot just use our theorem above. It is possible that our function could be differentiable even though the partial derivatives are not continuous. We must go to the definition of the derivative. What is the tangent plane at $(0, 0)$? It is

$$z \;=\; f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0) \;=\; 0.$$

Thus the function is differentiable if

$$\lim_{(x,y)\to(0,0)} \frac{(xy)^{2/3} - 0}{||(x,y) - (0,0)||}$$

exists and equals zero. This limit does not exist.

To see this, let's investigate several paths. Note the denominator is $\sqrt{x^2 + y^2}$. If we take the path $x = 0$ we get 0, which we also see along the path $y = 0$ or $y = x$ or even $y = mx$. One might then be led to think the limit exists. If we try $x = r\cos\theta$ and $y = r\sin\theta$, we find

$$\lim_{r\to 0} \frac{(r^2\cos\theta\sin\theta)^{2/3}}{r} \;=\; \lim_{r\to 0} r^{1/3}(\cos\theta\sin\theta)^{2/3} \;=\; 0.$$

Thus, it does seem as if the limit is zero; **unfortunately, there is a slight technical error in what we've done here and in class. It is a very subtle point, something I am not going to hold you responsible for on exams, but for completeness I will mention it here.** Technically, we are ***not*** considering all paths when we use polar coordinates; we are only checking along paths (x, y) where $x^2 + y^2 = r \to 0$. For this problem, consider the path $y = x^{1/6}$. Then

$$(xy)^{2/3} \ = \ (x \cdot x^{1/6})^{2/3} \ = \ (x^{7/6})^{2/3} \ = \ x^{14/18};$$

the limit will not exist.

For exams, I will *not* give you a problem such as this, but I want you to be aware of them.

(3) $f(x_1, \dots, x_n) = (x_1 x_2 \cdots x_n)^2$.
Solution: The partial derivatives are computed using the power rule (or the chain rule). We have

$$\frac{\partial f}{\partial x_1} \ = \ 2(x_1 \cdots x_n)^{2-1} \frac{\partial (x_1 x_2 \cdots x_n)}{\partial x_1} \ = \ 2(x_1 \cdots x_n) \cdot x_2 \cdots x_n.$$

Note $\frac{\partial f}{\partial x_1}$ exists and is continuous; the other partial derivatives are calculated similarly, and also seen to be continuous. Thus the function is differentiable.

(4) $f(x, y, z) = 1701 x^{24601} \log(1793 x^5 y^4)$.
Solution: As the logarithm is only defined for positive inputs, we must have $x > 0$ and $y \neq 0$. Note the partial derivatives exist and are continuous, and thus the function is differentiable. If we needed to compute the derivatives, it might be worthwhile to Thoreau the logarithm term, and note

$$\log(1793 x^5 y^4) \ = \ \log 1793 + 5 \log x + 4 \log y.$$

(5) $f(x, y) = \sin(x^2 + y^2)$.
Solution: This function is clearly differentiable. We have

$$\nabla f \ = \ \left(\cos(x^2 + y^2) \cdot 2x, \cos(x^2 + y^2) \cdot 2y\right) \ = \ 2\cos(x^2 + y^2)\,(x, y)\,;$$

the partial derivatives exist and are continuous.

(6) $f(x, y, z) = x^3 \cos(x) + y^3 \cos(y)$.
Solution: This function is differentiable; the partial derivatives are

$$\nabla f \ = \ (3x^2 \cos x - x^3 \sin x, 3y^2 \cos y - y^3 \sin y),$$

and these functions are continuous.

