MATH 105: MULTIVARIABLE CALC

INTRO/OBJECTIVES

↳ Building block course

↳ Study how things change/modeling

↳ Learn material **and techniques!**

↳ General math skills

- Good Notation
- Asking right question (motivation clip)
- Common techniques
- How to attack problems (brute force vs elegant)
- Building intuition/special cases : $\dfrac{p \pm pq}{p+q \pm 2pq}$

TYPES OF PROBLEMS

↳ Physics (forces, Newton's law)

↳ Economics (optimization)

↳ Finance (Monte-Carlo: dartboard)

↳ Geometry / Probability (areas/volumes)

↳ Approximation Theory (everything!)

↳ Discuss discovery of Neptune

↳ Library trip

COURSE MECHANICS

- Pace: faster than 104: start very fast, slower at end
  - learn by doing, daily HW
- Food / cell phones
- Read material before class (5% of grade!)
- 12 week vs 15 week
  - supplemental lectures
- Minimal "ugly" geometry
- Office hours / on-line schedule
  - strongly urge to visit
  - TAs

THEME: CONTROL OVER YOUR EDUCATION

GRADING

- HW 15%
- Midterms 40% (2 or 3, best 1 or 2)
- Final 45%
  - READING BEFORE CLASS! HONOR CODE: 5% (never unprepared business meeting)
  - SCRIBE OPTION: REPLACE EXAMS: LOWER DECREASES BY 5%
- QUIZ OPTION: SAME; PROJECT: SAME
  - CAN DO ~~BOTH~~ TWO FOR 10%, THREE FOR 15%, FOUR FOR 20%

Different people different styles! INTRO grading tries to reflect

General Thoughts on the Course

Again, lots of items could emphasize, need to make choices in a 12 week class. Always happy to meet individually or in small groups to discuss further.

- Building intuition on theorems and equations
- Importance of knowing defns, examples and counter-examples.
- Being prepared for class: unless you say otherwise, 5% of your grade is a 100 for reading sections BEFORE class; if you don't read honor bound to tell me, may miss two without penalty.
- Will minimize plotting in course: lots of programs on line to do this.

Below is a tentative list of homework problems for Math 105. The first set are the assigned problems and are to be handed in for a grade; the second set are optional. In general, HW is due each class, and we will typically cover on the order of one section a day. All problems are worth 10 points; I will drop whatever assignment helps your HW average the most. For the suggested problems: these are *not* to be handed in, though of course I and the TAs are happy to chat with you about them. The purpose of these problems is to provide both guided, additional practice for those who want it, as well as to post a few more challenging problems. I will add due dates as the semester progresses.

Homework problems for Math 105:

- Introduction: (1) What is wrong with the following argument (from Mathematical Fallacies, Flaws, and Flimflam - by Edward Barbeau): There is no point on the parabola $16y = x^2$ closest to (0,5). This is because the distance-squared from (0,5) to a point (x,y) on the parabola is $x^2 + (y-5)^2$. As $16y = x^2$ the distance-squared is $f(y) = 16y + (y-5)^2$. As $df/dy = 2y+6$, there is only one critical point, at $y = -3$; however, there is no x such that (x,-3) is on the parabola. Thus there is no shortest distance! (2) Compute the derivative of $\cos(\sin(3x^2 + 2x \ln x))$. Note that if you can do this derivative correctly, you should be fine for the course. (3) Let $f(x) = x^2 + 8x + 16$ and $g(x) = x^2+2x-8$. Compute the limits as x goes to 0, 3 and ∞ of $f(x)+g(x)$, $f(x)g(x)$ and $f(x)/g(x)$.
- Section 11.1: Page 823: #9, #18, #38, #42.
- Section 11.2: Page 833: #1, #39, and also find the cosine of the angle between **a** = <2, 5, -4> and **b** = <1, -2, -3>.
- Section 11.3: Page 842: #1, #5, #11, #12.
- Section 11.4: Page 849: #1, #2, #3, #22.
- Section 11.8: Page 893: #1, #26.
- Section 12.2: Page 908: #2, #4, #5, #27, #32.
- Section 12.3: Page 917: #1, #8, #10, #24, #38, #54 (hint: limit of sin(t)/t is 1 as t tends to 0).
- Section 12.4: Page 928: #1, #4, #5, #22, #25, #33, #36, #63 (is this surprising?).
- Section 12.5: Page 940: #5, #11, #29, #61.
- From my notes on the Method of Least Squares: Exercise 3.3, Exercise 3.9.
- Section 12.6: Page 949: #18, #23. Use Newton's Method to find a rational number that estimates the square-root of 5 correctly to at least 4 decimal places.
- Section 12.7: Page 960: #2, #5, #8, #34, #41.
- Section 12.8: Page 971: #3, #10, #11, #19, #21, #29.
- Section 12.9: Page 981: #1, #14 (note the symmetry), #19, #35, #51.
- Section 13.1: Page 1004: #15, #24, #25, #37.
- Section 13.2: Page 1011: #4, #11, #13, #25, #30.
- Section 13.3: Page 1018: #13, #42 (hint: notice that the region that matters lies above a circle).
- Section 13.4: Page 1026: #4, #13.
- Section 13.7: Page 1056: #37.
- Section 13.9: Page 1070: #2, #3, #14.
- From multivariable calculus (Cain and Herod): Page 10-7: #13. Page 10-8: #14, #15, #16. Page 10-8: #17. Page 10-10: #18, #19.
- Problems leading up to Green's Theorem TBD.

