Math 105: Double Integral Example

Below is my penance for my algebra mistakes in Section 2 today. The problem is to find

$$\int_{y=0}^{2} \int_{x=y/2}^{1} y e^{x^3}\,dx\,dy.$$

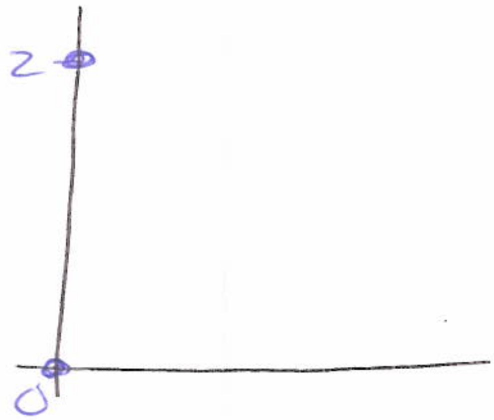
As we don't have a nice anti-derivative for e^{x^3}, we expect we'll have to switch orders.

The goal of these notes is to illustrate how to find the bounding curves for vertically and horizontally simple regions, how to write down the regions of integration, and how to evaluate the integrals.

The Calc III content is finding the bounding curves; evaluating the iterated integrals is Calc II.

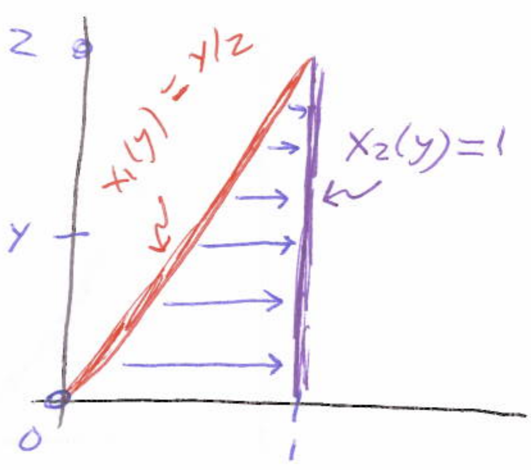
Integral: $\int_{y=0}^{2} \int_{x=y/2}^{1} y e^{x^3} \, dx \, dy$

Step 1: Find the region of integration (ie, draw picture)



(1) First we fix y; y runs from 0 to 2

As y is fixed, x is integrated over first, and we are thus looking for bounding curves $x_1(y)$, $x_2(y)$



(2) For a given y, x runs from y/2 to 1.

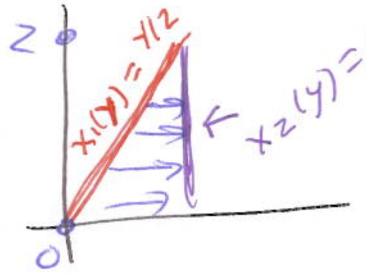
This tells us that

- $x_1(y) = y/2$
- $x_2(y) = 1$

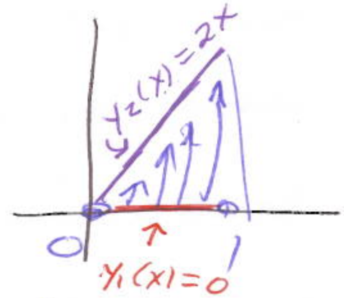
Note we are presenting the integration by regarding this as a horizontally simple region. We found $x_1(y)$ and $x_2(y)$ for fixed y.

Integral: $\int_{y=0}^{2} \int_{x=y/2}^{1} ye^{x^3}\,dx\,dy$

Step 3: Switch Orders of integration



$\Longrightarrow$ to $\Longrightarrow$



$$\int_{y=0}^{2} \int_{x=y/2}^{1} ye^{x^3}\,dx\,dy = \int_{x=0}^{1} \int_{y=y_1(x)}^{y_2(x)} ye^{x^3}\,dy\,dx$$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{2x} ye^{x^3}\,dy \right] dx$$