(7) $f(x, y) = xy \cos(1/y)$.
Solution: If f is not defined at the origin, then it is clearly differentiable at every point in its domain, as its partials exist and are continuous away from the origin. What if the function is defined at the origin? What should its value be? Using the squeeze theorem, we see that we may define the function at the origin (or, in fact, any time $y = 0$) to be 0, and with such a definition our function is continuous. Thus our definition is

$$f(x, y) \ = \ \begin{cases} xy\cos(1/y) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

If $y \neq 0$, the partial derivatives are

$$\frac{\partial f}{\partial x} = y\cos(1/y), \quad \frac{\partial f}{\partial y} = x\cos(1/y) + \frac{x\sin(1/y)}{y},$$

while if $y = 0$ the partial derivatives are

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0.$$

The partial derivatives exist but are not continuous (while $\frac{\partial f}{\partial x}$ is continuous when $y = 0$, $\frac{\partial f}{\partial y}$ is not). This does not prove that the function is not differentiable, though; it just means that we cannot appeal to our theorem that says the partial derivatives being continuous implies the function is differentiable. The tangent plane at $(0,0)$ is just $z = 0$. To be differentiable, we would need

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - 0}{||(x,y) - (0,0)||}$$

to exist and equal 0. This limit does exist, as it is either involves terms such as $\frac{xy\cos(1/y)}{\sqrt{x^2+y^2}}$ or $\frac{0}{\sqrt{x^2+y^2}}$. In both cases, we do get 0 as $(x,y) \to (0,0)$. Thus our function is differentiable, even though the partial derivatives are not continuous. **In other words, we have found an example of a function which is differentiable, but whose partial derivatives, while existing, are not continuous.**

Question 6: Find the tangent plane approximation to $f(x,y) = e^{xy} + 2\sin(x+y)\cos(x-y)$ at the point (x_0, y_0). Use the tangent plane plane to estimate $f(-.01, .02)$ by choosing (x_0, y_0) appropriately.

Solution: The tangent plane is given by

$$z \;=\; f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(y-y_0).$$

To estimate the value at $(-.01, .02)$, we take $(x_0, y_0) = (0,0)$. We have $f(0,0) = 1$ and

$$\begin{aligned}\frac{\partial f}{\partial x} &= ye^{xy} + 2\cos(x+y)\cos(x-y) - 2\sin(x+y)\sin(x-y)\\ \frac{\partial f}{\partial y} &= xe^{xy} + 2\cos(x+y)\cos(x-y) + 2\sin(x+y)\sin(x+y),\end{aligned}$$

which implies

$$\frac{\partial f}{\partial x}(0,0) \;=\; 2, \qquad \frac{\partial f}{\partial y}(0,0) \;=\; 2.$$

Therefore the tangent plane approximation to $f(-.01, .02)$ is

$$f(0,0) + \frac{\partial f}{\partial x}(0,0)(-.01-0) + \frac{\partial f}{\partial y}(0,0)(.02-0) \;=\; 1 + 2(-.01) + 2(.02) \;=\; 1.02,$$

which is quite close to $f(-.01, .02)$, which is approximately 1.0198.

Note one can avoid the product rule by recalling trig identities; in particular, as $2\sin(x+y)\cos(x-y) = \sin 2x + \sin 2y$, we can simplify the derivative.

Question 7: Parametrize the following curves, and find the tangent line approximation at the given point

(1) A circle of radius 5 centered at $(2, 3)$ going counter-clockwise starting at the point $(7, 3)$; find the tangent line at the point $(7, 3)$.

Solution: The answer is $c(t) = (2, 3) + (5\cos t, 5\sin t)$, or $c(t) = (2 + 5\cos t, 3 + 5\sin t)$. Note that when $t = 0$ we have $c(0) = (7, 3)$. The tangent line goes through the point $(7, 3)$ and is in the direction $c'(0)$. As $c'(t) = (-5\sin t, 5\cos t)$, we have $c'(0) = (0, 5)$, the tangent line is just

$$(x, y) \;=\; c(0) + tc'(0) \;=\; (7, 3) + t(0, 5) \;=\; (7, 3 + 5t).$$

To check that our curve $c(t)$ correctly parametrizes our circle, we must show

$$(x(t) - 2)^2 + (y(t) - 3)^2 \;=\; 5^2,$$

which follows from straightforward algebra.