Suggested problems for Math 105:

- Introduction: Extra Credit: (1) Let N be a large integer. How should we divide N into positive integers a_i such that the product of the a_i is as large as possible. Redo the problem when N and the a_i need not be integers. (2) Without using *any* computer, calculator or computing by brute force, determine which is larger: e^π or π^e. (In other words, find out which is larger *without* actually determining the values of e^π or π^e). If you're interested in formulas for π, see also my paper A probabilistic proof of Wallis' formula for π, which appeared in the American Mathematical Monthly (there are a lot of good articles in this magazine, many of which are accessible to freshmen). (3) Prove that the product of the slopes of two perpendicular lines in the plane that are not parallel to the coordinate axes is -1. What is the generalization of this to lines in three-dimensional space? What is the analogue of the product of the slopes of the line equaling -1?
- Section 11.1: Page 823: Is #38 true for all points (i.e., if you take any three points in the plane)?
- Section 11.2: Page 833: #59, #61.
- Section 11.3: Page 842: #7, #17a.
- Section 11.4: Page 849: #25, #54, #58, #60.
- Section 11.8: Page 893: #33, #53. Extra Credit: #55.
- Section 12.2: Page 908: #41, #43, #45.
- Section 12.3: Page 917: #41, #51, #55.
- Section 12.4: Page 981: #55, #57, #58, #68.
- Section 12.5: Page 940: #10, #17, #46.
- Section 12.6: Extra Credit: Let $f(x) = \exp(-1/x^2)$ if $|x| > 0$ and 0 if $x = 0$. Prove that $f^{(n)}(0) = 0$ (i.e., that all the derivatives at the origin are zero). Show this implies the Taylor series approximation to f(x) is the function which is identically zero. As f(x) = 0 only for x=0, this means the Taylor series (which converges for all x) only agrees with the function at x=0, a very unimpressive feat (as it is forced to agree there).
- Section 12.7: Page 960: #38, #53.
- Section 12.8: Page 971: #40, #41, #60.
- Section 12.9: Page 981: #36, #37, #47, #49, #62 (important).
- Section 13.1: Page 1004: #33.
- Section 13.2: Page 1011: #41, #44, #49.
- Section 13.3: Page 1018: #29.
- Section 13.4: Page 1026: #7, #34.
- Section 13.7: Page 1056: #47, #48. (Extra credit for solving both of these.)
- Section 13.9: Page 1070: #10, #28, #29.
- From multivariable calculus (Cain and Herod): Exercise 1 (page 10.3). Extra credit: Find a series where the ratio test provides no information on whether or not it converges but the root test says whether or not it converges or diverges.
- Problems leading up to Green's Theorem TBD.

INFORMATION ON READING BEFORE CLASS

Below are some comments to help you prepare for each class' lecture. For each section in the book, I'll mention what you should have read for class. In other words, what are the key points. When you come to class, you should have already read the section and have some sense of the definitions of the terms we'll study and the results we'll prove. This does not mean you should know the material well enough to give the lecture; it does mean that you should have a familiarity with the material so that when I lecture on the math, it won't be your first exposure to the terminology or results. Everyone processes and learns material in different ways; for me, I find it ***very*** hard to go to a lecture on a subject I'm unfamiliar with and get much out of it. I need to have some sense of what will happen, as otherwise I spend too much time absorbing the definitions, and then I fall behind. I'm hoping the bullet points below will help you in preparing for each lecture. If there is anything else I can do to assist, as always let me know (either email directly, or anonymously through mathephs@gmail.com, passsword 11235813).

