FOLLOWING ONLINE HANDOUT

SEQUENCES AND SERIES

GOAL: Many complex phenomena can be well-approx with a series expansion

EXAMPLES:
- Instantaneous interest
- Xeno's (Zeno?) Paradox
- Newton's Method
- Riemann Sums
  - ↳ many limits of sums have exact, closed form answer
    - ↳ proofs frequently follow by Induction

Mathematical Induction:

Given some statement $P(n)$ for $n \in \{0, 1, 2, \ldots\}$, to show $P(n)$ is true for all $n \in \{0, 1, 2, \ldots\}$ it suffices to show the following:

(1) Basis Step: $P(0)$ is true

(2) Inductive Step: Assuming $P(n)$ true Then $P(n+1)$ is true

Proof: $P(0)$ true

$\underline{P(0) \to P(1) \text{ true}}$ and so on and so on.

$P(1)$ true

$\underline{P(1) \to P(2) \text{ true}}$

$P(2)$ true

Notes on sequences and series

Sequence is a collection of terms

$$\{a_n\}_{n=0}^{\infty} = \{a_0, a_1, a_2, a_3 \ldots\}$$

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$$

Examples:

- $a_n = 1/n^2 \quad n = \cancel{a} 1, 2, 3 \ldots$

  $$\{a_n\}_{n=1}^{\infty} = \{1, 1/4, 1/9, 1/16, 1/25, \ldots\}$$

- $b_n = 1/n \quad n = 1, 2, 3, 4 \ldots$ (Harmonic sequence)

  $$\{b_n\}_{n=1}^{\infty} = \{1, 1/2, 1/3, 1/4, \ldots\}$$

- Fibonacci numbers:

  $a_0 = 1, \quad a_1 = 1, \quad a_{n+1} = a_n + a_{n-1}$

  Seq is $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

  $\{a_n\}_{n=1}^{\infty}$ with $a_n = n^2$

  ↳ would be $\{0, 1, 4, 9, 16, \ldots\}$

- Prime Euclid: OEIS:
- $3X+1$

Convergence of a sequence

Say a seq $\{a_n\}_{n=1}^{\infty}$ converges to L if $\lim_{n\to\infty} a_n = L$.

<u>Ex</u>: If $a_n = 1/n$ then the seq converges to 0 since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

<u>Ex</u>: If $a_n = 2^n$ then the limit does not exist and thus the sequence diverges:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^n = +\infty$$

↳ limit does not exist.

<u>Ex</u>: Let $a_n = \frac{3n^2 - 2n + 4}{4n^2 + 8n + 2}$. Does the seq converge?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2\left(3 - \frac{2}{n} + \frac{4}{n^2}\right)}{n^2\left(4 + \frac{8}{n} + \frac{2}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{3 - \frac{2}{n} + \frac{4}{n^2}}{4 + \frac{8}{n} + \frac{2}{n^2}}$$

as we have lim(numerator) = 3 and lim(denom) = 4, use quotient rule for limits:

Quotient rule for limits

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \frac{\lim_{n\to\infty} b_n}{\lim_{n\to\infty} c_n}$$

provided both limits $\lim_{n\to\infty} b_n$ and $\lim_{n\to\infty} c_n$ exist and are finite, and $\lim_{n\to\infty} c_n$ is not zero

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For our problem, just get 3/4

Alternatively, use L'Hopital

$$\to \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{6n-2}{8n+8}$$

$$= \lim_{n\to\infty} \frac{6}{8} = \frac{6}{8} = \frac{3}{4}$$

Note: can use L'Hopital so long as you have 0/0 or ∞/∞; once you no longer have this cannot use L'Hopital.

Ex: $a_n = \dfrac{n^2 + 2n^{3/2} - \log n}{n^3 + 1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n + 3n^{1/2} - \frac{1}{n}}{3n^2}$$

$$= \lim_{n\to\infty} \frac{2 + \frac{3}{2}n^{-1/2} + \frac{1}{n^2}}{6n}$$

$$= 0$$

Stop: No more L'Hopital as $0/\infty$

Note $0/\infty$ is okay: that's just 0

Defn of Series

A series is the sum of terms in a sequence. A partial sum is the sum of the first few terms:

n^{th} partial sum $S_n = \sum_{k=0}^{n} a_k$

Series is $\lim_{n\to\infty} S_n = \sum_{k=0}^{\infty} a_k$

Ex: If $a_n = (1/2)^n \quad n = 0, 1, 2, \ldots$

Then

$$\left.\begin{aligned} S_0 &= 1 \\ S_1 &= 1 + \tfrac{1}{2} \\ S_2 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} \\ S_3 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} \\ &\vdots \end{aligned}\right\} \text{Partial Sums}$$

$$S = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots \Big\} \text{ Series}$$

When does a series converge?