(2) A circle of radius 5 centered at $(2, 3)$ going counter-clockwise starting at the point $(5, 7)$; find the tangent line at the point $(5, 7)$.

Solution: This is similar to the previous problem; the difference is now we want to start at the point $(5, 7)$. Note that $(5, 7) - (2, 3) = (3, 4)$; as $\sqrt{3^2 + 4^2} = 5$, we do see that the point $(5, 7)$ is on the circle of radius 5 centered at $(2, 3)$. Let's consider our curve from the previous part: $c(t) = (2 + 5\cos t, 3 + 5\sin t)$. This traces out the right path, but it starts at the wrong point. Thus, consider

$$c(t) \;=\; (2 + 5\cos(t + t_0), 3 + 5\sin(t + t_0)).$$

If we take $t = 0$, we now have $c(0) = (2 + 5\cos t_0, 3 + 5\sin t_0)$; we just need to take $\cos t_0 = 3/5$ and $\sin t_0 = 4/5$, which means $\tan t_0 = 4/3$ or $t_0 = \arctan(4/3)$. The tangent line is

$$(x, y) \;=\; c(0) + c'(0)t.$$

As $c'(t) = (-5\sin(t + t_0), 5\cos(t + t_0))$, we have

$$c'(0) \;=\; (-5\sin t_0, 5\cos t_0) \;=\; (-4, 3),$$

leading to a tangent line of

$$(x, y) \;=\; (5, 7) + (-4, 3)t \;=\; (5 - 4t, 7 + 3t).$$

(3) The curve $y = e^x$ from $x = 1$ to $x = 10$; find the tangent line at the point $(2, e^2)$.

Solution: We may parametrize the curve by $c(t) = (t, e^t)$ for $1 \le t \le 10$. The point $(2, e^2)$ corresponds to $t = 2$. We have $c(2) = (2, e^2)$ and $c'(t) = (1, e^t)$ so $c'(2) = (1, e^2)$ and hence the tangent line is

$$(x, y) \;=\; c(2) + c'(2)t \;=\; (2, e^2) + (1, e^2)t \;=\; (2 + t, e^2 + e^2 t).$$

Question 8: Consider the parametrized curve $c(t) = (\cos t, 2\sin t)$. What path does this trace out in the plane? Is it periodic (i.e., does it repeat where it is), and if so, what is the period? Does a particle whose position is given by $c(t)$ move at constant speed? If not, when is it moving fastest?

Solution: This path traces out an ellipse. Note that $x(t) = \cos t$ and $y(t) = 2\sin t$. Thus

$$4x(t)^2 + y(t)^2 \ = \ 4\cos^2 t + 4\sin^2 t = 4.$$

It is periodic; every time t increases by 2π we return to the starting point. As

$$c'(t) \ = \ (-\sin t, 2\cos t),$$

we have

$$||c'(t)|| \ = \ \sqrt{\sin^2 t + 4\cos^2 t} \ = \ \sqrt{1 + 3\cos^2 t}.$$

Note the speed is not constant; the particle is traveling fastest when $\cos^2 t = 1$, or at time $0, \pi, 2\pi, \dots$.

Question 9: Find the derivative of $A(x,y,z) = (x^3y + y^2z + e^x)(\sin(x) + \cos(y) - z + 10)$.

Solution: There are two ways to do this. The fist is to multiply everything out and then take the derivative directly. The problem with that is the first factor has three terms and the second has four. Multiplying everything out would give 12 terms. It is thus better to use the product rule. Let the first factor be $f(x,y,z)$ and the second be $g(x,y,z)$. Then

$$(DA)(x,y,z) \ = \ (Df)(x,y,z)g(x) + f(x)(Dg)(x,y,z).$$

We thus find

$$\begin{aligned}(Df)(x,y,z) \ &= \ \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= \ \left(3x^2y + e^x, x^3 + 2yz, y^2\right),\end{aligned}$$

while

$$\begin{aligned}(Dg)(x,y,z) \ &= \ \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) \\ &= \ (\cos(x), -\sin(y), -1).\end{aligned}$$