Also, you may wish to look at some worked out examples before class that are similar to the HW. Examples from when I taught the class in 2010 are available online here, though we used a different book then and covered slightly different material; I will do many of these problems in class. The reason I want to do these is precisely because I have written up the solution. This way you can sit back a bit more and follow the example without worrying about writing everything down.

CHAPTER 11: Vectors, Curves and Surfaces in Space

- **Section 11.1: Vectors in the Plane**
  - Notation, definition of vectors and properties.
  - Proof of the Pythagorean formula (which is crucial in determining lengths).
- **Section 11.2: Three-Dimensional Vectors**
  - Know the definition of the dot product of two vectors, and the connection of that to the angle between two vectors.
- **Section 11.3: The Cross Product of Vectors**
  - Know the definition of determinants of 2x2 and 3x3 matrices, and how to compute these.
  - The determinant has much geometrical meaning, denoting the (signed) volume of the parallelpiped spanned by the rows (or columns).
  - Know the definition of the cross product and how to compute it, as well as some of its properties.
- **Section 11.4: Lines and Planes in Space**
  - Know the various formulas for writing the equation of a line.
  - There are several ways to write the equation of a plane; it's similar to writing the equation of a line: depending what information you are given, some ways are more convenient than others.
  - One easy way to find the equation of a plane is to know the normal direction. This is a great application of the cross product, as **v** x **w** is perpendicular to both **v** and **w**. Unfortunately we don't have the cross product in higher dimensions.
- **Section 11.8: Cylindrical and Spherical Coordinates**
  - Know the different formulas to convert from Cartesian to Cylindrical or Spherical coordinates.

CHAPTER 12: Partial Differentiations

- **Section 12.1: Introduction**

- Not much here except (what a surprise) that many functions in the real world depend on several variables.

- **Section 12.2: Functions of Several Variables**
  - First is when a function is defined on a domain (usually just making sure the denominator is non-zero).
  - The level set (of value c) of a function are all inputs that are mapped to c. Think of this as all points on a mountain that are the same height, or on a weathermap all places with the same temperature.
- **Section 12.3: Limits and Continuity**
  - Know the definition and basic properties of limits.
  - Caveats: certain operations are not defined: $\infty-\infty$, $0 * \infty$.
- **Section 12.4: Partial Derivatives**
  - Know the definition of how to take a partial derivative. Similar to one-variable calculus, we do not want to have to use this definition in practice, and thus want to modify our rules of one-variable differentiation to allow us to take derivatives here.
  - Know the formula for computing the tangent plane to $z = f(x,y)$ at a given point, so long as the partial derivatives exist at that point.
  - Iterated Partial Derivatives:
    - The definition of mixed partial derivatives: Given a function f, we can compute its partial derivatives, such as $\delta f/\delta x$ and $\delta f/\delta y$. We can then take the partial derivatives of the partial derivatives: $\delta(\delta f/\delta x)\delta y$ and $\delta(\delta f/\delta y)\delta x$. In the first, we first take the derivative with respect to x, and then take the derivative with respect to y; in the second, we take the derivatives in the other order. Does the order matter? We write $\delta^2 f/\delta y\delta x$ for $\delta(\delta f/\delta x)\delta y$; thus the derivative symbol on the far right of the denominator is the derivative we take first, and the symbol on the far left is what we take last.
    - The definition of C^2, the class of twice continuously differentiable functions: If the function is C^2, the mixed partial derivatives of second order (i.e., involving at most two derivatives) exist and are continuous. Just as C^1 functions had nice properties (being C^1 means the partial derivatives exist and are continuous, which implies the function is differentiable), being C^2 has nice properties.
    - Equality of Mixed Partial Derivatives: For a C^2 function, the order of differentiation does not matter; in other words, $\delta^2 f/\delta y\delta x = \delta^2 f/\delta x\delta y$.
    - Notation: f_x means $\delta f/\delta x$, f_{xy} means $(f_x)_y$, which is $\delta^2 f/\delta y\delta x$. Note that the order of subscripts is the opposite of the order of differentiation; fortunately if f is C^2 then the order does not matter!
    - Examples of partial differential equations: The rest of the section is devoted to examples of partial differential equations. Solving these in general are beyond the scope of this course; in fact, most are beyond the scope of humanity! One example is the Millenium Prize for the Navier-Stokes equation (i.e., solve this and earn \$1,000,000). You are not responsible for any of this material; it is provided in nice detail in this book for your interest, and to show you what you will see if you continue with mathematics.
- **Section 12.5: Multivariable Optimization Problems**
  - Advanced result: any continuous function on a nice region that includes the boundary attains its maximum and minimum.
  - Definition of local maximum / minimum: You should be comfortable with the definition of a local

