MATH 105: PRACTICE PROBLEMS FOR CHAPTER 12: SPRING 2013

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Question 1 : These problems deal with open sets. Open sets were covered in Spring 2010 but not that much in Spring 2011 or 2013, so do not worry about these problems if you're in 105 in Spring 2011 or 2013; these are included for general interest.

(1) Let $S = \{(x, y, z) : 3x^2 + 4y^2 + 5z^2 < 6\}$. Is S open?
(2) Let $S = \{(x, y) : x^2 - y^2 = 1\}$. Is S open?
(3) Let $S = \{(x_1, \dots, x_n) : x_1^2 + \cdots + x_n^2 < 1\}$. Is S open?
(4) Let $S = \{(x, y, z) : x^2 + y^2 \le z\}$. Is S open?
(5) Let $S = \{(x, y) : xy = 1\}$. Is S open?
(6) Let $S = \{(x, y) : x^2 + y^2 > 1\}$. Is S open?

Question 2 : Compute the following limits (if they exist), or prove they do not. Remember $\log x$ means the logarithm of x base e.

(1) $\lim_{x\to 1}(x^4 - 2x^3 + 3x^2 + 4x - 5)$.
(2) $\lim_{x\to 2} \sin(3x^2 - 12)$.
(3) $\lim_{x\to 2} \frac{\sin(3x^2-12)}{x-2}$.
(4) $\lim_{x\to 0} \frac{\log x}{x}$.
(5) $\lim_{x\to 0} \frac{x}{\log x}$.
(6) $\lim_{(x,y)\to(0,0)}(4xy\cos(xy) + x^2 - y^3)$.
(7) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2-1}{xy-1}$.
(8) $\lim_{(x,y)\to(1,1)} \frac{x^2y^2-1}{xy-1}$.
(9) $\lim_{(x,y)\to(0,0)} \frac{x^4-x^2y^2+y^4}{x^2+y^2+x^4y^4}$.
(10) $\lim_{(x,y)\to(0,0)} x^2y^3\cos\left(\frac{1}{x^2+y^2}\right)$.
(11) $\lim_{(x,y)\to(0,0)} \frac{x^2y^3\cos\left(\frac{1}{x^2+y^2}\right)}{x^2+y^2}$.
(12) $\lim_{(x,y,z)\to(0,0,0)} \frac{x^3+y^3+z^3}{x^2+y^2+z^2}$.

Question 3 : Plot the level sets of value c for each function below (do enough values of c so you can recognize the result).

(1) $f(x, y) = \sin(x + y)$.
(2) $f(x, y) = (x + y)\sin(x + y)$.
(3) $f(x, y) = x^2 - 4y^2$.
(4) $f(x, y) = x^2 + 4y$.

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(5) $f(x,y) = e^{\cos x}$.

Question 4 : Find the gradients of the following functions:

(1) $f(x,y,z) = xy + yz + zx$.
(2) $f(x,y) = x\cos(y) + y\cos(x)$.
(3) $f(x_1,\dots,x_n) = x_1x_2\cdots x_n$.
(4) $f(x,y,z) = 1701x^{24601}\log(1793x^5y^4)$.
(5) $f(x,y) = \sin(x^2+y^2)$.

Question 5: Determine which functions below are differentiable. To be differentiable the tangent plane is supposed to do an excellent job approximating the function. A sufficient condition to ensure the function is differentiable is that the partial derivatives all exist and are continuous. This concept was covered more in Spring 2010 than later years, so if you are taking this in Spring 2011 or later do not worry as much about this problem.

(1) $f(x,y,z) = (xyz)^{4/3}$.
(2) $f(x,y) = (xy)^{2/3}$.
(3) $f(x_1,\dots,x_n) = (x_1x_2\cdots x_n)^2$.
(4) $f(x,y,z) = 1701x^{24601}\log(1793x^5y^4)$.
(5) $f(x,y) = \sin(x^2+y^2)$.
(6) $f(x,y,z) = x^3\cos(x) + y^3\cos(y)$.
(7) $f(x,y) = xy\cos(1/y)$.

Question 6 : Find the tangent plane approximation to $f(x,y) = e^{xy} + 2\sin(x+y)\cos(x-y)$ at the point (x_0,y_0). Use the tangent plane plane to estimate $f(-.01,.02)$ by choosing (x_0,y_0) appropriately.

Question 7 : Parametrize the following curves, and find the tangent line approximation at the given point. *Note: we haven't done too much with parametrization yet in 2013; I'm shifting the order of the material a bit. A curve in 3-dimensional space is parametrized by a curve c if $c(t) = (x(t), y(t), z(t))$ traces out the curve.*

(1) A circle of radius 5 centered at $(2,3)$ going counter-clockwise starting at the point $(7,3)$; find the tangent line at the point $(7,3)$.

(2) A circle of radius 5 centered at $(2,3)$ going counter-clockwise starting at the point $(5,7)$; find the tangent line at the point $(5,7)$.

(3) The curve $y = e^x$ from $x = 1$ to $x = 10$; find the tangent line at the point $(2,e^2)$.

Question 8 : Consider the parametrized curve $c(t) = (\cos t, 2\sin t)$. What path does this trace out in the plane? Is it periodic (i.e., does it repeat where it is), and if so, what is the period? Does a particle whose position is given by $c(t)$ move at constant speed? If not, when is it moving fastest?

Question 9 : Find the derivative of $A(x, y, z) = (x^3 y + y^2 z + e^x)(\sin(y) + \cos(y) - z + 10)$.

Question 10 : Let $g(x, y, z) = (xy, yz, xz)$ and $f(u, v, w) = u^2 + v^2$. Set $A(x, y, z) = f(g(x, y, z))$. Compute DA.

Question 11 : Let $g(x, y, z) = xy^2z^3$. Compute the directional derivative of g at $(1, 1, 1)$ in the direction $\overrightarrow{v}$, where $\overrightarrow{v}$ is a unit vector in the direction $(3, 4, 12)$. In what direction is g increasing fastest?

Question 12 : Let $g(x, y, z) = e^x \cos \pi y + z \cos \pi x$. If possible, find the tangent plane to the level set of value 1 for $g(x, y, z)$ at $(x, y, z) = (1, 1, 1)$. If possible, find the tangent plane to the level set of value 0 for $g(x, y, z)$ at $(x, y, z) = (0, 1, 1)$.