Math 150: Multivariable Calculus: Steven J Miller, Spring 2020

COURSE DESCRIPTION: Applications of calculus in mathematics, science, economics, psychology, the social sciences, involve several variables. This course extends calculus to several variables: vectors, partial derivatives, multiple integrals. There is also a unit on infinite series, sometimes with applications to differential equations. This course is the right starting point for students who have seen differentiation and integration before. Students with the equivalent of advanced placement of AB 4, BC 3 or above should enroll in Mathematics 150. Prerequisites: Mathematics 140 *or* equivalent, such as satisfactory performance on an Advanced Placement Examination. *No enrollment limit (expected: 45). NOTE: We will be moving at a very fast pace. You should spend at least one if not two hours a day (every day!) on this course. I strongly encourage you to work in groups, and you should skim the reading before each class. We will not cover all the material in the book in class; you are responsible for reading the other examples at home.*

GRADING / HW: Homework 15%, Midterms 40% (there will be 2 or 3), Final 45%. Homework is to be handed in on time, stapled and legible; there will be HW due each class. *Late, messy or unstapled homework will not be graded.* I encourage you to work in small groups, but everyone must submit their own homework assignment. All exams are cumulative, the lowest midterm grade will be dropped. There is also another options, worth 5%. Doing that reduces everything else to 95% of your grade. Project: You may explore a topic in multivariable calculus in great detail and write it up.

FLIPPED SPECIAL: One of the greatest challenges with multivariable calculus is the large amount of material and the small amount of time to combat it; in high school one meets twice more per week and for more weeks. To combat this the class will be flipped. You are expected to read the relevant sections before lecture and watch the video from the 2014 class. If you have any questions about the material / items you want to see in class, you are to enter that on the googlesheet or email me. If you are not prepared for class you are to let me know; it is on the honor system, everyone can have three such days without penalty, so long as you let me know before class. Doing this will allow us to use classtime more effectively. Also, more importantly, it will help you cement a very important life skill: learning how to learn. One of the strongest items I can say in letters of recommendation is that a student can pick up material. Doing this will also give you enormous control and personalization of your education. **Doing this adjusts the grading as follows: the items above now are rescaled to count for 40%, and you will receive a participation grade of 92.5 (right on the A-/A boundary) worth 60% (with the provision that you must pass the final to pass the course).**

SYLLABUS GENERAL: The textbook is the seventh edition of Edwards and Penney: Calculus (Early Transcendentals). This should be the textbook used in Math 104. You may use either the 7th edition or the 6th; unfortunately, while the content is essentially the same, the page numbering and chapter labeling differ, and you are responsible for making sure you do the right problems (I'll try and make sure the problems are the same, but it is your responsibility to make sure you do the right ones). There will also be supplemental handouts. *Please read the relevant sections before class. This means you should be familiar with the definitions as well as what we are going to study; this does not mean you should be able to give the lecture.* You do not need a calculator for this class, though I strongly urge you to become familiar with either Matlab or Mathematica to plot some of the multi-dimensional objects. There are many good references on the web. You can access certain books online: Calculus in Vector Spaces (Lawrence J. Corwin, Robert Henry Szczarba) and Multivariable Calculus (Lawrence J. Corwin, Robert Henry Szczarba) are somewhat theoretical expositions. Another great source is Cain and Herod's book on multivariable calculus (which you can download in its entirety for free). If you have any concerns or suggestions for the course and would prefer to communicate them anonymously, you may email me by using the account ephsmath@gmail.com (the password is the first eight Fibonacci numbers, 011235813); sadly google often prevents new logins.

We will cover most of chapters 11, 12 and 13, and supplemental material on sequences and series. Here are the key points from the different sections.