max/min. Essentially, a point $\mathbf{x_0}$ is a local maximum if there is some ball centered at $\mathbf{x_0}$ such that $f(\mathbf{x_0})$ is at least as large as $f(\mathbf{x})$ for all other $\mathbf{x}$ in that ball. For example, $f(x,y) = y^2 \sin^2(xy)$ has a local minimum at $(x,y) = (0,0)$. Clearly $f(x,y)$ is never negative, and it is zero at $(0,0)$. Thus $(0,0)$ is a local minimum. Note that $f(x,0)$ is also zero for *any* choice of x. Thus to be a local minimum we don't need to be strictly less than all other nearby points.

  - First derivative test for local extrema: The generalization of the results from one-variable calculus; candidates for max/min are where the first derivative (the gradient) vanishes.
  - Important Example: The Method of Least Squares: We will give one of the most important applications of partial derivatives and optimization, the Method of Least Squares. This is a technique to allow us to find best fit parameters. Finding such values is central in numerous disciplines. Specifically, we have some data and we want to see if it fits our theory. If you have a data set you'd like analyzed, please let me know.
- **Method of Least Squares**
  - My notes on the Method of Least Squares
  - There are many different ways to choose how we measure errors. Different choices lead to different `best fit' values for parameters. The main advantage of measure errors by summing the squares of the deviations is that the tools of calculus and linear algebra are available.
- **Section 12.6: Increments and Linear Approximations**
  - The idea is to replace complicated functions with simpler ones that are easier to analyze (in many cases, one can get very good results just by using linear approximations).
  - Newton's method is one of the most important uses of the tangent line. The idea is based on the fact that locally any function is approximately linear.
- **Section 12.7: The Multivariable Chain Rule**
  - The most important part of this section is the statement of the chain rule. A good way to remember what goes where is through the atom graph.
- **Section 12.8: Directional Derivatives and the Gradient Vector**
  - The definition of the gradient. Note the gradient is the derivative of a function from R^n to R; it is a vector with n components, where the i^{th} component is the partial derivative of f in the direction of the i^{th} coordinate axis.
  - The definition of the directional derivative: This generalizes the partial derivatives we've discussed, and allows us to look at how a function is changing along an arbitrary line (but not an arbitrary curve). The definition even suggests a way to compute the directional derivative: use the chain rule.
  - The directional derivative of f in the direction of $\mathbf{v}$ at the point $\mathbf{x}$ is just the dot product of the gradient of f and v; in other words, the directional derivative is $(\nabla f)(\mathbf{x}) \cdot \mathbf{v}$.
  - Geometric interpretation of the gradient: the gradient points in the direction where f is increasing the fastest, and is perpendicular to level surfaces (we'll discuss this in much greater detail in class). These two items will be of great aid in optimization problems.
- **Section 12.9: Lagrange Multipliers and Constrained Optimization**
  - The method of Lagrange Multipliers: This is the key result: it says that if we want to find the extrema for a function f whose input $\mathbf{x}$ is the level set of some value for a function g (i.e., find the max/min of $f(\mathbf{x})$ given that $g(\mathbf{x}) = c$ for a fixed c), then this happens at points where the direction of the gradient of f is the same as the direction of the gradient of g. We may rewrite this condition and say that $\mathbf{x_0}$ is an extremum for f with our constraints if there is some number λ such that $(\nabla f)(\mathbf{x_0}) = \lambda\, (\nabla g)(\mathbf{x_0})$.
  - Caveats: Existence of solutions: Just because we've found candidates does not mean one of them

must work! Also, while the idea is straightforward, frequently the algebra needed to solve the problem can be tedious.
    - I will try and do several examples of applications of Lagrange multipliers.
- **Section 12.10: Critical Points of Functions of Two Variables**
    - Basically just be aware of Theorem 1, namely that there exist conditions to classify the nature of critical points. The formulas look quite strange, and will make a lot more sense after learning about eigenvalues in Linear Algebra.