A neccessary condition for $\sum_{n=0}^{\infty} a_n$ to converge is if each $a_n \to 0$ (ie, $\lim_{n\to\infty} a_n = 0$, ie, the sequence converges to 0).

<u>Note</u>: If $\lim_{n\to\infty} a_n \neq 0$, series diverges

If $\lim_{n\to\infty} a_n = 0$, series <u>may</u> or <u>may not</u> converge

Ex: Consider the following sequence:

$$\begin{array}{cccccccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ 1, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{3}, & \frac{1}{3}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \underbrace{\frac{1}{5}, \dots, \frac{1}{5}}_{5\text{ times}}, & \dots \end{array}$$

Clearly $\lim_{n\to\infty} a_n = 0$

Note Series diverges: $S_1 = 1$, $S_3 = 2$, $S_6 = 3$, $S_{10} = 4$, $S_{15} = 5$, $S_{21} = 6$, $S_{28} = 7, \dots$

Geometric Series

Let $|r| < 1$, let $a_n = r^n$

Then $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

More generally, $\displaystyle\sum_{n=0}^{\infty} c\, r^n = \frac{c}{1-r}$

Ex: $r = \frac{1}{2}$: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = 2$

$r = 2$: $1 + 2 + 4 + 8 + \cdots$ diverges

↳ geometric series formula is for $|r| < 1$
Cannot use here as $r > 1$

(Standard)
Proof: $S_n = 1 + r + r^2 + \cdots + r^{n-1} + r^n$

$$r S_n = \quad r + r^2 + \cdots + r^n + r^{n+1}$$

$$S_n - r S_n = 1 - r^{n+1}$$

$$(1-r) S_n = 1 - r^{n+1}$$

$$S_n = \frac{1}{1-r} - \frac{r^{n+1}}{1-r}$$

$$\lim_{n\to\infty} S_n = \frac{1}{1-r} - \frac{1}{1-r}\lim_{n\to\infty} r^{n+1} = \frac{1}{1-r}$$

└ tends to 0 as $|r| < 1$

Geometric Series

$$S_n = 1 + r + r^2 + \cdots + r^n$$

$$\underline{rS_n =} \quad \underline{r + r^2 + \cdots + r^n + r^{n+1}}$$

$$(1-r)S_n = 1 - r^{n+1}$$

$$\Rightarrow S_n = \frac{1-r^{n+1}}{1-r}$$

↳ if $|r|<1$ Then $\lim_{n\to\infty} S_n = \frac{1}{1-r}$ ∎

Alternate proof.

<u>Basketball Game:</u> A and B alternate with A getting a basket with prob p, B with prob q. First basket wins. Set $r = (1-p)(1-q)$, let $X = \text{Prob}(A \text{ wins})$, assume $0<r<1$.

$$\text{Prob}(A\text{ wins}) = X = p + rp + r^2p + r^3p + \cdots$$
$$X = p(1 + r + r^2 + \cdots)$$

$$\text{Prob}(A\text{ wins}) = X = p + rX \Rightarrow (1-r)X = p \text{ so } X = \frac{p}{1-r}$$

$$\text{Thus } X = p(1+r+r^2+\cdots) = \frac{p}{1-r}$$

$$\text{or } 1 + r + r^2 + \cdots = \frac{1}{1-r}$$ ∎

↳ Note: Expanded calculation:

$$\text{Prob}(A\text{ wins}) = X = \underset{\substack{\uparrow \\ \text{wins on first shot}}}{p} + \underbrace{(1-p)(1-q)p}_{\substack{\text{wins on second shot, so sequence} \\ \text{is miss, miss, hit}}} + \underbrace{(1-p)(1-q)(1-p)(1-q)p}_{\substack{\text{wins on third shot, so sequence of shots is} \\ \text{miss, miss, miss, miss, hit}}} + \cdots$$

Comparison and Squeeze Tests

Comparison Test: Sequence $\{b_n\}_{n=0}^{\infty}$ of positive terms whose sum converges, assume seq $\{a_n\}_{n=0}^{\infty}$ satisfies $|a_n| \leq b_n$. Then seq $\{a_n\}_{n=0}^{\infty}$ has a convergent sum, i.e., $\lim_{n\to\infty} \sum_{k=0}^{n} a_k$ exists and is finite

Example: Let $b_n = 1/2^n$ and $a_n = (-1)^n/2010 \cdot n!$
Then $\sum_{n=0}^{\infty} a_n$ exists (it actually equals $\frac{1}{2010e}$)

↳ Note: Can weaken comparison tests: all that matters is that have $|a_n| \leq b_n$ for all $n \geq n_0$

Squeeze Thm: Assume $a_n \leq b_n \leq c_n$ and $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} c_n$ exist and are finite. Then $\sum_{n=0}^{\infty} b_n$ exists and is finite.