CHAPTER 11: Vectors, Curves and Surfaces in Space

- **Section 11.1: Vectors in the Plane**
  - Notation, definition of vectors and properties.
  - Proof of the Pythagorean formula (which is crucial in determining lengths).
- **Section 11.2: Three-Dimensional Vectors**
  - Know the definition of the dot product of two vectors, and the connection of that to the angle between two vectors.
- **Section 11.3: The Cross Product of Vectors**
  - Know the definition of determinants of 2x2 and 3x3 matrices, and how to compute these.
  - The determinant has much geometrical meaning, denoting the (signed) volume of the parallelpiped spanned by the rows (or columns).
  - Know the definition of the cross product and how to compute it, as well as some of its properties.
- **Section 11.4: Lines and Planes in Space**
  - Know the various formulas for writing the equation of a line.
  - There are several ways to write the equation of a plane; it's similar to writing the equation of a line: depending what information you are given, some ways are more convenient than others.
  - One easy way to find the equation of a plane is to know the normal direction. This is a great application of the cross product, as **v** x **w** is perpendicular to both **v** and **w**. Unfortunately we don't have the cross product in higher dimensions.
- **Section 11.8: Cylindrical and Spherical Coordinates**
  - Know the different formulas to convert from Cartesian to Cylindrical or Spherical coordinates.

CHAPTER 12: Partial Differentiations

- **Section 12.1: Introduction**
  - Not much here except (what a surprise) that many functions in the real world depend on several variables.
- **Section 12.2: Functions of Several Variables**
  - First is when a function is defined on a domain (usually just making sure the denominator is non-zero).
  - The level set (of value c) of a function are all inputs that are mapped to c. Think of this as all points on a mountain that are the same height, or on a weathermap all places with the same temperature.
- **Section 12.3: Limits and Continuity**
  - Know the definition and basic properties of limits.
  - Caveats: certain operations are not defined: ∞-∞, 0 * ∞.
- **Section 12.4: Partial Derivatives**
  - Know the definition of how to take a partial derivative. Similar to one-variable calculus, we do not want to have to use this definition in practice, and thus want to modify our rules of one-variable differentiation to allow us to take derivatives here.
  - Know the formula for computing the tangent plane to z = f(x,y) at a given point, so long as the partial derivatives exist at that point.
  - Iterated Partial Derivatives:
    - The definition of mixed partial derivatives: Given a function f, we can compute its partial derivatives, such as δf/δx and δf/δy. We can then take the partial derivatives of the partial derivatives: δ(δf/δx)δy and δ(δf/δy)δx. In the first, we first take the derivative with respect to x, and then take the derivative with respect to y; in the second, we take the derivatives in the other order. Does the order matter? We write $\delta^2 f/\delta y\delta x$ for δ(δf/δx)δy; thus the derivative symbol on the far right of the denominator is the derivative we take first, and the symbol on the far left is what we take last.
    - The definition of C^2, the class of twice continuously differentiable functions: If the function is C^2, the mixed partial derivatives of second order (i.e., involving at most two derivatives) exist and are continuous. Just as C^1 functions had nice properties (being

C^1 means the partial derivatives exist and are continuous, which implies the function is differentiable), being C^2 has nice properties.

  - Equality of Mixed Partial Derivatives: For a C^2 function, the order of differentiation does not matter; in other words, $\delta^2 f/\delta y\delta x = \delta^2 f/\delta x\delta y$.
  - Notation: f_x means $\delta f/\delta x$, f_{xy} means $(f_x)_y$, which is $\delta^2 f/\delta y\delta x$. Note that the order of subscripts is the opposite of the order of differentiation; fortunately if f is C^2 then the order does not matter!
  - Examples of partial differential equations: The rest of the section is devoted to examples of partial differential equations. Solving these in general are beyond the scope of this course; in fact, most are beyond the scope of humanity! One example is the Millenium Prize for the Navier-Stokes equation (i.e., solve this and earn $1,000,000). You are not responsible for any of this material; it is provided in nice detail in this book for your interest, and to show you what you will see if you continue with mathematics.