CHAPTER 13: Multiple Integrals

- **General Comments:**
    - As many people have not seen a proof of the Fundamental Theorem of Calculus, I will prove this important result in full detail in class, and merely state what happens in several variables. We will loosely follow the book for this chapter. The reason is that, as we only have 12 weeks, we do not have time to delve fully into the theory of double and triple integrals. Instead, for this chapter we will concentrate on the applications, namely becoming proficient at computing these integrals.
- **Section 13.1: Double Integrals**
    - The definition of the double integral is very important; we'll discuss in great depth the corresponding framework in one-dimension. One can check the Fundamental Theorem of Calculus by using Mathematical Induction and limits to find the area under polynomial functions.
    - Any continuous function on a closed rectangle, such as [a,b] x [c,d] with a,b,c,d finite, is integrable. We will discuss the proof of a related, simpler statement. We will not prove this result in full generality, though the proof is in the book if you wish to read it / discuss it with me.
    - Be aware of the four properties of integrals (linearity, homogeneity, monotonicity and additivity). The proofs are similar to the proofs in the 1-dimensional case.
- **Section 13.2: Double Integrals over more general Regions**
    - Know the definition of the following terms: boundary, vertically simple, horizontally simple.
    - The main result is that the integral over the rectangle can be written as an iterated integral (remember the double integral is defined through boxes and limits).
    - Know the statement of Fubini's Theorem about when you can interchange orders of integration. We will not do the proof in class, though it is in the book.
- **Section 13.3: Area and Volume by Double Integrals**
    - Know the formulas to find volumes from integrating.
- **Section 13.4: Double Integrals in Polar Coordinates**
    - Know how to convert an integral in (x,y) space to one in (r,theta) space.
    - Unit analysis is a great guide, and suggests dx dy becomes r dr dtheta.
- **Section 13.6: Triple Integrals**
    - Essentially the same as double integrals.
- **Section 13.7: Integration in Cylindrical and Spherical Coordinates**
    - Know the change of variables from Cartesian to Cylindrical and Spherical.
    - Know how the volume element changes in each: r dr dtheta in the first, rho^2 sin(phi) dpho dphi dtheta in the second.
    - Know how to convert Cartesian integrals to Cylindrical or Spherical.
- **Special Topic: Monte Carlo Integration**
    - Monte Carlo Integration has been called one of the most useful results of 20th century mathematics. We'll discuss how it is done. It is an alternative to standard integration. Normally we look for anti-

derivatives; however, in the real world most functions we encounter do not have nice anti-derivatives; Monte Carlo Integration provides a way to approximate these integrals.

  - The lecture notes for this is not the book, but rather my lecture notes (the last three pages of my chapter 3 notes, namely pages 36-38).

- **Section 13.9: Change of Variables in Multiple Integrals**
  - Know the definition of Jacobian determinants.
  - Read the statement of the Change of Variables formula. We will not deal with this theorem in its full generality, but I want you to at least be aware of its statement. We will concentrate on several special cases: polar coordinates, cylindrical coordinates, and spherical coordinates.

CHAPTER 10 (Cain and Herod): SEQUENCES, SERIES AND ALL THAT: notes available here.

- **10.1: Introduction**
  - Just know that one motivation comes from Taylor series.
- **10.2: Sequences**
  - Know the definition of a sequence and some common examples.
  - Know what it means for a sequence to converge.
- **10.3: Series**
  - Know the definition of a series.
  - Know what it means for a series to converge.
  - Know the definition of the harmonic series.
  - Know the integral test.
- **10.4: More Series**
  - Know the definition of a positive series.
  - Know the comparison test for convergence (it's the method of this section; they don't call it the comparison test till the next section).
- **10.5: Even More Series**
  - Know the ratio test for convergence.
- **10.6: A Final Remark**
  - Know the alternating test for convergence.

From Line Integrals to Green's Theorem: TBD

Also, you may wish to look at some worked out examples before class that are similar to the HW. These are available online here, those these problems are from the 2010 version of the class, which used a different book; I will do many of these problems in class. The reason I want to do these is precisely because I have written up the solution. This way you can sit back a bit more and follow the example without worrying about writing everything down.