Putting all the pieces together, we find

$$\begin{aligned}(Dh)(x,y,z) \ &= \ \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}\right) \\ &= \ \left(3x^2y + e^x, x^3 + 2yz, y^2\right)(\sin(x) + \cos(y) - z + 10) \\ &\quad + (x^3y + y^2z + e^x)(\cos(x), -\sin(y), -1).\end{aligned}$$

Question 10 : Let $g(x,y,z) = (xy, yz, xz)$ and $f(u,v,w) = u^2 + v^2$. Set $A(x,y,z) = f(g(x,y,z))$. Compute DA.

Solution: Setting $g(x,y,z) = (g_1(x,y,z), g_2(x,y,z), g_3(x,y,z))$, we have

$$(Dg)(x,y,z) \ = \ \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} & \frac{\partial g_3}{\partial z} \end{pmatrix} \ = \ \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{pmatrix}.$$

Similarly we find

$$(Df)(u,v,w) \;=\; \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}\right) \;=\; (2u, 2v, 0),$$

which implies

$$(Df)(g(x,y,z)) \;=\; (Df)(xy, yz, xz) \;=\; (2xy, 2yz, 0).$$

The chain rule says $(DA)(x,y,z) = (Df)(g(x,y,z))(Dg)(x,y,z)$, so

$$(DA)(x,y,z) \;=\; (2xy, 2yz, 0)\begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{pmatrix} \;=\; (2xy^2, 2x^2y + 2yz^2, 2y^2z).$$

Question 11 : Let $g(x,y,z) = xy^2z^3$. Compute the directional derivative of g at $(1,1,1)$ in the direction $\overrightarrow{v}$, where $\overrightarrow{v}$ is a unit vector in the direction $(3,4,12)$. In what direction is g increasing fastest?

Solution: The length of $\overrightarrow{v}$ is $||\overrightarrow{v}|| = \sqrt{3^2+4^2+12^2} = \sqrt{9+16+144} = \sqrt{169} = 13$; thus a unit vector in this direction is $\overrightarrow{u} = (3/13, 4/13, 12/13)$. The gradient of g is simply

$$(\nabla g)(x,y,z) \;=\; (y^2z^3, 2xyz^3, 3xy^2z^2),$$

so

$$(\nabla g)(1,1,1) \;=\; (1,2,3).$$

The directional derivative is thus

$$(\nabla g)(1,1,1)\cdot(3/13, 4/13, 12/13) \;=\; (1,2,3)\cdot(3/13,4/13,12/13) \;=\; \frac{1\cdot 3 + 2\cdot 4 + 3\cdot 12}{13} \;=\; \frac{47}{13}.$$

Our function g is increasing fastest in the direction of its gradient, i.e., in the direction $(1,2,3)$.

Question 12 : Let $g(x,y,z) = e^x\cos\pi y + z\cos\pi x$. If possible, find the tangent plane to the level set of value 1 for $g(x,y,z)$ at $(x,y,z) = (1,1,1)$. If possible, find the tangent plane to the level set of value 0 for $g(x,y,z)$ at $(x,y,z) = (0,1,1)$.

Solution: It is not possible to find the tangent plane in the first case, as the specified point is not on the level set (specifically, $g(1,1,1) = e(-1) + 1(-1) = -(e+1) \neq 1$. For the second, the point is on the level set as $g(0,1,1) = 0$. We have

$$\nabla g \;=\; (e^x\cos\pi y - \pi z\sin\pi x, -\pi e^x\sin\pi y, \cos\pi x)$$

which implies

$$(\nabla g)(0,1,1) \;=\; (-1, 0, -1).$$

The equation of the tangent plane is

$$(\nabla g)(0,1,1)\cdot(x-0, y-1, z-1) \;=\; 0,$$

or

$$(-1,0,-1)\cdot(x-0, y-1, z-1) \;=\; 0,$$

which simplifies to

$$z \;=\; 1 - x.$$