$$= \int_{x=0}^{1} \left[\frac{y^2}{2} e^{x^3} \Big|_{y=0}^{2x} \right] dx$$

$$= \int_{x=0}^{1} 2x^2 e^{x^3}\,dx$$

Let $u=x^3$, $du=3x^2dx$ so $2x^2dx = \frac{2}{3}du$

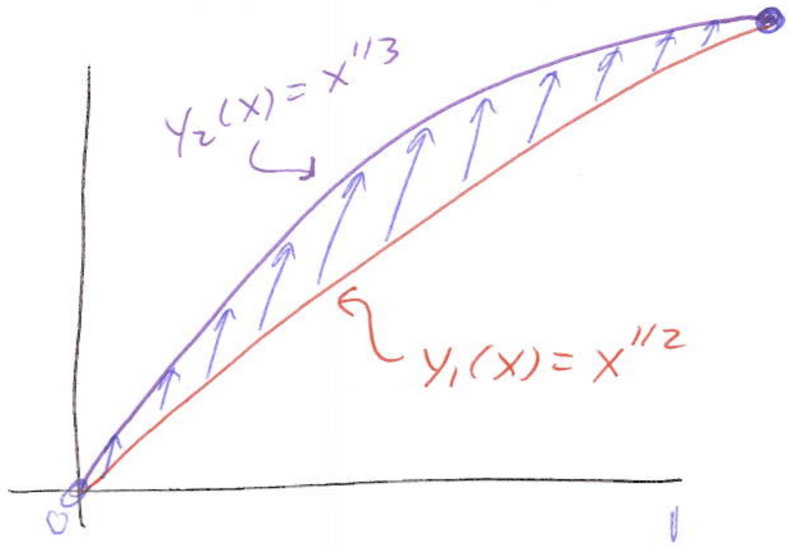
$x: 0 \to 1$ implies $u: 0 \to 1$

$$= \int_{u=0}^{1} e^u \cdot \frac{2}{3}du = \frac{2}{3}e^u \Big|_{u=0}^{1} = \frac{2}{3}e^1 - \frac{2}{3}e^0$$

$$= \frac{2}{3}e - \frac{2}{3}.$$

Example: Region b/w $X=Y^2$ and $X=Y^3$

Consider the region b/w the curves $X=Y^2$ and $X=Y^3$ from $X=0$ to $X=1$. Sketch, set up integration.



(1) Fix x from 0 to 1.

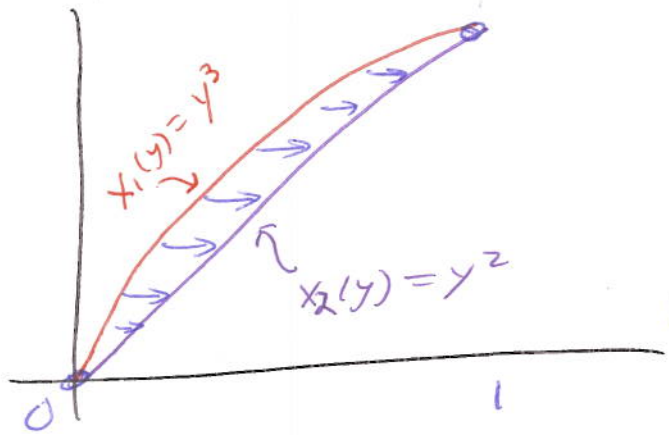
(2) Start at $y_1(x) = ?$

(3) go to $y_2(x) = ?$

Which curve is on top? We have $x=y^2$ and $x=y^3$ are the same as $y=x^{1/2}$ and $y=x^{1/3}$. If we take $x = 1/2^6$ then $x^{1/2} = 1/2^3 = 1/8$ and $x^{1/3} = 1/2^2 = 1/4$, so by looking at this special point we see the top curve is $y_2(x) = x^{1/3}$ and the bottom curve is $y_1(x) = x^{1/2}$.

$$\int_{x=0}^{1} \int_{y=x^{1/2}}^{x^{1/3}} f(x,y)\,dy\,dx.$$

Example (Cont): Region b/w $x=y^2$ and $x=y^3$



(1) Fix y from 0 to 1

(2) Start at $x_1(y)=?$

(3) Go to $x_2(y)=?$

We know from before that $x=y^3$ is the "top" curve as vertically simple; as horizontally simple it gives us the "left" or "start" curve.

$$\int_{y=0}^{1} \int_{x=y^3}^{y^2} f(x,y)\,dx\,dy.$$

Fubini's Thm: If f is "nice" and region of integration is finite can switch orders of integration, noting that may have to change the bounds of integration (if not a rectangle)