Ex: $\sum_{n=1}^{\infty} \frac{1}{4^n - 3^n}$ Converges

Reason: Claim n big:

$$4^n > 2 \cdot 3^n$$

$\hookrightarrow$ take logs: $n \log 4$ vs $n \log 3 + \log 2$

or $n \log \frac{4}{3}$ vs $\log 2$

or n vs $\frac{\log 2}{\log \frac{4}{3}}$

So if $n > \frac{\log 2}{\log \frac{4}{3}}$, win!

$$4^n > 2 \cdot 3^n \Rightarrow 4^n - 3^n > 2 \cdot 3^n - 3^n$$

or $4^n - 3^n > 3^n$

or $\frac{1}{3^n} > \frac{1}{4^n - 3^n}$ comparison test

CONVERGENCE TESTS

RATIO TEST: Consider sequence $\{a_n\}$ of positive terms.
The series $\sum_{m=0}^{N} a_m$ converges as $N \to \infty$ if
$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$; if $r > 1$ then the series diverges

Suppose $r<1$, let $\rho = r + \frac{1-r}{2} < 1$

Note that as $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$ and $\rho < r$, for all

n sufficiently large we have $\frac{a_{k+1}}{a_k} < \rho$

Thus $a_{k+1} < \rho a_k$

$$a_{k+2} < \rho a_{k+1} < \rho^2 a_k$$

$$a_{k+3} < \rho a_{k+2} < \rho^2 a_{k+1} < \rho^3 a_k$$

$$\vdots$$

$$a_{k+l} < \rho^l a_k$$

So
$$\sum_{m=0}^{N} a_m = \sum_{m=0}^{k-1} a_m + \sum_{m=k}^{N} a_m$$
$$= \sum_{m=0}^{k-1} a_m + \sum_{l=0}^{N-k} a_{k+l}$$
$$< \sum_{m=0}^{k-1} a_m + a_k \sum_{l=0}^{N-k} \rho^l$$

↳ Using the Comparison Test

Ex: Try $a_n = r^n$ with Ratio Test

Ex: Try $a_n = \frac{1}{n}$ or $\frac{1}{n^2}$ with Ratio Test
↳ sadly get $\rho = 1$ both times

Ex: Try $a_n = \frac{1}{n!}$: get $\rho = 0$

Proof: $\rho = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$= \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \cdot \frac{n!}{n!}$$

or use $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!}$$

$3! = 3 \cdot 2 \cdot 1$
$4! = 4 \cdot 3!$
$0! = 1$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0$$

~~[illegible]~~ ROOT TEST

ROOT TEST: LET $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive terms. Then the series $\lim_{N\to\infty} \sum_{n=0}^{N} a_n$ converges if $\lim_{n\to\infty} a_n^{1/n}$ converges to a $\rho < 1$ and diverges if $\rho > 1$.

Proof is similar to that of ratio test.

For n large, $a_n^{1/n} < r$ for some $r < 1$; holds for all large n

↳ Thus for n large, $a_n < r^n$ and win by Comparison Test

↳ if $\rho > 1$, similar argument shows diverge

↳ now $a_n^{1/n} > r > 1$ for all n large ∎

NOTE: RATIO AND ROOT TEST PROVIDE NO INFORMATION IF CORRESPONDING LIMIT IS 1

Example: p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \le 1$

↳ ratio: $\lim_{n\to\infty} \frac{1}{(n+1)^p} \Big/ \frac{1}{n^p} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^p = 1$$

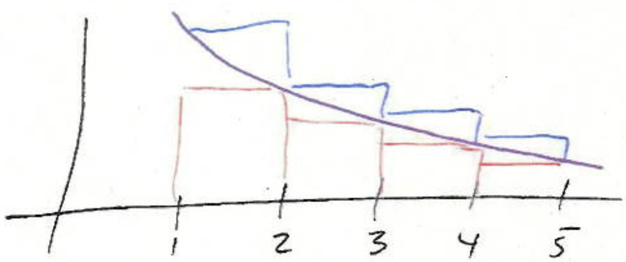
↳ root: also get 1

INTEGRAL TEST

INTEGRAL TEST: Sequence $\{a_n\}_{n=0}^{\infty}$ of non-neg terms, assume $f(n) = a_n$ and $f(x)$ is ~~continuous~~ decreasing and integrable. Then $\sum_{n=0}^{\infty} a_n$ and $\int_0^{\infty} f(x)\,dx$ either both converge or diverge; in many applications start at 1 and not 0.

