Geometric Random Variable

$\text{Prob}(X = n) = c \cdot p^n$ for $n \geq 0$ integer

Now $\sum_{n=0}^{\infty} c p^n = \frac{c}{1-p} \Rightarrow c = 1-p$

Thus $\text{Prob}(X = n) = (1-p)p^n$ $n \geq 0$ integer

Mean $= \mathbb{E}[X] = \sum_{n=0}^{\infty} n \cdot (1-p)p^n = (1-p)\sum_{n=0}^{\infty} n \cdot p^n$

↳ Differentiating identities:

$$\sum_{n=0}^{\infty} x^n = (1-x)^{-1} \text{ : apply } x\frac{d}{dx} \text{ to both sides}$$

$$\sum_{n=0}^{\infty} n x^n = x(1-x)^{-2}$$

Thus $\mathbb{E}[X] = (1-p) \cdot p(1-p)^{-2} = \frac{p}{1-p}$ = Runs Scored or Allowed

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Say $X = \text{Geom}(p)$ and $Y = \text{Geom}(q)$: What is $\text{Prob}(X > Y)$?

↳ $\sum_{n=1}^{\infty} \sum_{m=0}^{n-1} (1-p)p^n \cdot (1-q)q^m$

$= (1-p)(1-q)\sum_{n=1}^{\infty} p^n \cdot \frac{1-q^n}{1-q}$

$= (1-p)\sum_{n=1}^{\infty} p^n - (pq)^n$

$= (1-p)\left[\sum_{n=0}^{\infty} p^n - \sum_{n=0}^{\infty} (pq)^n\right]$

$= (1-p)\frac{1}{1-p} - (1-p)\frac{1}{1-pq} = 1 - \frac{1-p}{1-pq}$

What does This mean?

not equal to 50% if p is q!

So $\text{Prob}(X > Y)$ is $1 - \frac{1-p}{1-pq} = \frac{-pq+p}{1-pq} = \frac{p(1-q)}{1-pq}$

Similarly $\text{Prob}(Y > X)$ is $\frac{q(1-p)}{1-pq}$

Trouble: don't add up to 1! Arg!

↳ Conditional Probability: scores discrete, ignored $\text{prob}(X = Y)$

↳ can find directly: $\sum_{n=0}^{\infty} (1-p)p^n(1-q)q^n = \frac{(1-p)(1-q)}{1-pq}$

Thus $$\text{Prob}(X \text{ wins}) = \frac{\text{Prob}(X > Y)}{\text{Prob}(X > Y) + \text{Prob}(Y > X)}$$

So $$\text{Prob}(X \text{ wins}) = \frac{\frac{p(1-q)}{1-pq}}{\frac{p(1-q)}{1-pq} + \frac{q(1-p)}{1-pq}} = \frac{p(1-q)}{p(1-q) + q(1-p)}$$

Now RS = ave runs scored for X is $\frac{p}{1-p}$

RA = " " allowed by X is $\frac{q}{1-q}$ (what Y scores)

Multiply by $\frac{1}{(1-p)(1-q)} \Big/ \frac{1}{(1-p)(1-q)}$

$$\Rightarrow \text{Prob}(X \text{ wins}) = \frac{\frac{p}{1-p}}{\frac{p}{1-p} + \frac{q}{1-q}} = \frac{RS}{RS + RA}$$

Check: somewhat reasonable: 50% if $RS = RA$,

and > 50% if $RS > RA$.