- **Section 12.5: Multivariable Optimization Problems**
  - Advanced result: any continuous function on a nice region that includes the boundary attains its maximum and minimum.
  - Definition of local maximum / minimum: You should be comfortable with the definition of a local max/min. Essentially, a point $\mathbf{x_0}$ is a local maximum if there is some ball centered at $\mathbf{x_0}$ such that $f(\mathbf{x_0})$ is at least as large as $f(\mathbf{x})$ for all other $\mathbf{x}$ in that ball. For example, $f(x,y) = y^2 \sin^2(xy)$ has a local minimum at $(x,y) = (0,0)$. Clearly $f(x,y)$ is never negative, and it is zero at $(0,0)$. Thus $(0,0)$ is a local minimum. Note that $f(x,0)$ is also zero for *any* choice of x. Thus to be a local minimum we don't need to be strictly less than all other nearby points.
  - First derivative test for local extrema: The generalization of the results from one-variable calculus; candidates for max/min are where the first derivative (the gradient) vanishes.
  - Important Example: The Method of Least Squares: We will give one of the most important applications of partial derivatives and optimization, the Method of Least Squares. This is a technique to allow us to find best fit parameters. Finding such values is central in numerous disciplines. Specifically, we have some data and we want to see if it fits our theory. If you have a data set you'd like analyzed, please let me know.
- **Method of Least Squares**
  - My notes on the Method of Least Squares
  - There are many different ways to choose how we measure errors. Different choices lead to different `best fit' values for parameters. The main advantage of measure errors by summing the squares of the deviations is that the tools of calculus and linear algebra are available.
- **Section 12.6: Increments and Linear Approximations**
  - The idea is to replace complicated functions with simpler ones that are easier to analyze (in many cases, one can get very good results just by using linear approximations).
  - Newton's method is one of the most important uses of the tangent line. The idea is based on the fact that locally any function is approximately linear.
- **Section 12.7: The Multivariable Chain Rule**
  - The most important part of this section is the statement of the chain rule. A good way to remember what goes where is through the atom graph.
- **Section 12.8: Directional Derivatives and the Gradient Vector**
  - The definition of the gradient. Note the gradient is the derivative of a function from R^n to R; it is a vector with n components, where the i^{th} component is the partial derivative of f in the direction of the i^{th} coordinate axis.
  - The definition of the directional derivative: This generalizes the partial derivatives we've discussed, and allows us to look at how a function is changing along an arbitrary line (but not an arbitrary curve). The definition even suggests a way to compute the directional derivative: use the chain rule.
  - The directional derivative of f in the direction of $\mathbf{v}$ at the point $\mathbf{x}$ is just the dot product of the gradient of f and v; in other words, the directional derivative is $(\nabla f)(\mathbf{x}) \cdot \mathbf{v}$.
  - Geometric interpretation of the gradient: the gradient points in the direction where f is increasing the fastest, and is perpendicular to level surfaces (we'll discuss this in much greater detail in class). These two items will be of great aid in optimization problems.

- **Section 12.9: Lagrange Multipliers and Constrained Optimization**
  - The method of Lagrange Multipliers: This is the key result: it says that if we want to find the extrema for a function f whose input **x** is the level set of some value for a function g (i.e., find the max/min of f(**x**) given that g(**x**) = c for a fixed c), then this happens at points where the direction of the gradient of f is the same as the direction of the gradient of g. We may rewrite this condition and say that $\mathbf{x_0}$ is an extremum for f with our constraints if there is some number λ such that $(\nabla f)(\mathbf{x_0}) = \lambda (\nabla g)(\mathbf{x_0})$.
  - Caveats: Existence of solutions: Just because we've found candidates does not mean one of them must work! Also, while the idea is straightforward, frequently the algebra needed to solve the problem can be tedious.
  - I will try and do several examples of applications of Lagrange multipliers.
- **Section 12.10: Critical Points of Functions of Two Variables**
  - Basically just be aware of Theorem 1, namely that there exist conditions to classify the nature of critical points. The formulas look quite strange, and will make a lot more sense after learning about eigenvalues in Linear Algebra.