Note clearly can fail if sequence a_n is not decreasing

Proof: Squeeze Theorem



$$\sum_{n=2}^{\infty} a_n \le \int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} a_n$$

Example: p-series

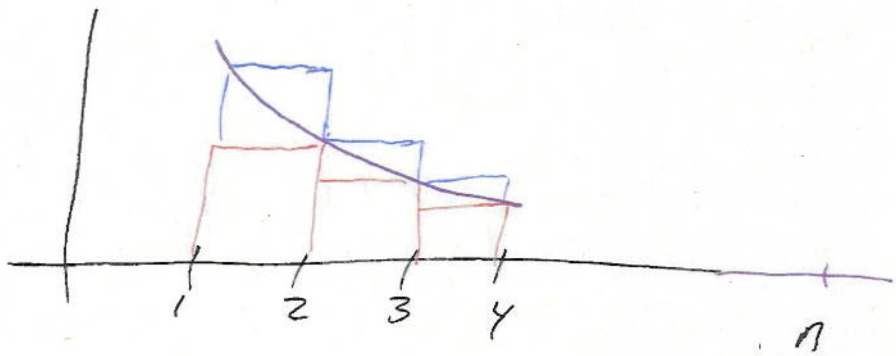
Convergence/divergence of $\sum_{n=1}^{\infty} 1/n^p$ from $\int_1^{\infty} 1/x^p\,dx$

↳ have $\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \left.\frac{x^{1-p}}{1-p}\right|_1^{\infty} & \text{if } p \neq 1 \\ \ln x \,\Big|_1^{\infty} & \text{if } p = 1 \end{cases}$

↳ Thus converges if $p > 1$, diverges if $p \le 1$

APPROXIMATING SUMS WITH INTEGRALS

Can use the integral test to approximate a sum with an integral, and have some control over the error.



Find $\sum_{k=2}^{n} \frac{1}{k} < \int_1^n \frac{1}{x}\,dx < \sum_{k=1}^{n} \frac{1}{k}$

so $\left(\sum_{k=1}^{n} \frac{1}{k}\right) - 1 < \log n < \sum_{k=1}^{n} \frac{1}{k}$

↳ Thus $\sum_{k=1}^{n} \log \frac{1}{k}$ is within 1 of $\log n$ (and a bit larger). Have $\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \cdots$

where $\gamma \approx .5772156$ is the Euler-Mascheroni constant.

<u>Other examples</u>: Stirling's formula for $n!$, very important in probability

<u>Generalization</u>: Better summation formulas (Simpson's rule, Euler-Maclaurin) that (over/under) estimate less do better.

ALTERNATING SERIES

We say a series is absolutely convergent if $\sum_{n=0}^{\infty} |a_n|$ exists and is finite. If $\sum_{n=0}^{\infty} a_n$ exists and is finite but $\sum_{n=0}^{\infty} |a_n|$ diverges (ie, is infinite) we say the sum is conditionally convergent.

↳ Example: $a_n = (-1)^n / n$ is only conditionally convergent

↳ Clear $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

↳ $$\sum_{n=1}^{\infty} a_n = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$$

$$= -1 + \frac{1}{2\cdot 3} + \frac{1}{4.5} + \frac{1}{6.7}$$

$$= -1 + \underbrace{\sum_{n=1}^{\infty} \frac{1}{2n(2n+1)}}_{\text{dominated by } \sum_{n=1}^{\infty} \frac{1}{4n^2} < \infty}.$$

note $-1, -1+\frac{1}{2}-\frac{1}{3}, \ldots < \sum_{n=1}^{\infty} a_n < -1+\frac{1}{2}, -1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}$

<u>THM</u>: If $\sum_{n=1}^{\infty} a_n$ is a alternating sum with $|a_n| \to 0$ then the sum converges

Rearrangement Thm: If $\sum_{n=1}^{\infty} a_n$ converges conditionally but not absolutely, can reorder the terms so that the sum converges to any number you wish!