CHAPTER 13: Multiple Integrals

- **General Comments:**
  - As many people have not seen a proof of the Fundamental Theorem of Calculus, I will prove this important result in full detail in class, and merely state what happens in several variables. We will loosely follow the book for this chapter. The reason is that, as we only have 12 weeks, we do not have time to delve fully into the theory of double and triple integrals. Instead, for this chapter we will concentrate on the applications, namely becoming proficient at computing these integrals.
- **Section 13.1: Double Integrals**
  - The definition of the double integral is very important; we'll discuss in great depth the corresponding framework in one-dimension. One can check the Fundamental Theorem of Calculus by using Mathematical Induction and limits to find the area under polynomial functions.
  - Any continuous function on a closed rectangle, such as [a,b] x [c,d] with a,b,c,d finite, is integrable. We will discuss the proof of a related, simpler statement. We will not prove this result in full generality, though the proof is in the book if you wish to read it / discuss it with me.
  - Be aware of the four properties of integrals (linearity, homogeneity, monotonicity and additivity). The proofs are similar to the proofs in the 1-dimensional case.
- **Section 13.2: Double Integrals over more general Regions**
  - Know the definition of the following terms: boundary, vertically simple, horizontally simple.
  - The main result is that the integral over the rectangle can be written as an iterated integral (remember the double integral is defined through boxes and limits).
  - Know the statement of Fubini's Theorem about when you can interchange orders of integration. We will not do the proof in class, though it is in the book.
- **Section 13.3: Area and Volume by Double Integrals**
  - Know the formulas to find volumes from integrating.
- **Section 13.4: Double Integrals in Polar Coordinates**
  - Know how to convert an integral in (x,y) space to one in (r,theta) space.
  - Unit analysis is a great guide, and suggests dx dy becomes r dr dtheta.
- **Section 13.6: Triple Integrals**
  - Essentially the same as double integrals.
- **Section 13.7: Integration in Cylindrical and Spherical Coordinates**
  - Know the change of variables from Cartesian to Cylindrical and Spherical.
  - Know how the volume element changes in each: r dr dtheta in the first, rho^2 sin(phi) dpho dphi dtheta in the second.
  - Know how to convert Cartesian integrals to Cylindrical or Spherical.
- **Special Topic: Monte Carlo Integration**
  - Monte Carlo Integration has been called one of the most useful results of 20th century mathematics. We'll discuss how it is done. It is an alternative to standard integration. Normally we

look for anti-derivatives; however, in the real world most functions we encounter do not have nice anti-derivatives; Monte Carlo Integration provides a way to approximate these integrals.
    - The lecture notes for this is not the book, but rather my lecture notes (the last three pages of my chapter 3 notes, namely pages 36-38).
- **Section 13.9: Change of Variables in Multiple Integrals**
    - Know the definition of Jacobian determinants.
    - Read the statement of the Change of Variables formula. We will not deal with this theorem in its full generality, but I want you to at least be aware of its statement. We will concentrate on several special cases: polar coordinates, cylindrical coordinates, and spherical coordinates.

CHAPTER 10 (Cain and Herod): SEQUENCES, SERIES AND ALL THAT: notes available here.

- **10.1: Introduction**
    - Just know that one motivation comes from Taylor series.
- **10.2: Sequences**
    - Know the definition of a sequence and some common examples.
    - Know what it means for a sequence to converge.
- **10.3: Series**
    - Know the definition of a series.
    - Know what it means for a series to converge.
    - Know the definition of the harmonic series.
    - Know the integral test.
- **10.4: More Series**
    - Know the definition of a positive series.
    - Know the comparison test for convergence (it's the method of this section; they don't call it the comparison test till the next section).
- **10.5: Even More Series**
    - Know the ratio test for convergence.
- **10.6: A Final Remark**
    - Know the alternating test for convergence.

From Line Integrals to Green's Theorem:

- One day introduction to Green’s theorem and its generalization.